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Bounds on third Hankel determinant for close-to-convex functions

J. K. Prajapat
Department of Mathematics,
Central University of Rajasthan, India
email: jkprajapat@curaj.ac.in

Deepak Bansal
Department of Mathematics,
Govt. College of Engineering and Technology, India
email: deepakbansal_79@yahoo.com

Alok Singh
Department of Mathematics,
Central University of Rajasthan, India
email: aloksingh@curaj.ac.in

Ambuj K. Mishra
Department of Mathematics,
GLA University, India
email: Ambuj_math@rediffmail.com

Abstract. In this paper, we have obtained upper bound on third Hankel determinant for the functions belonging to the class of close-to-convex functions.

1 Introduction

Let $\mathcal{H}(\mathbb{U})$ denote the class of functions which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$. Let $\mathcal{A}$ be the class of all functions $f \in \mathcal{H}(\mathbb{U})$ which are normalized by $f(0) = 0$, $f'(0) = 1$ and have the following form:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots, \quad z \in \mathbb{U}. \tag{1}$$

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We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of all functions in $\mathcal{A}$ which are also univalent in $\mathbb{U}$. Let $\mathcal{P}$ be the class of all functions $p \in \mathcal{H}(\mathbb{U})$ satisfying $p(0) = 1$ and $\Re(p(z)) > 0$. The function $p \in \mathcal{P}$ have the following form:

$$p(z) = 1 + c_1 z + c_2 z^2 + \dots, \quad z \in \mathbb{U}. \tag{2}$$

Further, a function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{S}^*$ of starlike functions in $\mathbb{U}$, if it satisfies the following inequality:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{U}. \tag{3}$$

Moreover, a function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{C}$ of close-to-convex functions in $\mathbb{U}$, if there exist a function $g \in \mathcal{S}^*$, such that the following inequality holds:

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > 0, \quad z \in \mathbb{U}. \tag{4}$$

The class of close-to-convex functions was introduced by Kaplan [9]. In [16], Noonan and Thomas studied the q^{th} Hankel determinants $H_q(n)$ of functions $f \in \mathcal{A}$ of the form (1) for $q \geq 1$, which is defined by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2(q-1)} \end{vmatrix} \qquad (a_1 = 1). \tag{5}$$

The Hankel determinants $H_q(n)$ have been investigated by several authors to study its rate of growth as $n \to \infty$ and to determine bounds on it for specific values of q and n. For example, Pommerenke [22] proved that the Hankel determinants of univalent functions satisfy $|H_q(n)| < Kn^{-(\frac{1}{2}+\beta)q+\frac{3}{2}}$ $(n = 1, 2, \dots,\ q = 2, 3, \dots)$, where $\beta > 1/4000$ and K depends only on q. Later, Hayman [8] proved that $|H_2(n)| < A\,n^{1/2}$ $(n = 1, 2, \dots;\ A$ is an absolute constant$)$ for areally mean univalent functions. Pommerenke [21] investigated the Hankel determinant of areally mean p-valent functions, univalent functions as well as of starlike functions. Ehrenborg studied Hankel determinant of the exponential polynomials [6] and Noor studied Hankel determinant for Bazilevic functions in [18] and for functions with bounded boundary rotations in [17, 19] also for close-to-convex functions in [20].

A classical theorem of Fekete and Szegö [7] considered the second Hankel determinant $H_2(1) = a_3 - a_2^2$ for univalent functions. They made an early study for the estimate of well known *Fekete-Szegö functional* $\left|a_3 - \mu a_2^2\right|$ when μ is real. Jenteng [12] investigated the sharp upper bound for second Hankel determinant $|H_2(2)| = |a_2 a_4 - a_3^2|$ for univalent functions whose derivative has positive real part. Recently, Lee *et al.* [13] have obtained bounds on $|H_2(2)|$ for functions belonging to the subclasses of Ma-Minda starlike and convex functions. Further Bansal [2] have obtained bounds on $|H_2(2)|$ for some new class of analytic functions. Recently, Babalola [1], Raza and Malik [24] and Bansal *et al.* [3] have studied third Hankel determinant $H_3(1)$, for various classes of analytic and univalent functions. In the present paper we investigate the upper bound on $|H_3(1)|$ for the functions belonging to the class of close-to-convex functions $\mathcal{K}$ defined by (4). To derive our results, we shall need the following Lemmas:

Lemma 1 (Carathéodory's Lemma [4], see also [5, p. 41]). *Let the function* $p \in \mathcal{P}$ *be given by the series then the sharp estimate* $|c_n| \leq 2, \ n = 1, 2, \cdots$ *holds. The inequality is sharp for each* n.

Lemma 2 (cf. [14, p. 254], see also [15]). *Let the function* $p \in \mathcal{P}$ *be given by* (2)*, then*

$$2c_2 = c_1^2 + x(4 - c_1^2)$$

for some x, $|x| \leq 1$, *and*

$$4c_3 = c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z$$

for some z, $|z| \leq 1$.

Lemma 3 ([5, p. 44]). *If* $f \in \mathcal{S}^*$ *be given by* (1)*, then* $|a_n| \leq n \ (n = 2, 3, \ldots)$. *Strict inequality holds for all* n *unless* f *is rotation of the Koebe function* $k(z) = z/(1 - z)^2$.

Lemma 4 ([23]). *If* $f \in \mathcal{C}$ *be given by* (1)*, then* $|a_n| \leq n \ (n = 2, 3, \ldots)$. *Equality holds for all* n *when* f *is rotation of the Koebe function.*

Lemma 5 ([10]). *If* $f \in \mathcal{S}^*$ *be given by* (1)*, then for any real number* μ*, we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, & \text{if} \quad \mu \leq \frac{1}{2} \\ 1, & \text{if} \quad \frac{1}{2} \leq \mu \leq 1 \\ 4\mu - 3, & \text{if} \quad \mu \geq 1. \end{cases}$$

Lemma 6 ([11]). *If* $f \in \mathcal{C}$ *be given by* (1)*, then* $|a_3 - a_2^2| \leq 1$. *There is a function in* $\mathcal{C}$ *such that equality holds.*

Lemma 7 ([12]). *If* $f \in \mathcal{S}^*$ *be given by* (1), *then* $|a_2a_4 - a_3^2| \leq 1$. *Equality is attended for the the Koebe function.*

Lemma 8 ([1]). *If* $f \in \mathcal{S}^*$ *be given by* (1), *then* $|a_2a_3 - a_4| \leq 2$. *Equality is attained by Koebe function.*

2 Main results

Our first main result is contained in the following theorem:

Theorem 1 *Let the function* $f \in \mathcal{C}$ *be given by* (1), *then*

$$|a_2a_3 - a_4| \leq 3. \tag{6}$$

Proof. Let the function $f \in \mathcal{C}$ be given by (6), then from the definition, we have

$$zf'(z) = g(z)p(z), \quad z \in \mathbb{U}, \tag{7}$$

for $p(z) \in \mathcal{P}$. The function $g(z)$ in (7) is a starlike function and let it have the form $g(z) = z + b_2z^2 + b_3z^3 + \ldots$. Substituting the valves of $f(z), g(z)$ and $p(z)$ and equating the coefficients, we get

$$2a_2 = b_2 + c_1 \tag{8}$$
$$3a_3 = b_3 + b_2c_1 + c_2 \tag{9}$$
$$4a_4 = b_4 + b_3c_1 + b_2c_2 + c_3. \tag{10}$$

Now

$$\begin{aligned}|a_2a_3 - a_4| &= \left|\frac{b_2 + c_1}{2}\frac{b_3 + b_2c_1 + c_2}{3} - \frac{b_4 + b_3c_1 + b_2c_2 + c_3}{4}\right| \\ &= \left|\frac{1}{4}(b_2b_3 - b_4) - \frac{c_1}{12}(b_3 - 2b_2^2) - \frac{1}{12}b_2b_3 + \frac{1}{6}b_2c_1^2\right. \\ &\qquad \left. + \left(\frac{c_1}{6} - \frac{b_2}{12}\right)c_2 - \frac{c_3}{4}\right|\end{aligned} \tag{11}$$

Substituting values of c_2 and c_3 by Lemma 2 in (11), we get

$$\begin{aligned}|a_2a_3 - a_4| = &\left|\frac{1}{4}(b_2b_3 - b_4) - \frac{c_1}{12}(b_3 - 2b_2^2) - \frac{1}{12}b_2b_3\right. \\ &+ \frac{1}{6}b_2c_1^2 + \left(\frac{c_1}{6} - \frac{b_2}{12}\right)\frac{c_1^2 + (4 - c_1^2)x}{2} \\ &\left.- \frac{c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z}{16}\right|\end{aligned}$$

$$= \left| \frac{1}{4}(b_2b_3 - b_4) - \frac{c_1}{12}(b_3 - 2b_2^2) - \frac{1}{12}b_2b_3 + \frac{1}{48}c_1^3 - \frac{1}{24}c_1(4-c_1^2)x + \frac{1}{8}b_2c_1^2 \right.$$
$$\left. - \frac{1}{24}b_2(4-c_1^2)x + \frac{1}{16}c_1(4-c_1^2)x^2 - \frac{1}{8}(4-c_1^2)(1-|x|^2)z \right|$$

By Lemma 1, we have $|c_1| \leq 2$. For convenience of notation, we take $c_1 = c$ and we may assume without loss of generality that $c \in [0, 2]$. Applying the triangle inequality with $\mu = |x|$ and using Lemma 3, Lemma 5 and Lemma 8, we obtain

$$\begin{aligned} |a_2a_3 - a_4| &\leq \frac{1}{4}|b_2b_3 - b_4| + \frac{1}{12}c|b_3 - 2b_2^2| + \frac{1}{12}|b_2||b_3| + \frac{1}{48}c^3 + \frac{1}{8}|b_2|c^2 \\ &\quad + \frac{1}{24}(4-c^2)(c+|b_2|)\mu + \frac{c}{16}(4-c^2)\mu^2 + \frac{1}{8}(4-c^2)(1-\mu^2) \\ &\leq \frac{3}{2} + \frac{5}{12}c + \frac{1}{8}c^2 + \frac{1}{48}c^3 + \frac{1}{24}(4-c^2)(c+2)\mu \\ &\quad + \frac{1}{16}(4-c^2)(c-2)\mu^2 = F_1(c,\mu). \end{aligned} \tag{12}$$

Differentiating $F_1(c, \mu)$ partially with respect to c, we have

$$\begin{aligned} \frac{\partial F_1}{\partial c} &= \frac{5}{12} + \frac{c}{4} + \frac{c^2}{16} + \frac{\mu}{24}(4 - 3c^2 - 4c) + \frac{\mu^2}{16}(4 - 3c^2 + 4c) \\ &= \frac{1}{12}(5 - \mu c^2) + \frac{c}{12}(3 - 2\mu) + \frac{c^2}{16} + \frac{\mu}{24}(4-c^2) + \frac{\mu^2}{16}(2-c)(3c+2) > 0, \end{aligned}$$

for $c \in [0, 2]$ and for any fixed μ with $\mu \in [0, 1]$. Therefore $F_1(c, \mu)$ is an increasing function of c on the closed interval $[0, 2]$, and hence $F_1(c, \mu)$ attained its maximum value at $c = 2$. Thus

$$\max_{0 \leq c \leq 2} F_1(c,\mu) = F_1(2,\mu) = G_1(\mu) \text{ (say)}. \tag{13}$$

From (12) and (13), we get $G_1(\mu) = 3$, which is independent of μ. Hence, the sharp upper bound of the functional $|a_2a_3 - a_4|$ can be obtained by setting $c = 2$ in (12), therefore

$$|a_2a_3 - a_4| \leq 3.$$

This completes the proof of Theorem 1. □

Theorem 2 *Let the function* $f \in \mathcal{C}$ *be given by* (1), *then*

$$H_2(2) = |a_2a_4 - a_3^2| \leq \frac{85}{36}. \tag{14}$$

Proof. Let $f \in \mathcal{C}$ of the form (1), then following the proof of Theorem 1, we get values of a_2, a_3 and a_4 given in (8)-(10). Using these values, we have

$$\begin{aligned}
|a_2a_4 - a_3^2| &= \left| \frac{b_2+c_1}{2} . \frac{b_4+b_3c_1+b_2c_2+c_3}{4} - \left(\frac{b_3+b_2c_1+c_2}{3}\right)^2 \right| \\
&= \left| \frac{1}{8}b_2b_4 - \frac{7}{72}b_2b_3c_1 + \frac{1}{8}b_2^2c_2 + \frac{1}{8}b_2c_3 + \frac{1}{8}b_3c_1^2 - \frac{7}{72}b_2c_1c_2 \right. \\
&\quad \left. + \frac{1}{8}b_4c_1 + \frac{1}{8}c_1c_3 - \frac{1}{9}b_3^2 - \frac{1}{9}b_2^2c_1^2 - \frac{1}{9}c_2^2 - \frac{2}{9}b_3c_2 \right| \\
&= \left| \frac{1}{8}(b_4 - b_2b_3)c_1 + \frac{1}{8}\left(b_3 - \frac{8}{9}b_2^2\right)c_1^2 + \frac{1}{8}(b_2b_4 - b_3^2) \right. \\
&\quad - \frac{2}{9}\left(b_3 - \frac{9}{16}b_2^2\right)c_2 + \frac{1}{36}b_2b_3c_1 \\
&\quad \left. + \frac{1}{8}b_2c_3 - \frac{7}{72}b_2c_1c_2 + \frac{1}{8}c_1c_3 + \frac{1}{72}b_3^2 - \frac{1}{9}c_2^2 \right|
\end{aligned}$$

Substituting the values of c_2 and c_3 from Lemma 2 in above equation, we have

$$\begin{aligned}
|a_2a_4 - a_3^2| &= \left| \frac{1}{8}(b_4 - b_2b_3)c_1 + \frac{1}{8}(b_3 - \frac{8}{9}b_2^2)c_1^2 + \frac{1}{8}(b_2b_4 - b_3^2) \right. \\
&\quad - \frac{1}{9}(b_3 - \frac{9}{16}b_2^2)(c_1^2 + x(4-c_1^2)) + \frac{1}{36}b_2b_3c_1 + \frac{1}{72}b_3^2 \\
&\quad - \frac{7}{144}b_2c_1(c_1^2 + x(4-c_1^2)) - \frac{1}{36}(c_1^2 + x(4-c_1^2))^2 \\
&\quad + \frac{1}{32}(b_2+c_1)[c_1^3 + 2c_1(4-c_1^2)x - c_1(4-c_1^2)x^2 \\
&\quad \left. + 2(1-|x|^2)(4-c_1^2)z] \right| \\
&= \left| \frac{1}{8}(b_4 - b_2b_3)c_1 + \frac{1}{8}(b_3 - \frac{8}{9}b_2^2)c_1^2 + \frac{1}{8}(b_2b_4 - b_3^2) \right. \\
&\quad - \frac{1}{9}(b_3 - \frac{9}{16}b_2^2)c_1^2 - \frac{1}{9}(b_3 - \frac{9}{16}b_2^2)(4-c_1^2)x + \frac{1}{36}b_2b_3c_1 \\
&\quad + \frac{1}{72}b_3^2 - \frac{5}{288}b_2c_1^3 + \frac{1}{288}{c_1}^4 + \frac{1}{72}b_2c_1(4-c_1^2)x + \frac{1}{144}c_1^2x(4-c_1^2) \\
&\quad - \frac{1}{36}x^2(4-c_1^2)^2 - \frac{1}{32}c_1b_2x^2(4-c_1^2) - \frac{1}{32}c_1^2(4-c_1^2)x^2 \\
&\quad \left. + \frac{1}{16}(b_2+c_1)(4-c_1^2)(1-|x|^2)z \right|
\end{aligned}$$

By Lemma 1, we have $|c_1| \leq 2$. For convenience of notation, we take $c_1 = c$ and we may assume without loss of generality that $c \in [0, 2]$. Applying the triangle inequality in above equation with $\mu = |x|$ and using Lemma 3, Lemma 5, Lemma 7 and Lemma 8, we obtain

$$\begin{aligned}
|a_2a_4 - a_3^2| &\leq \frac{1}{8}|b_4 - b_2b_3|c + \frac{1}{8}|b_3 - \frac{8}{9}b_2^2|c^2 + \frac{1}{8}|b_2b_4 - b_3^2| + \frac{1}{9}|b_3 \\
&\quad - \frac{9}{16}b_2^2|c^2 + \frac{1}{9}|b_3 - \frac{9}{16}b_2^2|(4-c^2)\mu + \frac{1}{36}|b_2||b_3|c + \frac{1}{72}|b_3|^2 \\
&\quad + \frac{5}{288}|b_2|c^3 + \frac{1}{288}c^4 + \frac{1}{72}|b_2|c(4-c^2)\mu + \frac{1}{144}c^2(4-c^2)\mu \\
&\quad + \frac{1}{36}(4-c^2)^2\mu^2 + \frac{1}{32}|b_2|c(4-c^2)\mu^2 + \frac{1}{32}c^2\mu^2(4-c^2) \\
&\quad + \frac{1}{16}(|b_2|+c)(4-c^2)(1-\mu^2) \\
&\leq \frac{1}{4}c + \frac{1}{8}c^2 + \frac{1}{8} + \frac{1}{9}c^2 + \frac{1}{9}(4-c^2)\mu + \frac{1}{6}c + \frac{1}{8} + \frac{5}{144}c^3 \\
&\quad + \frac{1}{288}c^4 + \frac{1}{36}c(4-c^2)\mu + \frac{1}{144}c^2(4-c^2)\mu + \frac{1}{36}(4-c^2)^2\mu^2 \\
&\quad + \frac{1}{16}c(4-c^2)\mu^2 + \frac{1}{32}c^2\mu^2(4-c^2) + \frac{1}{16}(2+c)(4-c^2)(1-\mu^2) \\
&= \frac{3}{4} + \frac{2}{3}c + \frac{1}{9}c^2 - \frac{1}{36}c^3 + \frac{1}{288}c^4 + \mu(4-c^2)\left(\frac{1}{9} + \frac{1}{36}c + \frac{1}{144}c^2\right) \\
&\quad + \frac{1}{288}(c^2-4)(4-c^2)\mu^2 = F_2(c,\mu)
\end{aligned} \tag{15}$$

Differentiating $F_2(c, \mu)$ in above equation with respect to μ, we get

$$\begin{aligned}
\frac{\partial F_2}{\partial \mu} &= \left(\frac{1}{9} + \frac{1}{36}c + \frac{1}{144}c^2\right)(4-c^2) + \frac{1}{144}(c^2-4)(4-c^2)\mu \\
&= \left(\frac{1}{36}(4-\mu) + \frac{1}{36}c + \frac{1}{144}c^2 + \frac{1}{144}\mu c^2\right)(4-c^2) > 0 \quad \text{for} \quad 0 \leq \mu \leq 1.
\end{aligned}$$

Therefore $F_2(c, \mu)$ is an increasing function of μ for $0 \leq \mu \leq 1$ and for any fixed c with $c \in [0, 2]$. Hence it attains maximum value at $\mu = 1$. Thus

$$\max_{0 \leq \mu \leq 1} F_2(c, \mu) = F_2(c, 1) = G_2(c) \ \ (\text{say}). \tag{16}$$

Therefore from (15) and (16), we have

$$G_2(c) = \frac{1}{144}(164 + 112c + 8c^2 - 8c^3 - c^4). \tag{17}$$

Now

$$G_2'(c) = \frac{1}{36}[28+4c-6c^2-c^3] = \frac{1}{36}[4+(6+c)(4-c^2)] > 0 \quad \text{for} \quad c \in [0,2].$$

This shows that $G_2(c)$ is an increasing function of c, hence it will attains maximum value at $c=2$. Therefore

$$\max_{0\le c\le 2} G_2(c) = G_2(2) = \frac{85}{36}.$$

Hence the upper bound on $|a_2a_4-a_3^2|$ can bee obtained by setting $\mu = 1$ and $c=2$ in (15) or $c=2$ in (17), therefore

$$|a_2a_4-a_3^2| \le \frac{85}{36}.$$

□

Theorem 3 *Let the function* $f \in \mathcal{C}$ *be given by* (1), *then*

$$|H_3(1)| \le \frac{289}{12}. \tag{18}$$

Proof. Let $f \in \mathcal{C}$ of the form (1), then by definition $H_3(1)$ is given by

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} \tag{19}$$
$$= a_3(a_2a_4-a_3^2) - a_4(a_4-a_2a_3) + a_5(a_3-a_2^2).$$

Using the triangle inequality in (19), we have

$$|H_3(1)| = |a_3||a_2a_4-a_3^2| + |a_4||a_4-a_2a_3| + |a_5||a_3-a_2^2|. \tag{20}$$

Now applying Lemma 4, Lemma 6, Theorem 1 and Theorem 2 in (20), we finally have the bound on $H_3(1)$ as desired. □

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References

[1] K. O. Babalola, On $H_3(1)$ Hankel determinant for some classes of univalent functions, *Inequal. Theory Appl.*, **6** (2007), 1–7.

[2] D. Bansal, Upper bound of second Hankel determinant for a new class of analytic functions, *Appl. Math. Lett.*, **26** (1) (2013), 103–107.

[3] D. Bansal, S. Maharana, J. K. Prajapat, Third order Hankel Determinant for certain univalent functions, *J. Korean Math. Soc.*, **52** (6) (2015), 1139–1148.

[4] C. Carathéodory, Über den variabilitätsbereich der Fourier'schen Konstanten Von Positiven harmonischen Funktionen, *Rend. Circ. Mat. Palermo,* **32** (1911), 193–217.

[5] P. L. Duren, *Univalent Functions*, vol. 259 of Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1983.

[6] R. Ehrenborg, The Hankel determinant of exponential polynomials, *Amer. Math. Monthly*, **107** (2000), 557–560.

[7] M. Fekete, G. Szegö, Eine Benberkung uber ungerada Schlichte funktionen, *J. London Math. Soc.*, **8** (1933), 85–89.

[8] W. K. Hayman, On second Hankel determinant of mean univalent functions, *Proc. Lond. Math. Soc.*, **18** (1968), 77–94.

[9] W. Kaplan, Close-to-convex schlicht functions, *Mich. Math. J.*, **1** (1952), 169–185.

[10] F. R. Keogh, E. P. Merkes, A coefficient inequality for certain classes of analytic functions, *Proc. Amer. Math. Soc.*, **20** (1969), 8–12.

[11] W. Koepf, On the Fekete-Szegö problem for close-to-convex functions, *Proc. Amer. Math. Soc.*, **101** (1987), 89–95.

[12] A. Janteng, S. Halim, M. Darus, Coefficient inequality for a function whose derivative has a positive real part, *J. Inequal. Pure Appl. Math.*, **7** (2) (2006), Article 50.

[13] S. K. Lee, V. Ravichandran, S. Supramaniam, Bounds for the second Hankel determinant of certain univalent functions, *J. Inequal. Appl.*, **2013** (2013), Article 281.

[14] R. J. Libera, E. J. Zlotkiewicz, coefficient bounds for the inverse of a function with derivative in $\mathcal{P}$, *Proc. Amer. Math. Soc.*, **87** (2) (1983), 251–257.

[15] R. J. Libera, E. J. Zlotkiewicz, Early coefficients of the inverse of a regular convex function, *Proc. Amer. Math. Soc.*, **85** (2) (1982), 225–230.

[16] J. W. Noonan, D. K. Thomas, On the second Hankel determinant of areally mean p-valent functions, *Trans. Amer. Math. Soc.*, **223** (2) (1976), 337–346.

[17] K. I. Noor, Hankel determinant problem for the class of function with bounded boundary rotation, *Rev. Roum. Math. Pures Et Appl.* **28** (1983), 731–739.

[18] K. I. Noor, S. A. Al-Bany, On Bazilevic functions, *Int. J. Math. Math. Sci.*, **10** (1) (1987), 79–88.

[19] K. I. Noor, On analytic functions related with function of bounded boundary rotation, *Comment. Math. Univ. St. Pauli*, **30** (2) (1981), 113–118.

[20] K. I. Noor, Higer order close-to-convex functions, *Math. Japon*, **37** (1) (1992), 1–8.

[21] C. Pommerenke, On the Hankel determinants of univalent functions, *Mathematika*, **14** (1967), 108–112.

[22] C. Pommerenke, On the coefficients and Hankel determinant of univalent functions, *J. Lond. Math. Soc.*, **41** (1966), 111-,112.

[23] M. O. Reade, On close-to-convex functions, *Mich. Math. J.*, **3** (1955-56), 59–62.

[24] M. Raza, S. N. Malik, Upper bound of the third Hankel determinant for a class of analytic functions related with Lemniscate of Bernoulli, *J. Inequal. Appl.*, **2013** (2013), Article 412.

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