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Several identities involving the falling and rising factorials and the Cauchy, Lah, and Stirling numbers

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Abstract. In the paper, the authors find several identities, including a new recurrence relation for the Stirling numbers of the first kind, involving the falling and rising factorials and the Cauchy, Lah, and Stirling numbers.

1 Notation and main results

It is known that, for $x \in \mathbb{R}$, the quantities

$$\langle x\rangle_n = \begin{cases} x(x-1)\cdots(x-n+1), & n \ge 1 \\ 1, & n = 0 \end{cases} = \prod_{\ell=0}^{n-1}(x-\ell) = \frac{\Gamma(x+1)}{\Gamma(x-n+1)}$$

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and

$$(x)_n = \begin{cases} x(x+1)\cdots(x+n-1), & n \ge 1 \\ 1, & n = 0 \end{cases} = \prod_{\ell=0}^{n-1}(x+\ell) = \frac{\Gamma(x+n)}{\Gamma(x)}$$

are respectively called the falling and rising factorials, where

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{\prod_{k=0}^{n}(z+k)}, \quad z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$$

is the classical gamma function, see [1, p. 255, 6.1.2]. For removable singularities of the ratio $\frac{\Gamma(x+m)}{\Gamma(x+n)}$ for $x \in \mathbb{R}$ and $m, n \in \mathbb{Z}$, please read [23, Theorem 1.1] and closely related references therein.

According to [4, pp. 293–294], there are two kinds of Cauchy numbers which may be defined respectively by

$$C_n = \int_0^1 \langle x \rangle_n \,\mathrm{d}\, x \quad \text{and} \quad c_n = \int_0^1 (x)_n \,\mathrm{d}\, x. \tag{1}$$

The Cauchy numbers C_n and c_n play important roles in some fields, such as approximate integrals, the Laplace summation formula, and difference-differential equations, and are also related to some famous numbers such as the Stirling, Bernoulli, and harmonic numbers. For recent conclusions on the Cauchy numbers, please read the papers [17, 18, 21, 30].

It is known that the coefficients expressing rising factorials $(x)_n$ in terms of falling factorials $\langle x \rangle_k$ are called the Lah numbers, denoted by $L(n,k)$. Precisely speaking,

$$(x)_n = \sum_{k=1}^{n} L(n,k)\langle x \rangle_k \quad \text{and} \quad \langle x \rangle_n = \sum_{k=1}^{n} (-1)^{n-k} L(n,k)(x)_k. \tag{2}$$

They can be computed by

$$L(n,k) = \binom{n-1}{k-1}\frac{n!}{k!}$$

and have an interesting meaning in combinatorics: they count the number of ways a set of n elements can be partitioned into k nonempty linearly ordered subsets. For more and recent results on the Lah numbers $L(n,k)$, please refer to [13, 15, 16].

The Stirling numbers of the first kind $s(n,k)$ may be generated by

$$\langle x\rangle_n=\sum_{k=0}^{n}s(n,k)x^k \quad\text{and}\quad (x)_n=\sum_{k=0}^{n}(-1)^{n-k}s(n,k)x^k. \tag{3}$$

The combinatorial meaning of the unsigned Stirling numbers of the first kind $(-1)^{n-k}s(n,k)$ can be interpreted as the number of permutations of $\{1,2,\dots,n\}$ with k cycles. Recently there are some new results on the Stirling numbers of the first kind $s(n,k)$ obtained in [17, 20, 21, 22].

An infinitely differentiable function f is said to be completely monotonic on an interval I if it satisfies $(-1)^n f^{(n)}(x)\ge 0$ for $x\in I$ and $n\ge 0$. See [38, Definition 1.3] and [40, Chapter XII]. An infinitely differentiable function $f: I\subseteq(-\infty,\infty)\to[0,\infty)$ is called a Bernstein function on I if its derivative $f'(t)$ is completely monotonic on I. See [38, Definition 3.1].

The class of completely monotonic functions may be characterized by [40, Theorem 12b] which reads that a necessary and sufficient condition that $f(x)$ should be completely monotonic for $0<x<\infty$ is that $f(x)=\int_0^\infty e^{-xt}\,d\,\alpha(t)$, where $\alpha(t)$ is non-decreasing and the integral converges for $0<x<\infty$. The Bernstein functions on $(0,\infty)$ can be characterized by the assertion that a function $f:(0,\infty)\to\mathbb{R}$ is a Bernstein function if and only if it admits the representation

$$f(x)=a+bx+\int_0^\infty\left(1-e^{-xt}\right)d\,\mu(t), \tag{4}$$

where $a,b\ge 0$ and μ is a Radon measure on $(0,\infty)$ satisfying $\int_0^\infty \min\{1,t\}\,d\,\mu(t)<\infty$. See [38, Theorem 3.2]. The triplet (a,b,μ) determines f uniquely and vice versa. The representing measure μ and the characteristic triplet (a,b,μ) from the expression (4) are often called the Lévy measure and the Lévy triplet of the Bernstein function f. The formula (4) is called the Lévy-Khintchine representation of f.

It was obtained inductively in [32, Lemma 2.1] that the derivatives of the functions

$$h_\alpha(t)=\left(1+\frac{1}{t}\right)^\alpha, \quad t>0, \quad \alpha\in(-1,1)$$

and

$$H_\alpha(t)=\frac{h_\alpha(t)}{\alpha}-\frac{h_{\alpha-1}(t)}{\alpha-1}$$

may be computed by

$$h_\alpha^{(i)}(t)=\frac{(-1)^i}{t^i(1+t)^i}\left(1+\frac{1}{t}\right)^\alpha\sum_{k=0}^{i-1}k!\binom{i}{k}\binom{i-1}{k}(\alpha)_{i-k}t^k \tag{5}$$

and

$$H_\alpha^{(i)}(t)=\frac{(-1)^i}{t^i(1+t)^{i+1}}\left(1+\frac{1}{t}\right)^\alpha\sum_{k=0}^{i-1}k!\binom{i+1}{k}\binom{i-1}{k}(\alpha)_{i-k}t^k,$$

for $i\in\mathbb{N}$. Consequently,

1. if $\alpha\in(0,1)$, the function $h_\alpha(t)$ is completely monotonic on $(0,\infty)$;
2. if $\alpha\in(-1,0)$, the function $h_\alpha(t)$ is a Bernstein function on $(0,\infty)$;
3. if $\alpha\in(0,1)$, the function $H_\alpha(t)$ is completely monotonic on $(0,\infty)$.

With the help of [32, Lemma 2.1], it was derived in [32, Theorem 1.1] that the weighted geometric mean

$$G_{x,y;\lambda}(t)=(x+t)^\lambda(y+t)^{1-\lambda}$$

is a Bernstein function of $t>-\min\{x,y\}$, where $\lambda\in(0,1)$ and $x,y\in\mathbb{R}$ with $x\neq y$. For more and detailed information on this topic, please refer to [2, 12, 22, 26, 32, 33, 34, 35, 36, 42] and closely related references therein.

In combinatorics, the Bell polynomials of the second kind (also called the partial Bell polynomials) $B_{n,k}(x_1,x_2,\dots,x_{n-k+1})$ are defined by

$$B_{n,k}(x_1,x_2,\dots,x_{n-k+1})=\sum_{\substack{1\le i\le n\\ \ell_i\in\{0\}\cup\mathbb{N}\\ \sum_{i=1}^n i\ell_i=n\\ \sum_{i=1}^n \ell_i=k}}\frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!}\prod_{i=1}^{n-k+1}\left(\frac{x_i}{i!}\right)^{\ell_i}$$

for $n\ge k\ge 0$, see [4, p. 134, Theorem A]. The Faà di Bruno formula may be described in terms of the Bell polynomials of the second kind $B_{n,k}$ by

$$\frac{d^n}{d\,x^n}f\circ g(x)=\sum_{k=0}^n f^{(k)}(g(x))B_{n,k}\big(g'(x),g''(x),\dots,g^{(n-k+1)}(x)\big),\tag{6}$$

see [4, p. 139, Theorem C].

The aims of this paper are, by virtue of the famous Faà di Bruno formula (6), to find a new form for derivatives of the function $h_\alpha(t)$, and then, by comparing this new form with (5), to derive some identities involving the falling and rising factorials and the Cauchy, Lah, and Stirling numbers.

Our main results may be summarized up as the following theorem.

Theorem 1 *For* $i \in \mathbb{N}$ *and* $\alpha \in \mathbb{R}$,

$$h_\alpha^{(i)}(t) = \frac{(-1)^i}{t^i(1+t)^i}\left(1+\frac{1}{t}\right)^\alpha \sum_{k=0}^{i-1}\left[\sum_{m=1}^{i-k}\binom{i-m}{k}L(i,m)\langle\alpha\rangle_m\right]t^k. \tag{7}$$

Consequently, the identities

$$(\alpha)_n = \frac{1}{k!\binom{n+k}{k}\binom{n+k-1}{k}}\sum_{m=1}^{n}\binom{n+k-m}{k}L(n+k,m)\langle\alpha\rangle_m, \tag{8}$$

$$c_n = \frac{1}{k!\binom{n+k}{k}\binom{n+k-1}{k}}\sum_{m=1}^{n}\binom{n+k-m}{k}L(n+k,m)C_m, \tag{9}$$

$$c_n = \frac{1}{k!\binom{n+k}{k}\binom{n+k-1}{k}}\sum_{\ell=1}^{n}\sum_{m=\ell}^{n}(-1)^{m-\ell}\binom{n+k-m}{k}L(n+k,m)L(m,\ell)c_\ell, \tag{10}$$

and

$$s(n,\ell) = \frac{(-1)^{n-\ell}}{k!\binom{n+k}{k}\binom{n+k-1}{k}}\sum_{m=\ell}^{n}\binom{n+k-m}{k}L(n+k,m)s(m,\ell) \tag{11}$$

hold for all $k,\ell \geq 0$ *and* $n \in \mathbb{N}$.

In next section, we will give a proof of Theorem 1. In the final section, we will list some remarks for explaining and interpreting the significance of identities obtained in Theorem 1.

2 Proof of Theorem 1

Now we are in a position to prove formulas or identities listed in Theorem 1.

The Bell polynomials of the second kind $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ satisfy

$$B_{n,k}\big(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}\big) = a^k b^n B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \tag{12}$$

and

$$B_{n,k}(1!, 2!, 3!, \dots, (n-k+1)!) = L(n,k), \tag{13}$$

see [4, p. 135], where a and b are any complex numbers.

Taking in (6) $f(u) = (1+u)^\alpha$ and $u = g(t) = \frac{1}{t}$, employing (12) and (13), interchanging the order of the double sum, and simplifying yield

$$
\begin{aligned}
h_\alpha^{(i)}(t) &= \sum_{k=1}^{i} \langle\alpha\rangle_k (1+u)^{\alpha-k} B_{i,k}\left(-\frac{1}{t^2}, \frac{2!}{t^3}, \dots, (-1)^{i-k+1}\frac{(i-k+1)!}{t^{i-k+2}}\right) \\
&= \sum_{k=1}^{i} \langle\alpha\rangle_k \left(1+\frac{1}{t}\right)^{\alpha-k} \frac{1}{t^k}\left(-\frac{1}{t}\right)^{i} B_{i,k}(1!, 2!, \dots, (i-k+1)!) \\
&= \frac{(-1)^i}{t^i(1+t)^i}\left(1+\frac{1}{t}\right)^{\alpha} \sum_{k=1}^{i} L(i,k)\langle\alpha\rangle_k (1+t)^{i-k} \\
&= \frac{(-1)^i}{t^i(1+t)^i}\left(1+\frac{1}{t}\right)^{\alpha} \sum_{m=0}^{i-1} L(i,i-m)\langle\alpha\rangle_{i-m} (1+t)^{m} \\
&= \frac{(-1)^i}{t^i(1+t)^i}\left(1+\frac{1}{t}\right)^{\alpha} \sum_{k=0}^{i-1} L(i,i-k)\langle\alpha\rangle_{i-k} \sum_{j=0}^{k}\binom{k}{j} t^j \\
&= \frac{(-1)^i}{t^i(1+t)^i}\left(1+\frac{1}{t}\right)^{\alpha} \sum_{j=0}^{i-1}\sum_{k=j}^{i-1} L(i,i-k)\langle\alpha\rangle_{i-k} \binom{k}{j} t^j \\
&= \frac{(-1)^i}{t^i(1+t)^i}\left(1+\frac{1}{t}\right)^{\alpha} \sum_{k=0}^{i-1}\sum_{q=k}^{i-1} L(i,i-q)\langle\alpha\rangle_{i-q} \binom{q}{k} t^k \\
&= \frac{(-1)^i}{t^i(1+t)^i}\left(1+\frac{1}{t}\right)^{\alpha} \sum_{k=0}^{i-1}\sum_{m=1}^{i-k} L(i,m)\langle\alpha\rangle_{m} \binom{i-m}{k} t^k.
\end{aligned}
$$

Comparing this with the formula (5) reveals

$$
k!\binom{i}{k}\binom{i-1}{k}(\alpha)_{i-k} = \sum_{m=1}^{i-k} L(i,m)\langle\alpha\rangle_m \binom{i-m}{k}. \tag{14}
$$

From this, the identity (8) follows immediately.

Integrating with respect to $\alpha \in (0,1)$ on both sides of (14) gives

$$
\sum_{m=1}^{i-k}\binom{i-m}{k} L(i,m)\int_0^1 \langle\alpha\rangle_m \, d\alpha = k!\binom{i}{k}\binom{i-1}{k}\int_0^1 (\alpha)_{i-k}\, d\,\alpha,
$$

that is, by (1),

$$
\sum_{m=1}^{i-k}\binom{i-m}{k} L(i,m) C_m = k!\binom{i}{k}\binom{i-1}{k} c_{i-k}.
$$

This can be rearranged as the identity (9).

Employing the second formula in (2) and the identity (14) acquires

$$\sum_{m=1}^{i-k}\binom{i-m}{k}L(i,m)\langle\alpha\rangle_m=\sum_{m=1}^{i-k}\binom{i-m}{k}L(i,m)\sum_{p=1}^{m}(-1)^{m-p}L(m,p)(\alpha)_p,$$

$$\sum_{m=1}^{i-k}\binom{i-m}{k}L(i,m)\sum_{p=1}^{m}(-1)^{m-p}L(m,p)(\alpha)_p=k!\binom{i}{k}\binom{i-1}{k}(\alpha)_{i-k},$$

$$\sum_{p=1}^{i-k}\sum_{m=p}^{i-k}\binom{i-m}{k}(-1)^{m-p}L(i,m)L(m,p)(\alpha)_p=k!\binom{i}{k}\binom{i-1}{k}(\alpha)_{i-k}.$$

Integrating on both sides of the above equality with respect to $\alpha\in(0,1)$ brings out

$$\sum_{p=1}^{i-k}\sum_{m=p}^{i-k}\binom{i-m}{k}(-1)^{m-p}L(i,m)L(m,p)\int_0^1(\alpha)_p\,\mathrm{d}\,\alpha \\ =k!\binom{i}{k}\binom{i-1}{k}\int_0^1(\alpha)_{i-k}\,\mathrm{d}\,\alpha,$$

that is,

$$\sum_{p=1}^{i-k}\sum_{m=p}^{i-k}\binom{i-m}{k}(-1)^{m-p}L(i,m)L(m,p)c_p=k!\binom{i}{k}\binom{i-1}{k}c_{i-k}.$$

This may be rearranged as (10).

Utilizing the formulas in (3) and (14) results in

$$\sum_{m=1}^{i-k}\binom{i-m}{k}L(i,m)\langle\alpha\rangle_m=\sum_{m=1}^{i-k}\binom{i-m}{k}L(i,m)\sum_{p=0}^{m}s(m,p)\alpha^p$$

$$=\sum_{m=1}^{i-k}\binom{i-m}{k}L(i,m)\sum_{p=1}^{m}s(m,p)\alpha^p=\sum_{p=1}^{i-k}\sum_{m=p}^{i-k}\binom{i-m}{k}L(i,m)s(m,p)\alpha^p$$

and

$$
\begin{aligned}
k!\binom{i}{k}\binom{i-1}{k}(\alpha)_{i-k} &= k!\binom{i}{k}\binom{i-1}{k}\sum_{p=0}^{i-k}(-1)^{i-k-p}s(i-k,p)\alpha^p \\
&= k!\binom{i}{k}\binom{i-1}{k}\sum_{p=1}^{i-k}(-1)^{i-k-p}s(i-k,p)\alpha^p \\
&= \sum_{p=1}^{i-k}k!\binom{i}{k}\binom{i-1}{k}(-1)^{i-k-p}s(i-k,p)\alpha^p.
\end{aligned}
$$

Equating coefficients of α^p in the above equations leads to

$$
\sum_{m=p}^{i-k}\binom{i-m}{k}L(i,m)s(m,p) = k!\binom{i}{k}\binom{i-1}{k}(-1)^{i-k-p}s(i-k,p)
$$

which may be reformulated as (11). The proof of Theorem 1 is complete.

3 Remarks

For explaining and interpreting the significance of formulas or identities obtained in Theorem 1, we are now list several remarks as follows.

Remark 1 *Because the sign of*

$$
\sum_{m=1}^{i-k}\binom{i-m}{k}L(i,m)\langle\alpha\rangle_m
$$

can not be made clear easily, the formula (7) *is much more complicated than the formula* (5)*. Concretely speaking, by virtue of the formula* (7)*, we can not obviously see the properties that* $h_\alpha(t)$ *for* $\alpha \in (0,1)$ *is a completely monotonic function on* $(0,\infty)$ *and that* $h_\alpha(t)$ *for* $\alpha \in (-1,0)$ *is a Bernstein function on* $(0,\infty)$*. This implies that [32, Lemma 2.1] is much more useful and significant.*

Remark 2 *The recurrence relation* (11) *is a new "horizontal" recurrence relation for the Stirling numbers of the first kind* $s(n,k)$*, because it is different from those "triangular", "horizontal", "vertical", and "diagonal" recurrence relations, listed or obtained in [4, pp. 214–215, Theorems A, B, and C] and [19, 20], for the Stirling numbers of the first kind* $s(n,k)$*.*

Remark 3 *It is a very interesting phenomenon that the variable* $k \geq 0$ *only appears in the right hand sides of* (8) *to* (11) *and that* k *can change anyway.*

Remark 4 *Comparing* (2) *with* (8) *reveals*

$$L(n,m) = \frac{\binom{n+k-m}{k}}{k!\binom{n+k}{k}\binom{n+k-1}{k}} L(n+k,m).$$

Remark 5 *When letting* $\alpha \to -1^+$, *the identity* (14) *becomes*

$$k!\binom{i}{k}\binom{i-1}{k}(-1)_{i-k} = \sum_{m=1}^{i-k}\binom{i-m}{k}L(i,m)\langle -1\rangle_m,$$

$$0 = \sum_{m=1}^{i-k}\binom{i-m}{k}L(i,m)(-1)^m m!.$$

In other words, the identity

$$\sum_{m=1}^{i-k}(-1)^m m!\binom{i-m}{k}L(i,m) = 0,$$

which may be reformulated as

$$\sum_{m=1}^{n}(-1)^m\binom{n+k-m}{k}\binom{n+k-1}{m-1} = 0,$$

holds for all $i > k+1 \geq 1$ *and* $n \in \mathbb{N}$.

Taking $\alpha = \pm\frac{1}{2}$ *in* (14) *respectively reveals*

$$\sum_{m=1}^{i-k}(-1)^{m+1}\frac{(2m-3)!!}{2^m}\binom{i-m}{k}L(i,m) = k!\frac{(2(i-k)-1)!!}{2^{i-k}}\binom{i}{k}\binom{i-1}{k}$$

and

$$\sum_{m=1}^{i-k}(-1)^{m+1}\frac{(2m-1)!!}{2^m}\binom{i-m}{k}L(i,m) = k!\frac{(2(i-k)-3)!!}{2^{i-k}}\binom{i}{k}\binom{i-1}{k},$$

which are equivalent to

$$\sum_{m=1}^{n}(-1)^m\frac{(2m-3)!!}{2^m}\binom{n+k-m}{k}L(n+k,m) \\ = -\frac{(2n-1)!!k!}{2^n}\binom{n+k}{k}\binom{n+k-1}{k}$$

and

$$\sum_{m=1}^{n}(-1)^m\frac{(2m-1)!!}{2^m}\binom{n+k-m}{k}L(n+k,m) \\ =-\frac{(2n-3)!!k!}{2^n}\binom{n+k}{k}\binom{n+k-1}{k},$$

holds for all $i+1>k\ge 0$ *and* $n\in\mathbb{N}$.

Remark 6 *Let* $u=u(x)$ *and* $v=v(x)\neq 0$ *be differentiable functions. In [3, p. 40], the formula*

$$\frac{d^n}{d\,x^n}\left(\frac{u}{v}\right)=\frac{(-1)^n}{v^{n+1}}\begin{vmatrix} u & v & 0 & \dots & 0 \\ u' & v' & v & \dots & 0 \\ u'' & v'' & 2v' & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ u^{(n-1)} & v^{(n-1)} & \binom{n-1}{1}v^{(n-2)} & \dots & v \\ u^{(n)} & v^{(n)} & \binom{n}{1}v^{(n-1)} & \dots & \binom{n}{n-1}v' \end{vmatrix} \tag{15}$$

for the n*th derivative of the ratio* $\frac{u(x)}{v(x)}$ *was listed. For easy understanding and convenient availability, we now reformulate the formula (15) as*

$$\frac{d^n}{d\,x^n}\left(\frac{u}{v}\right)=\frac{(-1)^n}{v^{n+1}}\left|A_{(n+1)\times 1}\quad B_{(n+1)\times n}\right|_{(n+1)\times(n+1)}, \tag{16}$$

where $|\cdot|_{(n+1)\times(n+1)}$ *denotes a determinant and the matrices*

$$A_{(n+1)\times 1}=(a_{i,1})_{0\le i\le n}$$

and

$$B_{(n+1)\times n}=(b_{i,j})_{0\le i\le n,0\le j\le n-1}$$

satisfy

$$a_{i,1}=u^{(i)}(x)\quad \textit{and}\quad b_{i,j}=\binom{i}{j}v^{(i-j)}(x)$$

under the conventions that $v^{(0)}(x)=v(x)$ *and that* $\binom{p}{q}=0$ *and* $v^{(p-q)}(x)\equiv 0$ *for* $p<q$. *See [39, Lemma 2.1].*

Applying $u(x)=(1+t)^\alpha$ *and* $v(x)=t^\alpha$ *into (16) yields*

$$a_{i,1}=[(1+t)^\alpha]^{(i)}=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-i+1)}(1+t)^{\alpha-i}$$

and

$$b_{i,j} = \binom{i}{j}(t^{\alpha})^{(i-j)} = \binom{i}{j}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-i+j+1)}t^{\alpha-i+j}$$

for $0 \le i \le n$ *and* $0 \le j \le n-1$*. As a result, a new and alternative form for derivatives of the functions* $h_\alpha(t)$ *and* $H_\alpha(t)$ *may be established.*

Remark 7 *In recent years, the first author and his coauthors obtained some new properties of the Bell, Bernoulli, Euler, Genocchi, Lah, Stirling numbers or polynomials in [6, 7, 8, 9, 10, 11, 14, 27, 37, 41].*

Remark 8 *In recent years, the first author and other mathematicians together considered the complete monotonicity and the Bernstein function properties in [5, 12, 24, 25, 28, 29, 31].*

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