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An explicit formula for derivative polynomials of the tangent function

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Abstract. In the paper, the authors derive an explicit formula for derivative polynomials of the tangent function, deduce an explicit formula for tangent numbers, pose an open problem about obtaining an alternative and explicit formula for derivative polynomials of the tangent function, and recommend some papers closely related to derivative polynomials of other elementary and applicable functions.

1 Introduction

It is not difficult to see that if f is a function whose derivative is a polynomial in f, that is, $f'(x) = P_1(f(x))$ for some polynomial P_1, then all the higher order derivatives of f are also polynomials in f, so we have a sequence of polynomials

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P_n defined by $f^{(n)}(x) = P_n(f(x))$ for $n \ge 0$. As usual, we call $P_n(u)$ the derivative polynomials of f. In fact, the polynomials P_n are determined by

$$P_0(u) = u, \quad P_{n+1}(u) = P_n'(u)P_1(u), \quad n \in \mathbb{N}.$$

For detailed information, please refer to [8, Section 2].

In 1945, Morley [10] observed that

$$\begin{gathered}(\tan x)' = 1 + \tan^2 x, \quad (\tan x)'' = 2\tan x + 2\tan^3 x, \\ (\tan x)''' = 2 + (2 + 2\cdot 3)\tan^2 x + 2\cdot 3\tan^4 x,\end{gathered} \tag{1}$$

a term $a_k \tan^k x$ in $(\tan x)^{(n)}$ gives $(\tan x)^{(n+1)}$ $ka_k \tan^{k-1} x + ka_k \tan^{k+1} x$, and then concluded that the coefficient of $\tan^{k-1} x$ in $(\tan x)^{(n+1)}$ is $(k-2)a_{k-2} + ka_k$, with $a_{k-2} = 0$ when $k \le 1$, and $a_k = 0$ when $k \ge n+2$.

In 1995, Hoffman [8, p. 25, (5)] obtained that the derivative polynomials P_n for the tangent function $\tan x$ defined by

$$\frac{d^n(\tan x)}{d\,x^n} = P_n(\tan x)$$

for $n \ge 0$ are polynomials of degree $n+1$ and satisfy the recurrence relation

$$P_{n+1}(u) = \sum_{k=0}^{n} \binom{n}{k} P_k(u)P_{n-k}(u) + \delta_{0n},$$

where

$$P_0(u) = u, \quad P_1(u) = 1 + u^2, \quad \text{and} \quad \delta_{ij} = \begin{cases} 0, & i \ne j; \\ 1, & i = j. \end{cases}$$

In [1, 9, 12, 26, 27, 32, 36], there are some explicit formulas and recurrence relations for the nth derivatives of trigonometric functions and other elementary functions. In [3, 4, 5, 20, 21, 26, 30, 33], there are some inequalities for trigonometric functions and other elementary functions. Specially, there are some explicit formulas and many other results on the nth derivative of the tangent function $\tan x$ in [11, 14].

Motivated by those results in [8, 10] and other references mentioned above, we are interested in the question: can one find explicit formulas for coefficients a_k of the derivative polynomials $P_n(u)$ for the tangent function $\tan x$?

The aim of this paper is to answer the above question. Our main results can be stated as the following theorem.

Theorem 1 *For $n \geq 0$, the derivative polynomials $P_n(u)$ of the tangent function $u = \tan x$ can be explicitly computed by*

$$P_n(u) = \sum_{k=0}^{\frac{1}{2}\left[n+\frac{1-(-1)^n}{2}\right]} a_{n,n+1-2k} u^{n+1-2k} \tag{2}$$

with

$$a_{2m-1,0} = (-1)^m \sum_{\ell=1}^{2m} (-1)^\ell 2^{2m-\ell} (\ell-1)! S(2m,\ell) \tag{3}$$

for $m \geq 1$ and

$$a_{n,n+1-2k} = (-1)^{k-1} \sum_{\ell=n+1-2k}^{n+1} (-1)^{n-\ell} 2^{n+1-\ell} (\ell-1)! \binom{\ell}{n+1-2k} S(n+1,\ell)$$

for $0 \leq k \leq \frac{1}{2}\left[n - \frac{1-(-1)^n}{2}\right]$, where $S(n,k)$ for $n \geq k \geq 1$ stand for the Stirling numbers of the second kind which can be generated by

$$\frac{(e^x-1)^k}{k!} = \sum_{n=k}^{\infty} S(n,k) \frac{x^n}{n!}, \quad k \in \mathbb{N}.$$

In Section 3 of this paper, we will pose an open problem about obtaining an alternative and explicit formula

$$a_{n,n-2m+1} = (n+1)! \sum_{\ell=0}^{m-1} (-1)^{m-1-\ell} b_{m,\ell} n^\ell, \quad n \geq 2, \quad 1 \leq m \leq \frac{1}{2}\left[n - \frac{1-(-1)^n}{2}\right] \tag{4}$$

for derivative polynomials $P_n(x)$ of the tangent function $\tan x$, where $b_{m,\ell}$ is a sequence to be determined.

In the final section of this paper, we give a consequence of Theorem 1 and recommend some papers closely related to derivative polynomials of other elementary and applicable functions.

2 Proof of Theorem 1

Now we start out to simply prove our Theorems 1 as follows.

In [36, Theorem 2.1] and [36, Corollaries 2.1 and 2.2], it was obtained that

$$(\tan x)^{(n)} = (-i)^{n+1}\sum_{k=1}^{n+1} 2^{n+1-k}(k-1)!S(n+1,k)(i\tan x-1)^k,$$

$$(\tan x)^{(n)} = (\tan x+i)\sum_{k=1}^{n}(2i)^{n-k}k!S(n,k)(\tan x-i)^k,$$

and

$$(\tan x)^{(n)} = \sum_{k=0}^{n+1}\Biggl[(-1)^{k+1}\cos\biggl(\frac{n+1+k}{2}\pi\biggr) \times \sum_{\ell=\max\{1,k\}}^{n+1}(-1)^{n-\ell}2^{n-\ell+1}(\ell-1)!S(n+1,\ell)\binom{\ell}{k}\Biggr]\tan^k x. \quad (5)$$

The identity (5) can be reformulated as

$$(\tan x)^{(n)} = -\cos\biggl(\frac{n+1}{2}\pi\biggr)\sum_{\ell=1}^{n+1}(-1)^{n-\ell}2^{n-\ell+1}(\ell-1)!S(n+1,\ell) + \sum_{k=1}^{n+1}\Biggl[(-1)^{k+1}\cos\biggl(\frac{n+1+k}{2}\pi\biggr)\sum_{\ell=k}^{n+1}(-1)^{n-\ell}2^{n-\ell+1}(\ell-1)!S(n+1,\ell)\binom{\ell}{k}\Biggr]\tan^k x.$$

Consequently, we arrives at

$$a_{2m-1,0} = -\cos\biggl(\frac{2m}{2}\pi\biggr)\sum_{\ell=1}^{2m}(-1)^{2m-\ell-1}2^{2m-\ell}(\ell-1)!S(2m,\ell) = (-1)^m\sum_{\ell=1}^{2m}(-1)^{\ell}2^{2m-\ell}(\ell-1)!S(2m,\ell)$$

for $m\ge1$ and

$$a_{n,n+1-2m} = (-1)^n\cos((n+1-m)\pi) \sum_{\ell=n+1-2m}^{n+1}(-1)^{n-\ell}2^{n-\ell+1}(\ell-1)!S(n+1,\ell)\binom{\ell}{n+1-2m} = (-1)^{m-1}\sum_{\ell=n+1-2m}^{n+1}(-1)^{n-\ell}2^{n+1-\ell}(\ell-1)!S(n+1,\ell)\binom{\ell}{n+1-2m}$$

for $0\le m\le \frac12\bigl[n-\frac{1-(-1)^n}{2}\bigr]$. The proof of Theorem 1 is thus complete.

3 An open problem

Now we would like to propose an open problem as follows.

The equation (2) means that

$$(\tan x)^{(n)} = \sum_{k=0}^{\frac{1}{2}\left[n+\frac{1-(-1)^n}{2}\right]} a_{n,n-2k+1} \tan^{n-2k+1} x. \tag{6}$$

Differentiating with respect to x on both sides of (6) gives

$$\begin{aligned}
(\tan x)^{(n+1)} &= \sum_{k=0}^{\frac{1}{2}\left[n+\frac{1-(-1)^n}{2}\right]} a_{n,n-2k+1}(n-2k+1)\tan^{n-2k} x\big(1+\tan^2 x\big)\\
&= \sum_{k=0}^{\frac{1}{2}\left[n+\frac{1-(-1)^n}{2}\right]} a_{n,n-2k+1}(n-2k+1)\tan^{n-2k} x\\
&\quad+ \sum_{k=0}^{\frac{1}{2}\left[n+\frac{1-(-1)^n}{2}\right]} a_{n,n-2k+1}(n-2k+1)\tan^{n-2k+2} x\\
&= \sum_{k=1}^{\frac{1}{2}\left[n+\frac{1-(-1)^n}{2}\right]+1} a_{n,n-2k+3}(n-2k+3)\tan^{n-2k+2} x\\
&\quad+ \sum_{k=0}^{\frac{1}{2}\left[n+\frac{1-(-1)^n}{2}\right]} a_{n,n-2k+1}(n-2k+1)\tan^{n-2k+2} x\\
= &\sum_{k=1}^{\frac{1}{2}\left[n+\frac{1-(-1)^n}{2}\right]} [a_{n,n-2k+3}(n-2k+3)+a_{n,n-2k+1}(n-2k+1)]\tan^{n-2k+2} x\\
&+a_{n,n+1}(n+1)\tan^{n+2} x + a_{n,\frac{1+(-1)^n}{2}}\frac{1+(-1)^n}{2}\tan^{\frac{(-1)^n-1}{2}} x.
\end{aligned}$$

Comparing this with

$$(\tan x)^{(n+1)} = \sum_{k=0}^{\frac{1}{2}\left[n+1+\frac{1+(-1)^n}{2}\right]} a_{n+1,n-2k+2}(\tan x)^{n-2k+2}$$

yields

$$a_{n+1,n+2} = a_{n,n+1}(n+1), \tag{7}$$

$$a_{n+1,\frac{1-(-1)^n}{2}} \tan^{\frac{1-(-1)^n}{2}} x = a_{n,\frac{1+(-1)^n}{2}} \frac{1+(-1)^n}{2} \tan^{\frac{(-1)^n-1}{2}} x, \tag{8}$$

and

$$a_{n+1,n-2k+2} = a_{n,n-2k+3}(n-2k+3) + a_{n,n-2k+1}(n-2k+1) \tag{9}$$

for $n \ge 1$ and $1 \le k \le \frac{1}{2}\left[n + \frac{1-(-1)^n}{2}\right]$.

The derivatives of the tangent function $\tan x$ in (1) means that $a_{0,1} = 1$, $a_{1,2} = 1$, $a_{2,3} = 2$, and $a_{3,4} = 2 \cdot 3$. Combining these values with (7) reveals that $a_{n,n+1} = n!$ for all $n \ge 0$.

The derivatives of the tangent function $\tan x$ in (1) also means that $a_{1,0} = 1$, $a_{2,1} = 2$, and $a_{3,0} = 2$. When $n = 2\ell$ for $\ell \ge 0$, the recurrence relation (8) becomes

$$a_{2\ell+1,0} = a_{2\ell,1}.$$

When $k = 1$, the recurrence relation (9) can be simplified as

$$a_{n+1,n} = a_{n,n+1}(n+1) + a_{n,n-1}(n-1) = a_{n,n-1}(n-1) + (n+1)!$$

for $n \ge 2$. From this recurrence relation, we acquire

$$a_{n,n-1} = \frac{1}{3}(n+1)!, \quad n \ge 2. \tag{10}$$

When $k = 2$, by (10), the recurrence relation (9) can be rearranged as

$$a_{n+1,n-2} = a_{n,n-1}(n-1) + a_{n,n-3}(n-3) = a_{n,n-3}(n-3) + (n-1)\frac{(n+1)!}{3}$$

for $n \ge 4$. Accordingly, it follows that

$$a_{n,n-3} = \frac{5n-8}{90}(n+1)!, \quad n \ge 4. \tag{11}$$

When $k = 3$, by (11), the recurrence relation (9) can be rewritten as

$$a_{n+1,n-4} = a_{n,n-3}(n-3) + a_{n,n-5}(n-5) = a_{n,n-5}(n-5) + (n-3)\frac{5n-8}{90}(n+1)!$$

for $n \ge 6$. Therefore, it follows that

$$a_{n,n-5} = \frac{35n^2 - 203n + 264}{5670}(n+1)!, \quad n \ge 6. \tag{12}$$

Similarly as above processing, we can procure that

$$a_{n,n-7} = \frac{175n^3 - 2205n^2 + 8654n - 10272}{340200}(n+1)!, \quad n \ge 8, \tag{13}$$

$$a_{n,n-9} = \frac{385n^4 - 8470n^3 + 66539n^2 - 217910n + 244704}{11226600}(n+1)!, \quad n \ge 10, \tag{14}$$

and the like. Accordingly, from (10), (11), (12), (13), and (14), we can conclude that

$$\begin{gathered} a_{n,n-2m+1} = (n+1)! \sum_{\ell=0}^{m-1} (-1)^{m-1-\ell} b_{m,\ell} n^{\ell}, \\ n \ge 2, \quad 1 \le m \le \frac{1}{2}\left[n - \frac{1-(-1)^n}{2}\right]. \end{gathered} \tag{15}$$

Substituting this conclusion into (9) leads to

$$\begin{aligned} (n+2)! \sum_{\ell=0}^{k-1} (-1)^{k-1-\ell} b_{k,\ell} (n+1)^{\ell} &= (n-2k+3)(n+1)! \sum_{\ell=0}^{k-2} (-1)^{k-2-\ell} b_{k-1,\ell} n^{\ell} \\ &+ (n-2k+1)(n+1)! \sum_{\ell=0}^{k-1} (-1)^{k-1-\ell} b_{k,\ell} n^{\ell}, \end{aligned}$$

$$\begin{aligned} \sum_{\ell=0}^{k-1} (-1)^{\ell+1} \left[(n+2)(n+1)^{\ell} - (n-2k+1)n^{\ell}\right] b_{k,\ell} \\ = (n-2k+3) \sum_{\ell=0}^{k-2} (-1)^{\ell} n^{\ell} b_{k-1,\ell}, \end{aligned}$$

where $n \ge 4$ and $2 \le k \le \frac{1}{2}\left[n - \frac{1-(-1)^n}{2}\right]$. Note that the sequence $b_{k,\ell}$ are independent of n.

To the best of our knowledge, we think that it is much difficult to explicitly determine the sequence $b_{m,\ell}$ in (15). Can one present a closed form for the sequence $b_{m,\ell}$ in (15)?

4 Remarks

Finally we comment on Theorem 1 and recommend some references closely related to derivative polynomials of other elementary and applicable functions.

Remark 1 *The expression* (3) *implies an explicit formula*

$$T_{2m-1} = (-1)^m \sum_{\ell=1}^{2m} (-1)^\ell 2^{2m-\ell} (\ell-1)! S(2m,\ell), \quad m \ge 1$$

for tangent numbers T_{2m-1} *which can be generated by*

$$\tan x = \sum_{k=1}^{\infty} T_{2k-1} \frac{x^{2k-1}}{(2k-1)!}, \quad |x| < \frac{\pi}{2}.$$

For more information on tangent numbers T_{2m-1}*, please refer to* [1, 11, 14, 36] *and the closely related references therein.*

Remark 2 *It is worthwhile to recommending the paper* [2] *which was found on 3 March 2017 by the authors.*

Remark 3 *Except the above-mentioned literature, there are other papers such as* [6, 7, 13, 15, 16, 17, 18, 19, 22, 23, 24, 25, 28, 29, 31, 34, 35, 36, 37] *and the closely related references therein to discuss derivative polynomials of other elementary and applicable functions.*

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