On a new subclass of bi-univalent functions satisfying subordinate conditions

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Abstract. In the present investigation, we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the function class $S_{\Sigma}(\lambda, \phi)$. The results presented in this paper improve or generalize the recent work of Magesh and Yamini [15].

1 Introduction and definitions

Let A denote the class of analytic functions in the unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\}$$

that have the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1)$$

Further, by S we shall denote the class of all functions in A which are univalent in U .

The Koebe one-quarter theorem [8] states that the image of U under every function f from S contains a disk of radius $\frac{1}{4}$. Thus every such univalent function has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z, \quad (z \in U)$$

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and

$$f\left(f^{-1}(w)\right)=w, \quad\left(|w|< r_0(f), \quad r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w)=w-a_2 w^2+\left(2 a_2^2-a_3\right) w^3-\left(5 a_2^3-5 a_2 a_3+a_4\right) w^4+\cdots .$$

A function $f(z) \in \mathcal{A}$ is said to be bi-univalent in $\mathcal{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathcal{U}$.

If the functions f and g are analytic in $\mathcal{U}$, then f is said to be subordinate to g , written as

$$f(z) \prec g(z), \quad(z \in \mathcal{U})$$

if there exists a Schwarz function $w(z)$, analytic in $\mathcal{U}$, with

$$w(0)=0 \quad \text { and } \quad|w(z)|<1 \quad(z \in \mathcal{U})$$

such that

$$f(z)=g(w(z)) \quad(z \in \mathcal{U}) .$$

Let Σ denote the class of bi-univalent functions defined in the unit disk $\mathcal{U}$. For a brief history and interesting examples in the class Σ , (see [20]).

Lewin [14] studied the class of bi-univalent functions, obtaining the bound 1.51 for modulus of the second coefficient $|a_2|$. Subsequently, Brannan and Clunie [5] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$. Netanyahu [16] showed that $\max |a_2| = \frac{4}{3}$ if $f(z) \in \Sigma$.

Brannan and Taha [4] introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses. $S^*(\alpha)$ and $K(\alpha)$ of starlike and convex function of order α ($0 < \alpha \leq 1$) respectively (see [16]). Thus, following Brannan and Taha [4], a function $f(z) \in \mathcal{A}$ is the class $S_\Sigma^*(\alpha)$ of strongly bi-starlike functions of order α ($0 < \alpha \leq 1$) if each of the following conditions is satisfied:

$$f \in \Sigma, \quad\left|\arg \left(\frac{zf'(z)}{f(z)}\right)\right|<\frac{\alpha \pi}{2} \quad(0<\alpha \leq 1, \quad z \in \mathcal{U})$$

and

$$\left|\arg \left(\frac{wg'(w)}{g(w)}\right)\right|<\frac{\alpha \pi}{2} \quad(0<\alpha \leq 1, \quad w \in \mathcal{U})$$

where g is the extension of f^{-1} to $\mathcal{U}$. The classes $S_{\Sigma}^*(\alpha)$ and $K_{\Sigma}(\alpha)$ of bi-starlike functions of order α and bi-convex functions of order α , corresponding to the function classes $S^*(\alpha)$ and $K(\alpha)$, were also introduced analogously. For each of the function classes $S_{\Sigma}^*(\alpha)$ and $K_{\Sigma}(\alpha)$, they found non-sharp estimates on the initial coefficients. Recently, many authors investigated bounds for various subclasses of bi-univalent functions ([1], [3], [7], [9], [13], [15], [20], [21], [22]).

Not much is known about the bounds on the general coefficient $|a_n|$ for $n \geq 4$. In the literature, the only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions ([2], [6], [10], [11], [12]). The coefficient estimate problem for each of $|a_n|$ ($n \in \mathbb{N} \setminus \{1, 2\}$; $\mathbb{N} = \{1, 2, 3, \dots\}$) is still an open problem.

In this paper, by using the method [17] different from that used by other authors, we obtain bounds for the coefficients $|a_2|$ and $|a_3|$ for the subclasses of bi-univalent functions considered Magesh and Yamini and get more accurate estimates than that given in [15].

2 Coefficient estimates

In the following, let ϕ be an analytic function with positive real part in $\mathcal{U}$, with $\phi(0) = 1$ and $\phi'(0) > 0$. Also, let $\phi(\mathcal{U})$ be starlike with respect to 1 and symmetric with respect to the real axis. Thus, ϕ has the Taylor series expansion

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots \quad (B_1 > 0). \quad (2)$$

Suppose that $u(z)$ and $v(w)$ are analytic in the unit disk $\mathcal{U}$ with $u(0) = v(0) = 0$, $|u(z)| < 1$, $|v(w)| < 1$, and suppose that

$$u(z) = b_1 z + \sum_{n=2}^{\infty} b_n z^n, \quad v(w) = c_1 w + \sum_{n=2}^{\infty} c_n w^n \quad (|z| < 1). \quad (3)$$

It is well known that

$$|b_1| \leq 1, \quad |b_2| \leq 1 - |b_1|^2, \quad |c_1| \leq 1, \quad |c_2| \leq 1 - |c_1|^2. \quad (4)$$

Next, the equations (2) and (3) lead to

$$\phi(u(z)) = 1 + B_1 b_1 z + (B_1 b_2 + B_2 b_1^2) z^2 + \dots, \quad |z| < 1 \quad (5)$$

and

$$\phi(v(w)) = 1 + B_1 c_1 w + (B_1 c_2 + B_2 c_1^2) w^2 + \dots, \quad |w| < 1. \quad (6)$$

Definition 1 A function $f \in \Sigma$ is said to be $S_\Sigma(\lambda, \phi)$, $0 \leq \lambda \leq 1$, if the following subordination hold

$$\frac{zf'(z) + (2\lambda^2 - \lambda)z^2f''(z)}{4(\lambda - \lambda^2)z + (2\lambda^2 - \lambda)zf'(z) + (2\lambda^2 - 3\lambda + 1)f(z)} \prec \phi(z)$$

and

$$\frac{wg'(w) + (2\lambda^2 - \lambda)w^2g''(w)}{4(\lambda - \lambda^2)w + (2\lambda^2 - \lambda)wg'(w) + (2\lambda^2 - 3\lambda + 1)g(w)} \prec \phi(w)$$

where $g(w) = f^{-1}(w)$.

Theorem 1 Let f given by (1) be in the class $S_\Sigma(\lambda, \phi)$. Then

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{|(12\lambda^4 - 28\lambda^3 + 15\lambda^2 + 2\lambda + 1)B_1^2 - (1 + 3\lambda - 2\lambda^2)^2B_2| + (1 + 3\lambda - 2\lambda^2)^2B_1}} \quad (7)$$

and

$$|a_3| \leq \begin{cases} \frac{B_1}{2(2\lambda^2 + 1)}; & \text{if } B_1 \leq \frac{(1 + 3\lambda - 2\lambda^2)^2}{2(2\lambda^2 + 1)} \\ \frac{|(12\lambda^4 - 28\lambda^3 + 15\lambda^2 + 2\lambda + 1)B_1^2 - (1 + 3\lambda - 2\lambda^2)^2B_2|B_1 + 2(2\lambda^2 + 1)B_1^3}{2(2\lambda^2 + 1)[|(12\lambda^4 - 28\lambda^3 + 15\lambda^2 + 2\lambda + 1)B_1^2 - (1 + 3\lambda - 2\lambda^2)^2B_2| + (1 + 3\lambda - 2\lambda^2)^2B_1]}; & \\ \text{if } B_1 > \frac{(1 + 3\lambda - 2\lambda^2)^2}{2(2\lambda^2 + 1)}. \end{cases} \quad (8)$$

Proof. Let $f \in S_\Sigma(\lambda, \phi)$, $0 \leq \lambda \leq 1$. Then there are analytic functions $u, v: \mathbb{U} \rightarrow \mathbb{U}$ given by (3) such that

$$\frac{zf'(z) + (2\lambda^2 - \lambda)z^2f''(z)}{4(\lambda - \lambda^2)z + (2\lambda^2 - \lambda)zf'(z) + (2\lambda^2 - 3\lambda + 1)f(z)} = \phi(u(z)) \quad (9)$$

and

$$\frac{wg'(w) + (2\lambda^2 - \lambda)w^2g''(w)}{4(\lambda - \lambda^2)w + (2\lambda^2 - \lambda)wg'(w) + (2\lambda^2 - 3\lambda + 1)g(w)} = \phi(v(w)) \quad (10)$$

where $g(w) = f^{-1}(w)$. Since

$$\begin{aligned} & \frac{zf'(z) + (2\lambda^2 - \lambda)z^2f''(z)}{4(\lambda - \lambda^2)z + (2\lambda^2 - \lambda)zf'(z) + (2\lambda^2 - 3\lambda + 1)f(z)} \\ &= 1 + (1 + 3\lambda - 2\lambda^2)a_2z \\ &+ \left[(12\lambda^4 - 28\lambda^3 + 11\lambda^2 + 2\lambda - 1)a_2^2 + (4\lambda^2 + 2)a_3 \right] z^2 + \dots \end{aligned}$$

and

$$\begin{aligned} & \frac{wg'(w) + (2\lambda^2 - \lambda) w^2 g''(w)}{4(\lambda - \lambda^2)w + (2\lambda^2 - \lambda)wg'(w) + (2\lambda^2 - 3\lambda + 1)g(w)} \\ &= 1 - (1 + 3\lambda - 2\lambda^2) a_2 w \\ & \quad + \left[(12\lambda^4 - 28\lambda^3 + 19\lambda^2 + 2\lambda + 3) a_2^2 - (4\lambda^2 + 2) a_3 \right] w^2 + \dots, \end{aligned}$$

it follows from (5), (6), (9) and (10) that

$$(1 + 3\lambda - 2\lambda^2) a_2 = B_1 b_1, \quad (11)$$

$$(12\lambda^4 - 28\lambda^3 + 11\lambda^2 + 2\lambda - 1) a_2^2 + (4\lambda^2 + 2) a_3 = B_1 b_2 + B_2 b_1^2, \quad (12)$$

and

$$-(1 + 3\lambda - 2\lambda^2) a_2 = B_1 c_1, \quad (13)$$

$$(12\lambda^4 - 28\lambda^3 + 19\lambda^2 + 2\lambda + 3) a_2^2 - (4\lambda^2 + 2) a_3 = B_1 c_2 + B_2 c_1^2. \quad (14)$$

From (11) and (13) we obtain

$$c_1 = -b_1. \quad (15)$$

By adding (14) to (12), further computations using (11) to (15) lead to

$$\left[2(12\lambda^4 - 28\lambda^3 + 15\lambda^2 + 2\lambda + 1) B_1^2 - 2(1 + 3\lambda - 2\lambda^2)^2 B_2 \right] a_2^2 = B_1^3 (b_2 + c_2). \quad (16)$$

(15) and (16), together with (4), give that

$$\left| (12\lambda^4 - 28\lambda^3 + 15\lambda^2 + 2\lambda + 1) B_1^2 - (1 + 3\lambda - 2\lambda^2)^2 B_2 \right| |a_2|^2 \leq B_1^3 (1 - |b_1|^2). \quad (17)$$

From (11) and (17) we get

$$|a_2| \leq \frac{B_1 \sqrt{B_1}}{\sqrt{\left| (12\lambda^4 - 28\lambda^3 + 15\lambda^2 + 2\lambda + 1) B_1^2 - (1 + 3\lambda - 2\lambda^2)^2 B_2 \right| + (1 + 3\lambda - 2\lambda^2)^2 B_1}}.$$

Next, in order to find the bound on $|a_3|$, by subtracting (14) from (12), we obtain

$$4(2\lambda^2 + 1) a_3 - 4(2\lambda^2 + 1) a_2^2 = B_1 (b_2 - c_2) + B_2 (b_1^2 - c_1^2). \quad (18)$$

Then, in view of (4) and (15) , we have

$$2 \left(2\lambda^2 + 1 \right) B_1 |a_3| \leq \left[2 \left(2\lambda^2 + 1 \right) B_1 - \left(1 + 3\lambda - 2\lambda^2 \right)^2 \right] |a_2|^2 + B_1^2.$$

Notice that (7), we get

$$|a_3| \leq \begin{cases} \frac{B_1}{2(2\lambda^2+1)}; & \text{if } B_1 \leq \frac{(1+3\lambda-2\lambda^2)^2}{2(2\lambda^2+1)} \\ \frac{|(12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1)B_1^2-(1+3\lambda-2\lambda^2)^2B_2|B_1+2(2\lambda^2+1)B_1^3}{2(2\lambda^2+1)\left[|(12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1)B_1^2-(1+3\lambda-2\lambda^2)^2B_2|+(1+3\lambda-2\lambda^2)^2B_1\right]}; & \\ \text{if } B_1 > \frac{(1+3\lambda-2\lambda^2)^2}{2(2\lambda^2+1)}. \end{cases}$$

□

Putting $\lambda = 0$ in Theorem 1, we have the following corollary.

Corollary 1 *Let f given by (1) be in the class $S_{\Sigma}^*(\phi)$. Then*

$$|a_2| \leq \frac{B_1 \sqrt{B_1}}{\sqrt{|B_1^2 - B_2|} + B_1}$$

and

$$|a_3| \leq \begin{cases} \frac{B_1}{2}; & \text{if } B_1 \leq \frac{1}{2} \\ \frac{|B_1^2 - B_2| B_1 + 2B_1^3}{2[|B_1^2 - B_2| + B_1]}; & \text{if } B_1 > \frac{1}{2}. \end{cases}$$

The estimates on the coefficients $|a_2|$ and $|a_3|$ of Corollary 1 are improvement of the estimates obtained in Corollary 2.1 in [19].

Corollary 2 *If let*

$$\phi(z) = \left(\frac{1+z}{1-z} \right)^\alpha = 1 + 2\alpha z + 2\alpha^2 z^2 + \dots \quad (0 < \alpha \leq 1),$$

then inequalities (7) and (8) become

$$|a_2| \leq \frac{2\alpha}{\sqrt{|20\lambda^4 - 44\lambda^3 + 25\lambda^2 - 2\lambda + 1|\alpha + (1 + 3\lambda - 2\lambda^2)^2}} \quad (19)$$

and

$$|a_3| \leq \begin{cases} \frac{\alpha}{2\lambda^2+1}; & \text{if } 0 < \alpha \leq \frac{(1+3\lambda-2\lambda^2)^2}{4(2\lambda^2+1)} \\ \frac{[20\lambda^4-44\lambda^3+25\lambda^2-2\lambda+1]+4(2\lambda^2+1)]\alpha^2}{(2\lambda^2+1)[20\lambda^4-44\lambda^3+25\lambda^2-2\lambda+1]\alpha+(1+3\lambda-2\lambda^2)^2}; & \text{if } \frac{(1+3\lambda-2\lambda^2)^2}{4(2\lambda^2+1)} < \alpha \leq 1. \end{cases} \quad (20)$$

The bounds on $|a_2|$ and $|a_3|$ given by (19) and (20) are more accurate than that given in Theorem 2.1 in [15].

We note that for $\lambda = 0$, the class $S_\Sigma(\lambda, \phi)$ reduces to the class of strongly bi-starlike functions of order α ($0 < \alpha \leq 1$) and denoted by $S_\Sigma^*(\alpha)$.

Putting $\lambda = 0$ in Corollary 2, we have the following corollary.

Corollary 3 Let f given by (1) be in the class $S_\Sigma^*(\alpha)$, ($0 < \alpha \leq 1$). Then

$$|a_2| \leq \frac{2\alpha}{\sqrt{\alpha+1}} \quad (21)$$

and

$$|a_3| \leq \begin{cases} \alpha; & \text{if } 0 < \alpha \leq \frac{1}{4} \\ \frac{5\alpha^2}{\alpha+1}; & \text{if } \frac{1}{4} < \alpha \leq 1. \end{cases} \quad (22)$$

The bounds on $|a_3|$ given by (22) is more accurate than that given by Remark 2.2 in [17] and Theorem 2.1 in [4].

Remark 1 The bounds on $|a_3|$ given by (22) is more accurate than that given in Corollary 2.3 in [18].

Corollary 4 If let

$$\phi(z) = \frac{1 + (1-2\alpha)z}{1-z} = 1 + 2(1-\alpha)z + 2(1-\alpha)z^2 + \dots \quad (0 < \alpha \leq 1),$$

then inequalities (7) and (8) become

$$|a_2| \leq \frac{2(1-\alpha)}{\sqrt{2(1-\alpha)(12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1)-(1+3\lambda-2\lambda^2)^2} + (1+3\lambda-2\lambda^2)^2} \quad (23)$$

and

$$|a_3| \leq \begin{cases} \frac{1-\alpha}{2\lambda^2+1}; & \text{if } \frac{4(2\lambda^2+1)-(1+3\lambda-2\lambda^2)^2}{4(2\lambda^2+1)} \leq \alpha < 1 \\ \frac{\left[\left| 2(1-\alpha)(12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1)-(1+3\lambda-2\lambda^2)^2 \right| + 4(1-\alpha)(2\lambda^2+1) \right] (1-\alpha)}{(2\lambda^2+1) \left[\left| 2(1-\alpha)(12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1)-(1+3\lambda-2\lambda^2)^2 \right| + (1+3\lambda-2\lambda^2)^2 \right]}; & \\ \text{if } 0 \leq \alpha < \frac{4(2\lambda^2+1)-(1+3\lambda-2\lambda^2)^2}{4(2\lambda^2+1)}. & \end{cases} \quad (24)$$

The bounds on $|a_2|$ and $|a_3|$ given by (23) and (24) are more accurate than that given in Theorem 3.1 in [15].

Putting $\lambda = 0$ in Corollary 4, we have the following corollary.

Corollary 5 Let f given by (1) be in the class $S_{\Sigma}^*(\alpha)$, $(0 \leq \alpha < 1)$. Then

$$|a_2| \leq \frac{2(1-\alpha)}{\sqrt{1+|1-2\alpha|}} \quad (25)$$

and

$$|a_3| \leq \begin{cases} 1-\alpha; & \text{if } \frac{3}{4} \leq \alpha < 1 \\ \frac{(1-\alpha)|1-2\alpha|+4(1-\alpha)^2}{1+|1-2\alpha|}; & \text{if } 0 \leq \alpha < \frac{3}{4}. \end{cases} \quad (26)$$

The bounds on $|a_3|$ given by (26) is more accurate than that given by Remark 2.2 in [17] and Theorem 3.1 in [4].

Remark 2 The bounds on $|a_3|$ given by (26) is more accurate than that given in Corollary 3.3 in [18].

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