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Certain non-linear differential polynomials sharing a non zero polynomial

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Abstract. In this paper with the notion of weighted sharing of values we study the uniqueness of nonlinear differential polynomials of meromorphic functions sharing a nonzero polynomial and obtain two results which improves and generalizes the results due to L. Liu [**Uniqueness of meromorphic functions and differential polynomials, Comput. Math. Appl., 56 (2008), 3236-3245.**] and P. Sahoo [**Uniqueness and weighted value sharing of meromorphic functions, Applied. Math. E-Notes., 11 (2011), 23-32.**].

1 Introduction, definitions and results

In this paper by meromorphic functions we shall always mean meromorphic functions in the complex plane.

Let f and g be two non-constant meromorphic functions and let a be a finite complex number. We say that f and g share a CM, provided that $f-a$ and $g-a$ have same zeros with same multiplicities. Similarly, we say that f and g share a IM, provided that $f-a$ and $g-a$ have same zeros ignoring

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multiplicities. In addition we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM.

We adopt the standard notations of value distribution theory (see [6]). We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. The notation $S(r)$ denotes any quantity satisfying $S(r) = o(T(r))$ as $r \longrightarrow \infty$, outside of a possible exceptional set of finite linear measure.

A meromorphic function $a(z)$ is called a small function with respect to f, provided that $T(r,a) = S(r,f)$.

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. Let $a(z)$ be a small function with respect to $f(z)$ and $g(z)$. We say that $f(z)$ and $g(z)$ share $a(z)$ CM (counting multiplicities) if $f(z)-a(z)$ and $g(z)-a(z)$ have same zeros with same multiplicities and we say that $f(z)$, $g(z)$ share $a(z)$ IM (ignoring multiplicities) if we do not consider the multiplicities.

Throughout this paper, we need the following definition.

$$\Theta(a;f) = 1 - \limsup_{r\longrightarrow\infty} \frac{\overline{N}(r,a;f)}{T(r,f)},$$

where a is a value in the extended complex plane.

In 1959, W. K. Hayman (see [6], Corollary of Theorem 9) proved the following theorem.

Theorem A *Let f be a transcendental meromorphic function and n (≥ 3) is an integer. Then $f^n f' = 1$ has infinitely many solutions.*

Fang and Hua [3], Yang and Hua [16] got a unicity theorem respectively corresponding Theorem A.

Theorem B *Let f and g be two non-constant entire (meromorphic) functions, $n \geq 6$ (≥ 11) be a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1c_2)^{n+1}c^2 = -1$ or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.*

Noting that $f^n(z)f^{'}(z) = \frac{1}{n+1}(f^{n+1}(z))^{'}$, Fang [4] considered the case of k-th derivative and proved the following results.

Theorem C *Let f and g be two non-constant entire functions, and let n, k be two positive integers with $n > 2k+4$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$ or $f \equiv tg$ for a constant t such that $t^n = 1$.*

Theorem D *Let* f *and* g *be two non-constant entire functions, and let* n, k *be two positive integers with* $n > 2k + 8$. *If* $(f^n(z)(f(z)-1))^{(k)}$ *and* $(g^n(z)(g(z)-1))^{(k)}$ *share* 1 *CM, then* $f(z) \equiv g(z)$.

In 2008, X. Y. Zhang and W. C. Lin [21] proved the following result.

Theorem E *Let* f *and* g *be two non-constant entire functions, and let* n, m *and* k *be three positive integers with* $n > 2k + m + 4$. *If* $[f^n(f-1)^m]^{(k)}$ *and* $[g^n(g-1)^m]^{(k)}$ *share* 1 *CM, then either* $f \equiv g$ *or* f *and* g *satisfy the algebraic equation* $R(f, g) = 0$, *where* $R(\omega_1, \omega_2) = \omega_1^n(\omega_1 - 1)^m - \omega_2^n(\omega - 1)^m$.

In 2001 an idea of gradation of sharing of values was introduced in ([7], [8]) which measures how close a shared value is to being share CM or to being shared IM. This notion is known as weighted sharing and is defined as follows.

Definition 1 [7, 8] *Let* k *be a nonnegative integer or infinity. For* $a \in \mathbb{C} \cup \{\infty\}$ *we denote by* $E_k(a; f)$ *the set of all* a*-points of* f, *where an* a*-point of multiplicity* m *is counted* m *times if* $m \leq k$ *and* $k + 1$ *times if* $m > k$. *If* $E_k(a; f) = E_k(a; g)$, *we say that* f, g *share the value* a *with weight* k.

The definition implies that if f, g share a value a with weight k then z_0 is an a-point of f with multiplicity m $(\leq k)$ if and only if it is an a-point of g with multiplicity m $(\leq k)$ and z_0 is an a-point of f with multiplicity m $(> k)$ if and only if it is an a-point of g with multiplicity n $(> k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k), then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

If $a(z)$ is a small function with respect to $f(z)$ and $g(z)$, we define that $f(z)$ and $g(z)$ share $a(z)$ IM or $a(z)$ CM or with weight l according as $f(z) - a(z)$ and $g(z) - a(z)$ share $(0, 0)$ or $(0, \infty)$ or $(0, l)$ respectively.

In 2008, L. Liu [12] proved the following.

Theorem F *Let* f *and* g *be two non-constant entire functions, and let* n, m *and* k *be three positive integers such that* $n > 5k + 4m + 9$. *If* $E_0(1, [f^n(f-1)^m]^{(k)}) = E_0(1, [g^n(g-1)^m]^{(k)})$ *then either* $f \equiv g$ *or* f *and* g *satisfy the algebraic equation* $R(f, g) = 0$, *where* $R(\omega_1, \omega_2) = \omega_1^n(\omega_1 - 1)^m - \omega_2^n(\omega_2 - 1)^m$.

Recently P. Sahoo [14] proved the following result.

Theorem G *Let* f *and* g *be two transcendental meromorphic functions and* n (≥ 1), k (≥ 1), m (≥ 0) *and* $l (\geq 0)$ *be four integers. Let* $[f^n(f-1)^m]^{(k)}$ *and* $[g^n(g-1)^m]^{(k)}$ *share* (b, l) *for a nonzero constant* b. *Then*

(1) *when* $m = 0$, *if* $f(z) \neq \infty$, $g(z) \neq \infty$ *and* $l \geq 2$, $n > 3k + 8$ *or* $l = 1$, $n > 5k + 10$ *or* $l = 0$, $n > 9k + 14$, *then either* $f \equiv tg$, *where* t *is a constant satisfying* $t^n = 1$, *or* $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, *where* c_1, c_2 *and* c *are three constants satisfying* $(-1)^k (c_1 c_2)^n (nc)^{2k} = b^2$,

(2) *when* $m = 1$ *and* $\Theta(\infty; f) > \frac{2}{n}$ *then either* $[f^n(f-1)]^{(k)}[g^n(g-1)]^{(k)} \equiv b^2$, *except for* $k = 1$ *or* $f \equiv g$, *provided one of* $l \geq 2$, $n > 3k + 11$ *or* $l = 1$, $n > 5k + 14$ *or*$l = 0$, $n > 9k + 20$ *holds; and*

(3) *when* $m \geq 2$, *and* $l \geq 2$, $n > 3k + m + 10$ *or* $l = 1$, $n > 5k + 2m + 12$ *or* $l = 0$, $n > 9k + 4m + 16$, *then either* $[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \equiv b^2$ *except for* $k = 1$ *or* $f \equiv g$ *or* f *and* g *satisfying the algebraic equation* $R(f, g) = 0$, *where*

$$R(\omega_1, \omega_2) = \omega_1^n (\omega_1 - 1)^m - \omega_2^n (\omega_2 - 1)^m.$$

It is quite natural to ask the following questions.
Question 1: Can lower bound of n be further reduced in Theorems F, G?
Question 2: Can one remove the condition $f \neq \infty$, $g \neq \infty$ when $m = 0$ in Theorem G?

In this paper, taking the possible answer of the above questions into background we obtain the following results which improve and generalize Theorems F, G.

Theorem 1 *Let* f *and* g *be two transcendental meromorphic functions and let* $p(z)$ *be a nonzero polynomial with* $\deg(p) = l$. *Suppose* $[f^n(f-1)^m]^{(k)} - p$ *and* $[g^n(g-1)^m]^{(k)} - p$ *share* $(0, k_1)$, *where* $n(\geq 1)$, $k(\geq 1)$, $m(\geq 0)$ *are three integers. Now when one of the following conditions holds:*

(i) $k_1 \geq 2$ *and* $n > 3k + m + 8 (= s_2)$;

(ii) $k_1 = 1$ *and* $n > 4k + \frac{3m}{2} + 9 (= s_1)$;

(iii) $k_1 = 0$ *and* $n > 9k + 4m + 14 (= s_0)$;

then the following conclusions occur

(1) *when* $m = 0$, *then either* $f \equiv tg$, *where* t *is a constant satisfying* $t^n = 1$, *or if* $p(z)$ *is not a constant and* $n > \max\{s_i, 2k + 2l - 1\}$, $i = 0, 1, 2$, *then* $f(z) = c_1 e^{cQ(z)}$, $g(z) = c_2 e^{-cQ(z)}$, *where* $Q(z) = \int_0^z p(z)dz$, c_1, c_2 *and* c *are constants such that* $(nc)^2 (c_1 c_2)^n = -1$, *if* $p(z)$ *is a nonzero constant* b, *then* $f(z) = c_3 e^{dz}$, $g(z) = c_4 e^{-dz}$, *where* c_3, c_4 *and* d *are constants such that* $(-1)^k (c_3 c_4)^n (nd)^{2k} = b^2$;

(2) *when* $m = 1$ *and* $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n}$, *then either* $[f^n(f-1)]^{(k)}[g^n(g-1)]^{(k)} \equiv p^2$, *except for* $k = 1$ *or* $f \equiv g$;

(3) *when* $m \geq 2$, *then either* $[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \equiv p^2$ *except for* $k = 1$ *or* $f \equiv g$ *or* f *and* g *satisfying the algebraic equation* $R(f, g) = 0$, *where*

$$R(\omega_1, \omega_2) = \omega_1^n(\omega_1 - 1)^m - \omega_2^n(\omega_2 - 1)^m.$$

In addition, when f *and* g *share* $(\infty, 0)$, *then the possibility* $[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \equiv p^2$ *does not occur for* $m \geq 1$.

Remark 1 *When* f *and* g *share* ∞ *IM then the conditions (i), (ii) and (iii) of Theorem 1 will be replaced by respectively* $l \geq 2$ *and* $n > 3k + m + 7$, $l = 1$ *and* $n > 4k + \frac{3m}{2} + 8$ *and* $l = 0$ *and* $n > 9k + 4m + 13$.

Theorem 2 *Let* f *and* g *be two transcendental entire functions and let* $p(z)$ *be a nonzero polynomial with* $\deg(p) = l$. *Suppose* $[f^n(f-1)^m]^{(k)} - p$ *and* $[g^n(g-1)^m]^{(k)} - p$ *share* $(0, k_1)$, *where* $n\ (\geq 1)$, $k\ (\geq 1)$, $m\ (\geq 0)$ *are three integers. Now when one of the following conditions holds:*

(i) $k_1 \geq 2$ *and* $n > 2k + m + 4 (= s_2)$;

(ii) $k_1 = 1$ *and* $n > \frac{5k+3m+9}{2} (= s_1)$;

(iii) $k_1 = 0$ *and* $n > 5k + 4m + 7 (= s_0)$;

then the following conclusions occur

(1) *when* $m = 0$, *then either* $f \equiv tg$, *where* t *is a constant satisfying* $t^n = 1$, *or if* $p(z)$ *is not a constant and* $n > \max\{s_i, k + 2l\}$, $i = 0, 1, 2$, *then* $f(z) = c_1 e^{cQ(z)}$, $g(z) = c_2 e^{-cQ(z)}$, *where* $Q(z) = \int_0^z p(z)dz$, c_1, c_2 *and* c *are constants such that* $(nc)^2(c_1c_2)^n = -1$,
if $p(z)$ *is a nonzero constant* b, *then* $f(z) = c_3 e^{dz}$, $g(z) = c_4 e^{-dz}$, *where* c_3, c_4 *and* d *are constants such that* $(-1)^k(c_3c_4)^n(nd)^{2k} = b^2$;

(2) *when* $m = 1$ *then* $f \equiv g$;

(3) *when* $m \geq 2$, *then either* $f \equiv g$ *or* f *and* g *satisfying the algebraic equation* $R(f, g) = 0$, *where*

$$R(\omega_1, \omega_2) = \omega_1^n(\omega_1 - 1)^m - \omega_2^n(\omega_2 - 1)^m.$$

We now explain some definitions and notations which are used in the paper.

Definition 2 [10] *Let* p *be a positive integer and* $a \in \mathbb{C} \cup \{\infty\}$.

(i) $N(r, a; f \mid \geq p)$ $(\overline{N}(r, a; f \mid \geq p))$ *denotes the counting function (reduced counting function) of those* a*-points of* f *whose multiplicities are not less than* p.

(ii) $N(r, a; f \mid \leq p)$ $(\overline{N}(r, a; f \mid \leq p))$ *denotes the counting function (reduced counting function) of those* a*-points of* f *whose multiplicities are not greater than* p.

Definition 3 {11, cf.[18]} *For* $a \in \mathbb{C} \cup \{\infty\}$ *and a positive integer* p *we denote by* $N_p(r, a; f)$ *the sum* $\overline{N}(r, a; f) + \overline{N}(r, a; f \mid \geq 2) + \ldots \overline{N}(r, a; f \mid \geq p)$. *Clearly* $N_1(r, a; f) = \overline{N}(r, a; f)$.

Definition 4 *Let* $a, b \in \mathbb{C} \cup \{\infty\}$. *Let* p *be a positive integer. We denote by* $\overline{N}(r, a; f \mid \geq p \mid g = b)$ $(\overline{N}(r, a; f \mid \geq p \mid g \neq b))$ *the reduced counting function of those* a*-points of* f *with multiplicities* $\geq p$, *which are the* b*-points (not the* b*-points) of* g.

Definition 5 {cf.[1], 2} *Let* f *and* g *be two non-constant meromorphic functions such that* f *and* g *share the value* 1 *IM. Let* z_0 *be a* 1*-point of* f *with multiplicity* p, *a* 1*-point of* g *with multiplicity* q. *We denote by* $\overline{N}_L(r, 1; f)$ *the counting function of those* 1*-points of* f *and* g *where* $p > q$, *by* $N_E^{1)}(r, 1; f)$ *the counting function of those* 1*-points of* f *and* g *where* $p = q = 1$ *and by* $\overline{N}_E^{(2}(r, 1; f)$ *the counting function of those* 1*-points of* f *and* g *where* $p = q \geq 2$, *each point in these counting functions is counted only once. In the same way we can define* $\overline{N}_L(r, 1; g)$, $N_E^{1)}(r, 1; g)$, $\overline{N}_E^{(2}(r, 1; g)$.

Definition 6 {cf.[1], 2} *Let* k *be a positive integer. Let* f *and* g *be two non-constant meromorphic functions such that* f *and* g *share the value* 1 *IM. Let* z_0 *be a* 1*-point of* f *with multiplicity* p, *a* 1*-point of* g *with multiplicity* q. *We denote by* $\overline{N}_{f>k}\ (r, 1; g)$ *the reduced counting function of those* 1*-points of* f *and* g *such that* $p > q = k$. $\overline{N}_{g>k}\ (r, 1; f)$ *is defined analogously.*

Definition 7 *[7, 8] Let* f, g *share a value* a *IM. We denote by* $\overline{N}_*(r, a; f, g)$ *the reduced counting function of those* a*-points of* f *whose multiplicities differ from the multiplicities of the corresponding* a*-points of* g.

Clearly $\overline{N}_*(r, a; f, g) \equiv \overline{N}_*(r, a; g, f)$ *and* $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

2 Lemmas

Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. We denote by H the function as follows:

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right). \tag{1}$$

Lemma 1 [15] *Let f be a non-constant meromorphic function and let $a_n(z)(\not\equiv 0)$, $a_{n-1}(z)$, ... , $a_0(z)$ be meromorphic functions such that $T(r,a_i(z))=S(r,f)$ for $i=0,1,2,...,n$. Then*

$$T(r,a_nf^n+a_{n-1}f^{n-1}+...+a_1f+a_0)=nT(r,f)+S(r,f).$$

Lemma 2 [20] *Let f be a non-constant meromorphic function, and p, k be positive integers. Then*

$$N_p\left(r,0;f^{(k)}\right)\le T\left(r,f^{(k)}\right)-T(r,f)+N_{p+k}(r,0;f)+S(r,f), \tag{2}$$

$$N_p\left(r,0;f^{(k)}\right)\le k\overline{N}(r,\infty;f)+N_{p+k}(r,0;f)+S(r,f). \tag{3}$$

Lemma 3 [9] *If $N(r,0;f^{(k)}\mid f\neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity, then*

$$N(r,0;f^{(k)}\mid f\neq 0)\le k\overline{N}(r,\infty;f)+N(r,0;f\mid<k)+k\overline{N}(r,0;f\mid\ge k)+S(r,f).$$

Lemma 4 [11] *Let f_1 and f_2 be two non-constant meromorphic functions satisfying $\overline{N}(r,0;f_i)+\overline{N}(r,\infty;f_i)=S(r;f_1,f_2)$ for $i=1,2$. If $f_1^sf_2^t-1$ is not identically zero for arbitrary integers s and $t(|s|+|t|>0)$, then for any positive ε, we have*

$$N_0(r,1;f_1,f_2)\le\varepsilon T(r)+S(r;f_1,f_2),$$

where $N_0(r,1;f_1,f_2)$ denotes the deduced counting function related to the common 1-points of f_1 and f_2 and $T(r)=T(r,f_1)+T(r,f_2)$, $S(r;f_1,f_2)=o(T(r))$ as $r\longrightarrow\infty$ possibly outside a set of finite linear measure.

Lemma 5 [6] *Suppose that f is a non-constant meromorphic function, $k\ge 2$ is an integer. If*

$$N(r,\infty,f)+N(r,0;f)+N(r,0;f^{(k)})=S(r,\frac{f'}{f}),$$

then $f(z)=e^{az+b}$, where $a\neq 0$, b are constants.

Lemma 6 [5] *Let* $f(z)$ *be a non-constant entire function and let* $k \geq 2$ *be a positive integer. If* $f(z)f^{(k)}(z) \neq 0$*, then* $f(z) = e^{az+b}$*, where* $a \neq 0, b$ *are constant.*

Lemma 7 [19] *Let* f *be a non-constant meromorphic function, and let* k *be a positive integer. Suppose that* $f^{(k)} \not\equiv 0$*, then*

$$N(r, 0; f^{(k)}) \leq N(r, 0; f) + k\overline{N}(r, \infty; f) + S(r, f).$$

Lemma 8 *Let* f *and* g *be two non-constant meromorphic functions. Let* n (≥ 1)*,* k (≥ 1) *and* m (≥ 0) *be three integers such that* $n > 3k + m + 1$*. If* $[f^n(f-1)^m]^{(k)} \equiv [g^n(g-1)^m]^{(k)}$*, then* $f^n(f-1)^m \equiv g^n(g-1)^m$*.*

Proof. We have $[f^n(f-1)^m]^{(k)} \equiv [g^n(g-1)^m]^{(k)}$. Integrating we get

$$[f^n(f-1)^m]^{(k-1)} \equiv [g^n(g-1)^m]^{(k-1)} + c_{k-1}.$$

If possible suppose $c_{k-1} \neq 0$. Now in view of Lemma 2 for $p = 1$ and using second fundamental theorem we get

$$\begin{aligned}
& (n+m)T(r, f) \\
\leq\ & T(r, [f^n(f-1)^m]^{(k-1)}) - \overline{N}(r, 0; [f^n(f-1)^m]^{(k-1)}) + N_k(r, 0; f^n(f-1)^m) \\
& + S(r, f) \\
\leq\ & \overline{N}(r, 0; [f^n(f-1)^m]^{(k-1)}) + \overline{N}(r, \infty; f) + \overline{N}(r, c_{k-1}; [f^n(f-1)^m]^{(k-1)}) \\
& -\overline{N}(r, 0; [f^n(f-1)^m]^{(k-1)}) + N_k(r, 0; f^n(f-1)^m) + S(r, f) \\
\leq\ & \overline{N}(r, \infty; f) + \overline{N}(r, 0; [g^n(g-1)^m]^{(k-1)}) + k\overline{N}(r, 0; f) + N(r, 0; (f-1)^m) \\
& + S(r, f) \\
\leq\ & (k+1+m)\ T(r, f) + (k-1)\overline{N}(r, \infty; g) + N_k(r, 0; g^n(g-1)^m) + S(r, f) \\
\leq\ & (k+1+m)\ T(r, f) + k\ \overline{N}(r, \infty; g) + k\ \overline{N}(r, 0; g) + N(r, 0; (g-1)^m) \\
& +S\ (r, f) \\
\leq\ & (k+1+m)\ T(r, f) + (2k+m)\ T(r, g) + S(r, f) + S(r, g) \\
\leq\ & (3k+2m+1)\ T(r) + S(r).
\end{aligned}$$

Similarly we get

$$(n+m)\ T(r, g) \leq (3k+2m+1)\ T(r) + S(r).$$

Combining these we get

$$(n-m-3k-1)\ T(r) \leq S(r),$$

which is a contradiction since $n > 3k + m + 1$. Therefore $c_{k-1} = 0$ and so

$$[f^n(f-1)^m]^{(k-1)} \equiv [g^n(g-1)^m]^{(k-1)}.$$

Proceeding in this way we obtain

$$[f^n(f-1)^m]' \equiv [g^n(g-1)^m]'.$$

Integrating we get

$$f^n(f-1)^m \equiv g^n(g-1)^m + c_0.$$

If possible suppose $c_0 \neq 0$. Now using second fundamental theorem we get

$$\begin{aligned}
& (n+m)T(r,f) \\
\leq\ & \overline{N}(r,0;f^n(f-1)^m) + \overline{N}(r,\infty;f^n(f-1)^m) + \overline{N}(r,c_0;f^n(f-1)^m) + S(r,f) \\
\leq\ & \overline{N}(r,0;f) + mT(r,f) + \overline{N}(r,\infty;f) + \overline{N}(r,0;g^n(g-1)^m) + S(r,f) \\
\leq\ & (m+1)\ T(r,f) + \overline{N}(r,\infty;f) + \overline{N}(r,0;g) + m\ T(r,g) + S(r,f) \\
\leq\ & (3+2m)\ T(r) + S(r).
\end{aligned}$$

Similarly we get

$$(n+m)\ T(r,g) \leq (3+2m)\ T(r) + S(r).$$

Combining these we get

$$(n-3-m)\ T(r) \leq S(r),$$

which is a contradiction since $n > 4 + m$. Therefore $c_0 = 0$ and so

$$f^n(f-1)^m \equiv g^n(g-1)^m.$$

This proves the Lemma. □

Lemma 9 *Let* f, g *be two transcendental meromorphic functions, let* $n(\geq 1)$, $m(\geq 0)$ *and* $k(\geq 1)$ *be three integers with* $n > k + 2$. *If* $[f^n(f-1)^m]^{(k)} - p$ *and* $[g^n(g-1)^m]^{(k)} - p$ *share* $(0,0)$, *where* $p(z)$ *is a non zero polynomial, then* $T(r,f) = O(T(r,g))$ *and* $T(r,g) = O(T(r,f))$.

Proof. In view of Lemmas 1, 2 for $p=1$ and using second fundamental theorem for small function (see [17]) we get

$$\begin{aligned}
& (n+m)T(r,f) = T(r,f^n(f-1)^m)+O(1)\\
\leq\ & T(r,[f^n(f-1)^m]^{(k)}) - \overline{N}(r,0;[f^n(f-1)^m]^{(k)}) + N_{k+1}(r,0;f^n(f-1)^m)\\
& + S(r,f)\\
\leq\ & \overline{N}(r,0;[f^n(f-1)^m]^{(k)}) + \overline{N}(r,\infty;f) + \overline{N}(r,p;[f^n(f-1)^m]^{(k)})\\
& -\overline{N}(r,0;[f^n(f-1)^m]^{(k)}) + N_{k+1}(r,0;f^n(f-1)^m) + (\varepsilon+o(1))T(r,f)\\
\leq\ & \overline{N}(r,\infty;f) + \overline{N}(r,p;[f^n(f-1)^m]^{(k)}) + (k+1)\overline{N}(r,0;f) + N(r,0;(f-1)^m)\\
& +(\varepsilon+o(1))T(r,f)\\
\leq\ & (k+2+m)\ T(r,f) + \overline{N}(r,p;[g^n(g-1)^m]^{(k)}) + (\varepsilon+o(1))T(r,f)\\
\leq\ & (k+2+m)\ T(r,f) + (k+1)(n+m)\ T(r,g) + (\varepsilon+o(1))T(r,f),
\end{aligned}$$

i.e.,

$$(n-k-2)\ T(r,f) \leq (k+1)(n+m)\ T(r,g) + (\varepsilon+o(1))T(r,f),$$

for all $\varepsilon>0$. Take $\varepsilon<1$. Since $n>k+2$, we have $T(r,f)=O(T(r,g))$. Similarly we have $T(r,g)=O(T(r,f))$. This completes the proof. □

Lemma 10 *Let* f*,* g *be two transcendental meromorphic functions and let* $F=\frac{[f^n(f-1)^m]^{(k)}}{p}$, $G=\frac{[g^n(g-1)^m]^{(k)}}{p}$*, where* $p(z)$ *is a non zero polynomial and* $n(\geq 1)$*,* $k(\geq 1)$ *and* $m(\geq 0)$ *are three integers such that* $n>3k+m+3$*. If* $H\equiv 0$*, then* $[f^n(f-1)^m]^{(k)}-p$ *and* $[g^n(g-1)^m]^{(k)}-p$ *share* $(0,\infty)$ *as well as one of the following conclusions occur*

(i) $[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)}\equiv p^2$*;*

(ii) $f^n(f-1)^m\equiv g^n(g-1)^m$.

Proof. Let $P(w)=(w-1)^m$. Then $F=\frac{[f^nP(f)]^{(k)}}{p}$ and $G=\frac{[g^nP(g)]^{(k)}}{p}$. Since $H\equiv 0$, by integration we get

$$\frac{1}{F-1}\equiv\frac{BG+A-B}{G-1}, \tag{4}$$

where A,B are constants and $A\neq 0$. From (4) it is clear that F and G share $(1,\infty)$. We now consider following cases.

Case 1. Let $B\neq 0$ and $A\neq B$.

If $B=-1$, then from (4) we have

$$F\equiv\frac{-A}{G-A-1}.$$

Therefore

$$\overline{N}(r, A+1; G) = \overline{N}(r, \infty; F) = \overline{N}(r, \infty; f) + \overline{N}(r, 0; p).$$

So in view of Lemmas 1, 2 and the second fundamental theorem we get

$$\begin{aligned}
& (n+m)\ T(r, g) \\
\leq\ & T(r, G) + N_{k+1}(r, 0; g^n P(g)) - \overline{N}(r, 0; G) + S(r, g) \\
\leq\ & \overline{N}(r, \infty; G) + \overline{N}(r, 0; G) + \overline{N}(r, A+1; G) + N_{k+1}(r, 0; g^n P(g)) \\
& - \overline{N}(r, 0; G) + S(r, g) \\
\leq\ & \overline{N}(r, \infty; g) + N_{k+1}(r, 0; g^n P(g)) + \overline{N}(r, \infty; f) + S(r, g) \\
\leq\ & \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + N_{k+1}(r, 0; g^n) + N_{k+1}(r, 0; P(g)) + S(r, g) \\
\leq\ & \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + (k+1)\overline{N}(r, 0; g) + T(r, P(g)) + S(r, g) \\
\leq\ & T(r, f) + (k+2+m)\ T(r, g) + S(r, f) + S(r, g).
\end{aligned}$$

Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r, f) \leq T(r, g)$ for $r \in I$.
So for $r \in I$ we have

$$(n-k-3)\ T(r, g) \leq S(r, g),$$

which is a contradiction since $n > k+3$.
If $B \neq -1$, from (4) we obtain that

$$F - (1 + \frac{1}{B}) \equiv \frac{-A}{B^2[G + \frac{A-B}{B}]}.$$

So

$$\overline{N}(r, \frac{(B-A)}{B}; G) = \overline{N}(r, \infty; F) = \overline{N}(r, \infty; f) + \overline{N}(r, 0; p).$$

Using Lemmas 1, 2 and the same argument as used in the case when $B = -1$ we can get a contradiction.
Case 2. Let $B \neq 0$ and $A = B$.
If $B = -1$, then from (4) we have

$$FG \equiv 1,$$

i.e.,

$$[f^n P(f)]^{(k)} [g^n P(g)]^{(k)} \equiv p^2,$$

i.e.,

$$[f^n(f-1)^m][g^n(g-1)^m] \equiv p^2.$$

If $B \neq -1$, from (4) we have

$$\frac{1}{F} \equiv \frac{BG}{(1+B)G-1}.$$

Therefore

$$\overline{N}(r, \frac{1}{1+B}; G) = \overline{N}(r, 0; F).$$

So in view of Lemmas 1, 2 and the second fundamental theorem we get

$$\begin{aligned}
& (n+m)\, T(r,g) \\
\leq\ & T(r,G) + N_{k+1}(r,0;g^nP(g)) - \overline{N}(r,0;G) + S(r,g) \\
\leq\ & \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r, \frac{1}{1+B}; G) + N_{k+1}(r.0;g^nP(g)) \\
& - \overline{N}(r,0;G) + S(r,g) \\
\leq\ & \overline{N}(r,\infty;g) + (k+1)\overline{N}(r,0;g) + T(r,P(g)) + \overline{N}(r,0;F) + S(r,g)
\end{aligned}$$

$$\begin{aligned}
\leq\ & \overline{N}(r,\infty;g) + (k+1)\overline{N}(r,0;g) + T(r,P(g)) + (k+1)\overline{N}(r,0;f) + T(r,P(f)) \\
& + k\overline{N}(r,\infty;f) + S(r,f) + S(r,g) \\
\leq\ & (k+2+m)\, T(r,g) + (2k+1+m)\, T(r,f) + S(r,f) + S(r,g).
\end{aligned}$$

So for $r \in I$ we have

$$(n-3k-3-m)\, T(r,g) \leq S(r,g),$$

which is a contradiction since $n > 3k+3+m$.
Case 3. Let $B = 0$. From (4) we obtain

$$F \equiv \frac{G+A-1}{A}. \tag{5}$$

If $A \neq 1$, then from (5) we obtain

$$\overline{N}(r, 1-A; G) = \overline{N}(r, 0; F).$$

We can similarly deduce a contradiction as in Case 2. Therefore $A = 1$ and from (5) we obtain

$$F \equiv G,$$

i.e.,

$$[f^nP(f)]^{(k)} \equiv [g^nP(g)]^{(k)}.$$

Then by Lemma 8 we have

$$f^nP(f) \equiv g^nP(g), \tag{6}$$

i.e.,

$$f^n(f-1)^m \equiv g^n(g-1)^m.$$

□

Lemma 11 *Let* f, g *be two transcendental meromorphic functions,* $p(z)$ *be a non-zero polynomial with* $\deg(p(z)) = l$, n, k *be two positive integers. Let* $[f^n]^{(k)} - p$ *and* $[g^n]^{(k)} - p$ *share* $(0, \infty)$. *Suppose* $[f^n]^{(k)}[g^n]^{(k)} \equiv p^2$,

(i) *if* $p(z)$ *is not a constant and* $n > 2k + 2l - 1$, *then* $f(z) = c_1e^{cQ(z)}$, $g(z) = c_2e^{-cQ(z)}$, *where* $Q(z) = \int_0^z p(z)dz$, c_1, c_2 *and* c *are constants such that* $(nc)^2(c_1c_2)^n = -1$,

(ii) *if* $p(z)$ *is a nonzero constant* b *and* $n > 2k$, *then* $f(z) = c_3e^{cz}$, $g(z) = c_4e^{-cz}$, *where* c_3, c_4 *and* c *are constants such that* $(-1)^k(c_3c_4)^n(nc)^{2k} = b^2$.

Proof. Suppose

$$[f^n]^{(k)}[g^n]^{(k)} \equiv p^2. \tag{7}$$

We consider the following cases.
Case 1: Let $\deg(p(z)) = l(\geq 1)$.
Let z_0 be a zero of f with multiplicity q. Then z_0 be a zero of $[f^n]^{(k)}$ with multiplicity $nq - k$. Now one of the following possibilities holds.
(i) z_0 will be neither a zero of $[g^n]^{(k)}$ nor a pole of g; (ii) z_0 will be a zero of g; (iii) z_0 will be a zero of $[g^n]^{(k)}$ but not a zero of g and (iv) z_0 will be a pole of g.
We now explain only the above two possibilities (i) and (iv) because other two possibilities follow from these.
For the possibility (i): Note that since $n \geq 2k + 2l$, we must have

$$nq - k \geq n - k \geq k + 2l. \tag{8}$$

Thus z_0 must be a zero of $[f^n]^{(k)}$ with multiplicity at least $k+2l$. But we see from (7) that z_0 must be a zero of $p^2(z)$ with multiplicity atmost $2l$. Hence we arrive at a contradiction and so f has no zero in this case.
For the possibility (iv): Let z_0 be a pole of g with multiplicity q_1. Clearly z_0 will be pole of $[g^n]^{(k)}$ with multiplicity nq_1+k. Obviously $q > q_1$, or else z_0 is a pole of $p(z)$, which is a contradiction since $p(z)$ is a polynomial. Clearly $nq-k \geq nq_1+k$. Now

$$nq-k = nq_1+k$$

implies that

$$n(q-q_1) = 2k. \tag{9}$$

Since $n \geq 2k+2l$, we get a contradiction from (9). Hence we must have

$$nq-k > nq_1+k.$$

This shows that z_0 is a zero of $p(z)$ and we have $N(r,0;f) = O(\log r)$. Similarly we can prove that $N(r,0;g) = O(\log r)$. Thus in general we can take $N(r,0;f)+N(r,0;g) = O(\log r)$.
We know that

$$N(r,\infty;[f^n]^{(k)}) = n\,N(r,\infty;f) + k\,\overline{N}(r,\infty;f).$$

Also by Lemma 7 we have

$$\begin{aligned} N(r,0;[g^n]^{(k)}) &\leq n\,N(r,0;g) + k\,\overline{N}(r,\infty;g) + S(r,g) \\ &\leq k\,\overline{N}(r,\infty;g) + O(\log r) + S(r,g). \end{aligned}$$

From (7) we get

$$N(r,\infty;[f^n]^{(k)}) = N(r,0;[g^n]^{(k)}),$$

i.e.,

$$n\,N(r,\infty;f) + k\,\overline{N}(r,\infty;f) \leq k\,\overline{N}(r,\infty;g) + O(\log r) + S(r,g). \tag{10}$$

Similarly we get

$$n\,N(r,\infty;g) + k\,\overline{N}(r,\infty;g) \leq k\,\overline{N}(r,\infty;f) + O(\log r) + S(r,f). \tag{11}$$

Since f and g are transcendental, it follows that

$$S(r,f) + O(\log r) = S(r,f), \quad S(r,g) + O(\log r) = S(r,g).$$

Combining (10) and (11) we get

$$N(r,\infty;f)+N(r,\infty;g)=S(r,f)+S(r,g).$$

By Lemma 9 we have $S(r,f)=S(r,g)$ and so we obtain

$$N(r,\infty;f)=S(r,f),\quad N(r,\infty;g)=S(r,g). \tag{12}$$

Let

$$F_1=\frac{[f^n]^{(k)}}{p},\quad G_1=\frac{[g^n]^{(k)}}{p}. \tag{13}$$

Note that $T(r,F_1)\leq n(k+1)T(r,f)+S(r,f)$ and so $T(r,F_1)=O(T(r,f))$. Also by Lemma 2 one can obtain $T(r,f)=O(T(r,F_1))$. Hence $S(r,F_1)=S(r,f)$. Similarly we get $S(r,G_1)=S(r,g)$. Also

$$F_1G_1\equiv 1. \tag{14}$$

If $F_1\equiv cG_1$, where c is a nonzero constant, then F_1 is a constant and so f is a polynomial, which contradicts our assumption. Hence $F_1\not\equiv cG_1$ and so in view of (14) we see that F_1 and G_1 share $(-1,0)$.
Now by Lemma 7 we have

$$N(r,0;F_1)\leq n\,N(r,0;f)+k\,\overline{N}(r,\infty;f)+S(r,f)\leq S(r,F_1).$$

Similarly we have

$$N(r,0;G_1)\leq n\,N(r,0;g)+k\,\overline{N}(r,\infty;g)+S(r,g)\leq S(r,G_1).$$

Also we see that

$$N(r,\infty;F_1)=S(r,F_1),\quad N(r,\infty;G_1)=S(r,G_1).$$

Here it is clear that $T(r,F_1)=T(r,G_1)+O(1)$. Let

$$f_1 = \frac{F_1}{G_1}.$$

and

$$f_2=\frac{F_1-1}{G_1-1}.$$

Clearly f_1 is non-constant. If f_2 is a nonzero constant then F_1 and G_1 share (∞, ∞) and so from (14) we conclude that F_1 and G_1 have no poles. Next we suppose that f_2 is non-constant. Also we see that

$$F_1 = \frac{f_1(1-f_2)}{f_1-f_2}, \quad G_1 = \frac{1-f_2}{f_1-f_2}.$$

Clearly

$$T(r, F_1) \leq 2[T(r, f_1) + T(r, f_2)] + O(1)$$

and

$$T(r, f_1) + T(r, f_2) \leq 4T(r, F_1) + O(1).$$

These give $S(r, F_1) = S(r; f_1, f_2)$. Also we see that

$$\overline{N}(r, 0; f_i) + \overline{N}(r, \infty; f_i) = S(r; f_1, f_2)$$

for $i = 1, 2$.
Next we suppose $\overline{N}(r, -1; F_1) \neq S(r, F_1)$, otherwise F_1 will be a constant. Also we see that

$$\overline{N}(r, -1; F_1) \leq N_0(r, 1; f_1, f_2).$$

Thus we have

$$T(r, f_1) + T(r, f_2) \leq 4\ N_0(r, 1; f_1, f_2) + S(r, F_1).$$

Then by Lemma 4 there exist two integers s and $t(|s| + |t| > 0)$ such that

$$f_1^s f_2^t \equiv 1,$$

i.e.,

$$\left[\frac{F_1}{G_1}\right]^s \left[\frac{F_1 - 1}{G_1 - 1}\right]^t \equiv 1. \tag{15}$$

We now consider following cases.
Case (i) Let $s = 0$ and $t \neq 0$. Then from (15) we get

$$(F_1 - 1)^t \equiv (G_1 - 1)^t.$$

This shows that F_1 and G_1 share (∞, ∞) and so from (14) we conclude that F_1 and G_1 have no poles.
Case (ii) Suppose $s \neq 0$ and $t = 0$. Then from (15) we get

$$F_1^s \equiv G_1^s$$

and so we arrive at a contradiction from (14).
Case (iii): Suppose $s > 0$ and $t = -t_1$, where $t_1 > 0$. Then we have

$$\left[\frac{F_1}{G_1}\right]^s \equiv \left[\frac{F_1 - 1}{G_1 - 1}\right]^{t_1}. \tag{16}$$

If possible suppose F_1 has a pole. Let z_{p_1} be a pole of F_1 of multiplicity p_1. Then from (14) we see that z_{p_1} must be a zero of G_1 of multiplicity p_1. Now from (16) we get $2s = t_1$ and so

$$\left[\frac{F_1}{G_1}\right]^s \equiv \left[\frac{F_1 - 1}{G_1 - 1}\right]^{2s}.$$

This implies that

$$F_1^{s-1} + F_1^{s-2}G_1 + F_1^{s-3}G_1^2 + \ldots + F_1G_1^{s-2} + G_1^{s-1} \equiv G_1^s \frac{(F_1 - 1)^{2s} - (G_1 - 1)^{2s}}{(G_1 - 1)^{2s}(F_1 - G_1)}. \tag{17}$$

If z_p is a zero of $F_1 - 1$ with multiplicity p then the Taylor expansion of $F_1 - 1$ about z_p is

$$F_1 - 1 = a_p(z - z_p)^p + a_{p+1}(z - z_p)^{p+1} + \ldots\ldots, \quad a_p \neq 0.$$

Since $F_1 - 1$ and $G_1 - 1$ share $(0, \infty)$,

$$G_1 - 1 = b_p(z - z_p)^p + b_{p+1}(z - z_p)^{p+1} + \ldots\ldots, \quad b_p \neq 0.$$

Let

$$\Phi_1 = \frac{F_1'}{F_1} - \frac{G_1'}{G_1} \quad \text{and} \quad \Phi_2 = \left(\frac{F_1'}{F_1}\right)^{2s} - \left(\frac{G_1'}{G_1}\right)^{2s}. \tag{18}$$

Since $F_1 \not\equiv cG_1$, where c is a nonzero constant, it follows that $\Phi_1 \not\equiv 0$ and $\Phi_2 \not\equiv 0$. Also

$$T(r, \Phi_1) = S(r, F_1) \quad \text{and} \quad T(r, \Phi_2) = S(r, F_1).$$

From (18) we find

$$\overline{N}_{(2}(r, 1; F_1) = \overline{N}_{(2}(r, 1; G_1) \leq N(r, 0; \Phi_1) = S(r, F_1).$$

Let $p = 1$. If $a_1 = b_1$, then by an elementary calculation gives that $\Phi_1(z) = O((z - z_1)^k)$, where k is a positive integer. This proves that z_1 is a zero of Φ_1. Next we suppose $a_1 \neq b_1$, but $a_1^{2s} = b_1^{2s}$. Then by an elementary calculation we get $\Phi_2(z) = O((z - z_1)^q)$ where q is a positive integer. This proves that z_1

is a zero of Φ_2.
Finally we suppose $a_1 \neq b_1$ and $a_1^{2s} \neq b_1^{2s}$. Therefore from (17) we arrive at a contradiction. Hence

$$N_{1)}(r,1;F_1) = N_{1)}(r,1;G_1) = S(r,F_1).$$

But this is impossible as $\overline{N}(r,1;F_1) \sim T(r,F_1)$ and $\overline{N}(r,1;G_1) \sim T(r,G_1)$. Hence F_1 has no pole. Similarly we can prove that G_1 also has no poles.
Case (iv): Suppose either $s>0$ and $t>0$ or $s<0$ and $t<0$. Then from (15) one can easily prove that F_1 and G_1 have no poles. Consequently from (14) we see that F_1 and G_1 have no zeros. We deduce from (13) that both f and g have no pole.
Since F_1 and G_1 have no zeros and poles, we have

$$F_1 \equiv e^{\gamma_1} G_1,$$

i.e.,

$$[f^n]^{(k)} \equiv e^{\gamma_1} [g^n]^{(k)},$$

where γ_1 is a non-constant entire function. Then from (7) we get

$$[f^n]^{(k)} \equiv c e^{\frac{1}{2}\gamma_1} p, \quad [g^n]^{(k)} \equiv c e^{-\frac{1}{2}\gamma_1} p, \tag{19}$$

where $c = \pm 1$. Since $N(r,0;f) = O(\log r)$ and $N(r,0;g) = O(\log r)$, so we can take

$$f(z) = P_1(z) e^{\alpha_1(z)}, \quad g(z) = Q_1(z) e^{\beta_1(z)}, \tag{20}$$

P_1, Q_1 are nonzero polynomials, α_1, β_1 are two non-constant entire functions. If possible suppose that $P_1(z)$ is not a constant. Let z_1 be a zero of f with multiplicity t. Then z_1 must be a zero of $[f^n]^{(k)}$ with multiplicity $nt-k$. Note that $nt-k \geq n-k \geq k+2l$, as $n \geq 2k+2l$. Clearly z_1 must be a zero of $p^2(z)$ with multiplicity at least $k+2l$, which is impossible since z_1 can be a zero of $p^2(z)$ with multiplicity at most $2l$. Hence $P_1(z)$ is a constant. Similarly we can prove that $Q_1(z)$ is a constant. So we can rewrite f and g as follows

$$f = e^{\alpha}, \quad g = e^{\beta}. \tag{21}$$

We deduce from (7) and (21) that either both α and β are transcendental entire functions or both α and β are polynomials. We now consider following cases.
Subcase 1.1: Let $k \geq 2$.

First we suppose both α and β are transcendental entire functions.
Note that

$$S(r, n\alpha) = S(r, \frac{[f^n]'}{f^n}), \quad S(r, n\beta) = S(r, \frac{[g^n]'}{g^n}).$$

Moreover we see that

$$N(r, 0; [f^n]^{(k)}) \leq N(r, 0; p^2) = O(\log r).$$

$$N(r, 0; [g^n]^{(k)}) \leq N(r, 0; p^2) = O(\log r).$$

From these and using (21) we have

$$N(r, \infty; f^n) + N(r, 0; f^n) + N(r, 0; [f^n]^{(k)}) = S(r, n\alpha) = S(r, \frac{[f^n]'}{f^n}) \tag{22}$$

and

$$N(r, \infty; g^n) + N(r, 0; g^n) + N(r, 0; [g^n]^{(k)}) = S(r, n\beta) = S(r, \frac{[g^n]'}{g^n}). \tag{23}$$

Then from (22), (23) and Lemma 5 we must have

$$f(z) = e^{az+b}, \quad g(z) = e^{cz+d}, \tag{24}$$

where $a \neq 0$, b, $c \neq 0$ and d are constants. But these types of f and g do not agree with the relation (7).
Next we suppose α and β are both polynomials.
Clearly $\alpha + \beta \equiv C$ and $\deg(\alpha) = \deg(\beta)$. Also $\alpha' \equiv \beta'$. If $\deg(\alpha) = \deg(\beta) = 1$, then we again get a contradiction from (7).
Next we suppose $\deg(\alpha) = \deg(\beta) \geq 2$.
We deduce from (21) that

$$\begin{aligned}
&(f^n)' = n\alpha' e^{n\alpha}\\
&(f^n)'' = [n^2(\alpha')^2 + n\alpha'']e^{n\alpha}\\
&(f^n)''' = [n^3(\alpha')^3 + 3n^2\alpha'\alpha'' + n\alpha''']e^{n\alpha}\\
&(f^n)^{(iv)} = [n^4(\alpha')^4 + 6n^3(\alpha')^2\alpha'' + 3n^2(\alpha'')^2 + 4n^2\alpha'\alpha''' + n\alpha^{(iv)}]e^{n\alpha}\\
&(f^n)^{(v)} = [n^5(\alpha')^5 + 10n^4(\alpha')^3\alpha'' + 15n^3\alpha'(\alpha'')^2 + 10n^3(\alpha')^2\\
&\alpha''' + 10n^2\alpha''\alpha''' + 5n^2\alpha'\alpha^{(iv)} + n\alpha^{(v)}]e^{n\alpha}\\
&\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots\\
&[f^n]^{(k)} = [n^k(\alpha')^k + K(\alpha')^{k-2}\alpha'' + P_{k-2}(\alpha')]e^{n\alpha},
\end{aligned}$$

where K is a suitably positive integer and $P_{k-2}(\alpha^{'})$ is a differential polynomial in $\alpha^{'}$.
Similarly we get

$$\begin{aligned}
[g^n]^{(k)} &= [n^k(\beta^{'})^k + K(\beta^{'})^{k-2}\beta^{''} + P_{k-2}(\beta^{'})]e^{n\beta} \\
&= [(-1)^k n^k(\alpha^{'})^k - K(-1)^{k-2}(\alpha^{'})^{k-2}\alpha^{''} + P_{k-2}(-\alpha^{'})]e^{n\beta}.
\end{aligned}$$

Since $\deg(\alpha) \geq 2$, we observe that $\deg((\alpha^{'})^k) \geq k\deg(\alpha^{'})$ and so $(\alpha^{'})^{k-2}\alpha^{''}$ is either a nonzero constant or $\deg((\alpha^{'})^{k-2}\alpha^{''}) \geq (k-1)\deg(\alpha^{'})-1$. Also we see that

$$\deg\left((\alpha^{'})^k\right) > \deg\left((\alpha^{'})^{k-2}\alpha^{''}\right) > \deg\left(P_{k-2}(\alpha^{'})\right) (\text{or } \deg\left(P_{k-2}(-\alpha^{'})\right)).$$

From (19), it is clear that the polynomials

$$n^k(\alpha^{'})^k + K(\alpha^{'})^{k-2}\alpha^{''} + P_{k-2}(\alpha^{'})$$

and

$$(-1)^k n^k(\alpha^{'})^k - K(-1)^{k-2}(\alpha^{'})^{k-2}\alpha^{''} + P_{k-2}(-\alpha^{'})$$

must be identical but this is impossible for $k \geq 2$. Actually the terms $n^k(\alpha^{'})^k + K(\alpha^{'})^{k-2}\alpha^{''}$ and $(-1)^k n^k(\alpha^{'})^k - K(-1)^{k-2}(\alpha^{'})^{k-2}\alpha^{''}$ can not be identical for $k \geq 2$.

Subcase 2: Let $k = 1$. Then from (7) we get

$$AB\alpha^{'}\beta^{'}e^{n(\alpha+\beta)} \equiv p^2, \tag{25}$$

where $AB = n^2$. Let $\alpha+\beta = \gamma$. Suppose that α and β are both transcendental entire functions. From (25) we know that γ is not a constant since in that case we get a contradiction. Then from (25) we get

$$AB\alpha^{'}(\gamma^{'} - \alpha^{'})e^{n\gamma} \equiv p^2. \tag{26}$$

We have $T(r,\gamma^{'}) = m(r,\gamma^{'}) \leq m(r, \frac{(e^{n\gamma})^{'}}{e^{n\gamma}}) + O(1) = S(r, e^{n\gamma})$. Thus from (26) we get

$$\begin{aligned}
T(r, e^{n\gamma}) &\leq T(r, \frac{p^2}{\alpha^{'}(\gamma^{'} - \alpha^{'})}) + O(1) \\
&\leq T(r,\alpha^{'}) + T(r,\gamma^{'} - \alpha^{'}) + O(\log r) + O(1) \\
&\leq 2\,T(r,\alpha^{'}) + S(r,\alpha^{'}) + S(r,e^{n\gamma}),
\end{aligned}$$

which implies that $T(r,e^{n\gamma}) = O(T(r,\alpha^{'}))$ and so $S(r,e^{n\gamma})$ can be replaced by $S(r,\alpha^{'})$. Thus we get $T(r,\gamma^{'}) = S(r,\alpha^{'})$ and so $\gamma^{'}$ is a small function with respect to $\alpha^{'}$. In view of (26) and by the second fundamental theorem for small functions we get

$$\begin{aligned} T(r,\alpha^{'}) &\leq \overline{N}(r,\infty;\alpha^{'}) + \overline{N}(r,0;\alpha^{'}) + \overline{N}(r,0;\alpha^{'}-\gamma^{'}) + S(r,\alpha^{'}) \\ &\leq O(\log r) + S(r,\alpha^{'}), \end{aligned}$$

which shows that $\alpha^{'}$ is a polynomial and so α is a polynomial, which contradicts that α is a transcendental entire function. Next suppose without loss of generality that α is a polynomial and β is a transcendental entire function. Thus γ is transcendental. So in view of (26) we can obtain

$$\begin{aligned} nT(r,e^{\gamma}) &\leq T(r,\frac{p^2}{\alpha^{'}(\gamma^{'}-\alpha^{'})}) + O(1) \\ &\leq T(r,\alpha^{'}) + T(r,\gamma^{'}-\alpha^{'}) + S(r,e^{\gamma}) \\ &\leq T(r,\gamma^{'}) + S(r,e^{\gamma}) = S(r,e^{\gamma}), \end{aligned}$$

which leads a contradiction. Thus α and β are both polynomials. Also from (25) we can conclude that $\alpha+\beta \equiv C$ for a constant C and so $\alpha^{'}+\beta^{'} \equiv 0$. Again from (25) we get $n^2e^{nC}\alpha^{'}\beta^{'} \equiv p^2$. By computation we get

$$\alpha^{'} = cp, \quad \beta^{'} = -cp. \tag{27}$$

Hence

$$\alpha = cQ + b_1, \quad \beta = -cQ + b_2, \tag{28}$$

where $Q(z) = \int_0^z p(z)dz$ and b_1, b_2 are constants. Finally f and g take the form

$$f(z) = c_1e^{cQ(z)}, \quad g(z) = c_2e^{-cQ(z)},$$

where c_1, c_2 and c are constants such that $(nc)^2(c_1c_2)^n = -1$.
Case 2: Let $p(z)$ be a nonzero constant b. Since $n > 2k$, one can easily prove that f and g have no zeros. Now proceeding in the same way as done in proof of **Case 1** we get $f = e^{\alpha}$ and $g = e^{\beta}$, where α and β are two non-constant entire functions.
We now consider following two subcases:
Subcase 2.1: Let $k \geq 2$.
We see that $f^n(z)[f^n(z)]^{(k)} \neq 0$ and $g^n(z)[g^n(z)]^{(k)} \neq 0$. Then by Lemma 6 we must have

$$f(z) = e^{az+b}, \quad g(z) = e^{cz+d}, \tag{29}$$

where $a \neq 0$, b, $c \neq 0$ and d are constants. But from (7) we see that $a+c=0$.
Subcase 2.1: Let $k=1$.
Considering **Subcase 1.2** one can easily get

$$f(z)=e^{az+b}, \quad g(z)=e^{cz+d}, \tag{30}$$

where $a \neq 0$, b, $c \neq 0$ and d are constants. Finally f and g take the form

$$f(z)=c_3e^{dz}, \quad g(z)=c_4e^{-dz},$$

where c_3, c_4 and d are nonzero constants such that $(-1)^k(c_3c_4)^n(nd)^{2k}=b^2$. This completes the proof. □

Lemma 12 *Let f, g be two transcendental meromorphic functions, let n, m and k be three positive integers such that $n>k$. If f and g share $(\infty,0)$ then $[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \not\equiv p^2$, where $p(z)$ is a non zero polynomial.*

Proof. Suppose

$$[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \equiv p^2. \tag{31}$$

Since f and g share $(\infty,0)$ we have from (31) that f and g are transcendental entire functions. So we can take

$$f(z)=h(z)e^{\alpha(z)}, \tag{32}$$

where h is a nonzero polynomial and α is a non-constant entire function. We know that $(w-1)^m = a_mw^m + a_{m-1}w^{m-1} + \ldots + a_0$, where $a_i = (-1)^{m-i}\,{}^mC_{m-i}$, $i=0,1,2,\ldots,m$. Since $f=he^{\alpha}$, then by induction we get

$$(a_if^{n+i})^{(k)} = t_i(\alpha',\alpha'',\ldots,\alpha^{(k)},h,h',\ldots,h^{(k)})e^{(n+i)\alpha}, \tag{33}$$

where $t_i(\alpha',\alpha'',\ldots,\alpha^{(k)},h,h',\ldots,h^{(k)})$ $(i=0,1,2,\ldots,m)$ are differential polynomials in
$\alpha',\alpha'',\ldots,\alpha^{(k)},h,h',\ldots,h^{(k)}$. Obviously

$$t_i(\alpha',\alpha'',\ldots,\alpha^{(k)},h,h',\ldots,h^{(k)}) \not\equiv 0,$$

for $i=0,1,2,\ldots,m$ and $[f^n(f-1)^m]^{(k)} \not\equiv 0$. Now from (31) and (33) we obtain

$$\overline{N}(r,0;t_me^{m\alpha(z)}+\ldots+t_0) \leq N(r,0;p^2) = S(r,f). \tag{34}$$

Since α is an entire function, we obtain $T(r,\alpha^{(j)}) = S(r,f)$ for $j = 1,2,\dots,k$. Hence $T(r,t_i) = S(r,f)$ for $i = 0,1,2,\dots,m$. So from (34) and using second fundamental theorem for small functions (see [17]), we obtain

$$\begin{aligned} mT(r,f) &= T(r,t_m e^{m\alpha}+\ldots+t_1e^{\alpha})+S(r,f) \\ &\leq \overline{N}(r,0;t_m e^{m\alpha}+\ldots+t_1e^{\alpha})+\overline{N}(r,0;t_m e^{m\alpha}+\ldots+t_1e^{\alpha}+t_0) \\ &\quad + S(r,f) \\ &\leq \overline{N}(r,0;t_m e^{(m-1)\alpha}+\ldots+t_1)+S(r,f) \\ &\leq (m-1)T(r,f)+S(r,f), \end{aligned}$$

which is a contradiction. This completes the Lemma. □

Lemma 13 *Let* f *and* g *be two non-constant meromorphic functions and* $\alpha(\not\equiv 0,\infty)$ *be small function of* f *and* g*. Let* n*,* m *and* k *be three positive integers such that* $n \geq m+3$*. Then*

$$[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \not\equiv \alpha^2, \quad \text{for} \quad k=1.$$

Proof. We omit the proof since it can be proved in the line of the proof of Lemma 3 [14]. □

Lemma 14 [1] *If* f,g *be two non-constant meromorphic functions such that they share* $(1,1)$*. Then*

$$2\overline{N}_L(r,1;f)+2\overline{N}_L(r,1;g)+\overline{N}_E^{(2}(r,1;f)-\overline{N}_{f>2}(r,1;g) \leq N(r,1;g)-\overline{N}(r,1;g).$$

Lemma 15 [2] *Let* f*,* g *share* $(1,1)$*. Then*

$$\overline{N}_{f>2}(r,1;g) \leq \frac{1}{2}\overline{N}(r,0;f)+\frac{1}{2}\overline{N}(r,\infty;f)-\frac{1}{2}N_0(r,0;f')+S(r,f),$$

where $N_0(r,0;f')$ *is the counting function of those zeros of* f' *which are not the zeros of* $f(f-1)$*.*

Lemma 16 [2] *Let* f *and* g *be two non-constant meromorphic functions sharing* $(1,0)$*. Then*

$$\begin{aligned} &\overline{N}_L(r,1;f)+2\overline{N}_L(r,1;g)+\overline{N}_E^{(2}(r,1;f)-\overline{N}_{f>1}(r,1;g)-\overline{N}_{g>1}(r,1;f) \\ \leq\ & N(r,1;g)-\overline{N}(r,1;g). \end{aligned}$$

Lemma 17 [2] *Let* f*,* g *share* $(1,0)$*. Then*

$$\overline{N}_L(r,1;f) \leq \overline{N}(r,0;f)+\overline{N}(r,\infty;f)+S(r,f)$$

Lemma 18 [2] *Let* f*,* g *share* $(1, 0)$*. Then*

$$\begin{aligned} &(i) \quad \overline{N}_{f>1}(r, 1; g) \leq \overline{N}(r, 0; f) + \overline{N}(r, \infty; f) - N_0(r, 0; f^{'}) + S(r, f) \\ &(ii) \quad \overline{N}_{g>1}(r, 1; f) \leq \overline{N}(r, 0; g) + \overline{N}(r, \infty; g) - N_0(r, 0; g^{'}) + S(r, g). \end{aligned}$$

3 Proof of the Theorem

Proof of Theorem 1. Let $F = \frac{[f^n P(f)]^{(k)}}{p}$ and $G = \frac{[g^n P(g)]^{(k)}}{p}$, where $P(w) = (w-1)^m$. It follows that F and G share $(1, k_1)$ except for the zeros of $p(z)$.
Case 1 Let $H \not\equiv 0$.
Subcase 1.1 $k_1 \geq 1$.
From (1) it can be easily calculated that the possible poles of H occur at (i) multiple zeros of F and G, (ii) those 1 points of F and G whose multiplicities are different, (iii) poles of F and G, (iv) zeros of $F^{'}(G^{'})$ which are not the zeros of $F(F-1)(G(G-1))$.
Since H has only simple poles we get

$$\begin{aligned} N(r, \infty; H) \leq\ & \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + \overline{N}_*(r, 1; F, G) + \overline{N}(r, 0; F| \geq 2) \\ & + \overline{N}(r, 0; G| \geq 2) + \overline{N}_0(r, 0; F^{'}) + \overline{N}_0(r, 0; G^{'}), \end{aligned} \tag{35}$$

where $\overline{N}_0(r, 0; F^{'})$ is the reduced counting function of those zeros of $F^{'}$ which are not the zeros of $F(F-1)$ and $\overline{N}_0(r, 0; G^{'})$ is similarly defined.
Let z_0 be a simple zero of $F-1$ but $p(z_0) \neq 0$. Then z_0 is a simple zero of $G-1$ and a zero of H. So

$$N(r, 1; F| = 1) \leq N(r, 0; H) \leq N(r, \infty; H) + S(r, f) + S(r, g). \tag{36}$$

While $k_1 \geq 2$, using (35) and (36) we get

$$\begin{aligned} & \overline{N}(r, 1; F) \\ & \leq N(r, 1; F| = 1) + \overline{N}(r, 1; F| \geq 2) \leq \overline{N}(r, \infty; f) \\ & \quad + \ \overline{N}(r, \infty; g) + \overline{N}(r, 0; F| \geq 2) + \overline{N}(r, 0; G| \geq 2) + \overline{N}_*(r, 1; F, G) \\ & \quad + \ \overline{N}(r, 1; F| \geq 2) + \overline{N}_0(r, 0; F^{'}) + \overline{N}_0(r, 0; G^{'}) + S(r, f) + S(r, g). \end{aligned} \tag{37}$$

Now in view of Lemma 3 we get

$$\begin{aligned}
&\overline{N}_0(r,0;G') + \overline{N}(r,1;F \mid\geq 2) + \overline{N}_*(r,1;F,G)\\
&\leq \overline{N}_0(r,0;G') + \overline{N}(r,1;F|\geq 2) + \overline{N}(r,1;F|\geq 3)\\
&\quad = \overline{N}_0(r,0;G') + \overline{N}(r,1;G|\geq 2) + \overline{N}(r,1;G|\geq 3) \qquad (38)\\
&\leq \overline{N}_0(r,0;G') + N(r,1;G) - \overline{N}(r,1;G)\\
&\leq N(r,0;G' \mid G\neq 0) \leq \overline{N}(r,0;G) + \overline{N}(r,\infty;g) + S(r,g),
\end{aligned}$$

Hence using (37), (38), Lemmas 1 and 2 we get from second fundamental theorem that

$$\begin{aligned}
&(n+m)T(r,f)\\
&\leq T(r,F) + N_{k+2}(r,0;f^nP(f)) - N_2(r,0;F) + S(r,f)\\
&\leq \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,1;F) + N_{k+2}(r,0;f^nP(f)) - N_2(r,0;F)\\
&\quad - N_0(r,0;F')\\
&\leq 2\,\overline{N}(r,\infty,f) + \overline{N}(r,\infty;g) + \overline{N}(r,0;F) + N_{k+2}(r,0;f^nP(f))\\
&\quad + \overline{N}(r,0;F|\geq 2) + \overline{N}(r,0;G|\geq 2) + \overline{N}(r,1;F|\geq 2) + \overline{N}_*(r,1;F,G)\\
&\quad + \overline{N}_0(r,0;G') - N_2(r,0;F) + S(r,f) + S(r,g)\\
&\leq 2\,\{\overline{N}(r,\infty;f) + \overline{N}(r,\infty;g)\} + N_{k+2}(r,0;f^nP(f)) + N_2(r,0;G) \qquad (39)\\
&\quad + S(r,f) + S(r,g)\\
&\leq 2\,\{\overline{N}(r,\infty;f) + \overline{N}(r,\infty;g)\} + N_{k+2}(r,0;f^nP(f)) + k\,\overline{N}(r,\infty;g)\\
&\quad + N_{k+2}(r,0;g^nP(g)) + S(r,f) + S(r,g)\\
&\leq 2\,\{\overline{N}(r,\infty;f) + \overline{N}(r,\infty;g)\} + (k+2)\,\overline{N}(r,0;f) + T(r,P(f))\\
&\quad + (k+2)\,\overline{N}(r,0;g) + T(r,P(g)) + k\,\overline{N}(r,\infty;g) + S(r,f) + S(r,g)\\
&\leq (k+4+m)\,T(r,f) + (2k+4+m)\,T(r,g) + S(r,f) + S(r,g)\\
&\leq (3k+8+2m)\,T(r) + S(r).
\end{aligned}$$

In a similar way we can obtain

$$(n+m)\,T(r,g) \leq (3k+8+2m)\,T(r) + S(r). \qquad (40)$$

Combining (39) and (40) we see that

$$(n+m)\,T(r) \leq (3k+8+2m)\,T(r) + S(r),$$

i.e.,

$$(n-3k-8-m)\,T(r) \leq S(r). \qquad (41)$$

Since $n > 3k + 8 + m$, (41) leads to a contradiction.
While $k_1 = 1$, using Lemmas 3, 14, 15, (35) and (36) we get

$$\begin{aligned}
&\overline{N}(r,1;F)\\
&\leq N(r,1;F|=1) + \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}_E^{(2}(r,1;F)\\
&\leq \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + \overline{N}(r,0;F|\geq 2) + \overline{N}(r,0;G|\geq 2)\\
&\quad + \overline{N}_*(r,1;F,G) + \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}_E^{(2}(r,1;F) + \overline{N}_0(r,0;F^{'})\\
&\quad + \overline{N}_0(r,0;G^{'}) + S(r,f) + S(r,g)\\
&\leq \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + \overline{N}(r,0;F|\geq 2) + \overline{N}(r,0;G|\geq 2)\\
&\quad + 2\overline{N}_L(r,1;F) + 2\overline{N}_L(r,1;G) + \overline{N}_E^{(2}(r,1;F) + \overline{N}_0(r,0;F^{'})\\
&\quad + \overline{N}_0(r,0;G^{'}) + S(r,f) + S(r,g)\\
&\leq \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + \overline{N}(r,0;F|\geq 2) + \overline{N}(r,0;G|\geq 2)\\
&\quad + \overline{N}_{F>2}(r,1;G) + N(r,1;G) - \overline{N}(r,1;G) + \overline{N}_0(r,0;F^{'}) + \overline{N}_0(r,0;G^{'})\\
&\quad + S(r,f) + S(r,g)\\
&\leq \frac{3}{2}\,\overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + \overline{N}(r,0;F|\geq 2) + \frac{1}{2}\,\overline{N}(r,0;F)\\
&\quad + \overline{N}(r,0;G|\geq 2) + N(r,1;G) - \overline{N}(r,1;G) + \overline{N}_0(r,0;G^{'})\\
&\quad + \overline{N}_0(r,0;F^{'}) + S(r,f) + S(r,g)\\
&\leq \frac{3}{2}\,\overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + \overline{N}(r,0;F|\geq 2) + \frac{1}{2}\,\overline{N}(r,0;F)\\
&\quad + \overline{N}(r,0;G|\geq 2) + N(r,0;G^{'}|G\neq 0) + \overline{N}_0(r,0;F^{'}) + S(r,f) + S(r,g)\\
&\leq \frac{3}{2}\,\overline{N}(r,\infty;f) + 2\overline{N}(r,\infty;g) + \overline{N}(r,0;F|\geq 2) + \frac{1}{2}\,\overline{N}(r,0;F)\\
&\quad + N_2(r,0;G) + \overline{N}_0(r,0;F^{'}) + S(r,f) + S(r,g).
\end{aligned} \tag{42}$$

Hence using (42), Lemmas 1 and 2 we get from second fundamental theorem that

$$\begin{aligned}
&(n+m)T(r,f)\\
&\leq T(r,F) + N_{k+2}(r,0;f^nP(f)) - N_2(r,0;F) + S(r,f)\\
&\leq \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,1;F) + N_{k+2}(r,0;f^nP(f)) - N_2(r,0;F)\\
&\quad - N_0(r,0;F^{'})\\
&\leq \frac{5}{2}\,\overline{N}(r,\infty,f) + 2\overline{N}(r,\infty;g) + N_2(r,0;F) + \frac{1}{2}\overline{N}(r,0;F)
\end{aligned} \tag{43}$$

$$
\begin{aligned}
&+N_{k+2}(r,0;f^nP(f))+N_2(r,0;G)-N_2(r,0;F)+S(r,f)+S(r,g)\\
\le\ &\frac{5}{2}\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+N_{k+2}(r,0;f^nP(f))+\frac{1}{2}\overline{N}(r,0;F)\\
&+N_2(r,0;G)+S(r,f)+S(r,g)\\
\le\ &\frac{5}{2}\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+N_{k+2}(r,0;f^nP(f))+k\,\overline{N}(r,\infty;g)\\
&+N_{k+2}(r,0;g^nP(g))+\frac{1}{2}\{k\overline{N}(r,\infty;f)\\
&+N_{k+1}(r,0;f^nP(f))\}+S(r,f)+S(r,g)\\
\le\ &\frac{5+k}{2}\,\overline{N}(r,\infty;f)+(k+2)\overline{N}(r,\infty;g)+\frac{3k+5}{2}\,\overline{N}(r,0;f)\\
&+\frac{3}{2}\,T(r,P(f))+(k+2)\,\overline{N}(r,0;g)+T(r,P(g))+S(r,f)+S(r,g)\\
\le\ &\left(2k+5+\frac{3m}{2}\right)\,T(r,f)+(2k+4+m)\,T(r,g)+S(r,f)+S(r,g)\\
\le\ &\left(4k+9+\frac{5m}{2}\right)\,T(r)+S(r).
\end{aligned}
$$

In a similar way we can obtain

$$
(n+m)\,T(r,g)\le\left(4k+9+\frac{5m}{2}\right)\,T(r)+S(r). \tag{44}
$$

Combining (43) and (44) we see that

$$
\left(n-4k-9-\frac{3m}{2}\right)\,T(r)\le S(r). \tag{45}
$$

Since $n>4k+9+\frac{3m}{2}$, (45) leads to a contradiction.
Subcase 1.2 $k_1=0$. Here (36) changes to

$$
N_E^{1)}(r,1;F\mid=1)\le N(r,0;H)\le N(r,\infty;H)+S(r,F)+S(r,G). \tag{46}
$$

Using Lemmas 3, 16, 17, 18, (35) and (46) we get

$$
\begin{aligned}
&\overline{N}(r,1;F)\\
&\le N_E^{1)}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)\\
&\le \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F\mid\ge 2)+\overline{N}(r,0;G\mid\ge 2)\\
&\quad+\overline{N}_*(r,1;F,G)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)+\overline{N}_0(r,0;F')
\end{aligned} \tag{47}
$$

$$
\begin{aligned}
&+ \overline{N}_0(r,0;G^{'}) + S(r,f) + S(r,g)\\
&\le \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + \overline{N}(r,0;F| \ge 2) + \overline{N}(r,0;G| \ge 2)\\
&\quad + 2\overline{N}_L(r,1;F) + 2\overline{N}_L(r,1;G) + \overline{N}_E^{(2}(r,1;F)\\
&\quad + \overline{N}_0(r,0;F^{'}) + \overline{N}_0(r,0;G^{'}) + S(r,f) + S(r,g)\\
&\le \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + \overline{N}(r,0;F| \ge 2) + \overline{N}(r,0;G| \ge 2)\\
&\quad + \overline{N}_{F>1}(r,1;G) + \overline{N}_{G>1}(r,1;F) + \overline{N}_L(r,1;F) + N(r,1;G) - \overline{N}(r,1;G)\\
&\quad + \overline{N}_0(r,0;F^{'}) + \overline{N}_0(r,0;G^{'}) + S(r,f) + S(r,g)\\
&\le 3\,\overline{N}(r,\infty;f) + 2\overline{N}(r,\infty;g) + N_2(r,0;F) + \overline{N}(r,0;F) + N_2(r,0;G)\\
&\quad + N(r,1;G) - \overline{N}(r,1;G) + \overline{N}_0(r,0;G^{'}) + \overline{N}_0(r,0;F^{'})\\
&\quad + S(r,f) + S(r,g)\\
&\le 3\,\overline{N}(r,\infty;f) + 2\overline{N}(r,\infty;g) + N_2(r,0;F) + \overline{N}(r,0;F) + N_2(r,0;G)\\
&\quad + N(r,0;G^{'}|G \ne 0) + \overline{N}_0(r,0;F^{'}) + S(r,f) + S(r,g)\\
&\le 3\overline{N}(r,\infty;f) + 3\overline{N}(r,\infty;g) + N_2(r,0;F) + \overline{N}(r,0;F) + N_2(r,0;G)\\
&\quad + \overline{N}(r,0;G) + \overline{N}_0(r,0;F^{'}) + S(r,f) + S(r,g).
\end{aligned}
$$

Hence using (47), Lemmas 1 and 2 we get from second fundamental theorem that

$$
\begin{aligned}
&(n+m)T(r,f)\\
&\le T(r,F) + N_{k+2}(r,0;f^nP(f)) - N_2(r,0;F) + S(r,f)\\
&\le \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,1;F) + N_{k+2}(r,0;f^nP(f)) - N_2(r,0;F)\\
&\quad - N_0(r,0;F^{'})\\
&\le 4\overline{N}(r,\infty,f) + 3\overline{N}(r,\infty;g) + N_2(r,0;F) + 2\,\overline{N}(r,0;F)\\
&\quad + N_{k+2}(r,0;f^nP(f)) + N_2(r,0;G) + \overline{N}(r,0;G) - N_2(r,0;F)\\
&\quad + S(r,f) + S(r,g)\\
&\le 4\overline{N}(r,\infty;f) + 3\overline{N}(r,\infty;g) + N_{k+2}(r,0;f^nP(f)) + 2\,\overline{N}(r,0;F)\\
&\quad + N_2(r,0;G) + \overline{N}(r,0;G) + S(r,f) + S(r,g)\\
&\le 4\overline{N}(r,\infty;f) + 3\overline{N}(r,\infty;g) + N_{k+2}(r,0;f^nP(f)) + 2k\overline{N}(r,\infty;f)\\
&\quad + 2\,N_{k+1}(r,0;f^nP(f)) + k\,\overline{N}(r,\infty;g) + N_{k+2}(r,0;g^nP(g))\\
&\quad + k\overline{N}(r,\infty;g) + N_{k+1}(r,0;g^nP(g)) + S(r,f) + S(r,g)
\end{aligned}
\tag{48}
$$

$$\leq (2k+4)\ \overline{N}(r,\infty;f) + (2k+3)\overline{N}(r,\infty;g) + (3k+4)\overline{N}(r,0;f)$$
$$+\,3T(r,P(f)) + (2k+3)\ \overline{N}(r,0;g) + 2T(r,P(g)) + S(r,f) + S(r,g)$$
$$\leq (5k+8+3m)\ T(r,f) + (4k+6+2m)\ T(r,g) + S(r,f) + S(r,g)$$
$$\leq (9k+14+5m)\ T(r) + S(r).$$

In a similar way we can obtain

$$(n+m)\ T(r,g) \leq (9k+14+5m)\ \ T(r) + S(r). \tag{49}$$

Combining (48) and (49) we see that

$$(n-9k-14-4m)\ \ T(r) \leq S(r). \tag{50}$$

Since $n > 9k+14+4m$, (50) leads to a contradiction.
Case 2. Let $H \equiv 0$. Then by Lemma 10 we get either

$$f^n(f-1)^m \equiv g^n(g-1)^m \tag{51}$$

or

$$[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \equiv p^2. \tag{52}$$

We now consider following two subcases.
Subcase 2.1: Let $m = 0$.
Now from (51) we get $f^n \equiv g^n$ and so $f \equiv tg$, where t is a constant satisfying $t^n = 1$.
Also from (52) we get

$$[f^n]^{(k)}[g^n]^{(k)} \equiv p^2.$$

Then by Lemma 11 we get the conclusion (1).
Subcase 2.2: Let $m \geq 1$.
Applying Lemma 13, from (52) we see that

$$[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \not\equiv p^2,$$

for $k = 1$.
In addition, when f and g share $(\infty, 0)$, then by Lemma 12 we must have

$$[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \not\equiv p^2.$$

Next we consider the relation (51) and let $h = \frac{g}{f}$.

First we suppose that h is non-constant.
For $m=1$: Then from (51) we get $f\equiv\frac{1-h^n}{1-h^{n+1}}$, i.e.,

$$f\equiv(\frac{h^n}{1+h+h^2+\ldots+h^n}-1).$$

Hence by Lemma 1 we get

$$T(r,f)=T(r,\sum_{j=0}^{n}\frac{1}{h^j})+O(1)=n\,T(r,\frac{1}{h})+S(r,h)=n\,T(r,h)+S(r,h).$$

Similarly we have $T(r,g)=nT(r,h)+S(r,h)$. Therefore $S(r,f)=S(r,g)=S(r,h)$.
Also it is clear that

$$\sum_{j=1}^{n}\overline{N}(r,u_j;h)\leq\overline{N}(r,\infty;f),$$

where $u_j=exp(\frac{2j\pi i}{n+1})$ and $j=1,2,\ldots,n$.
Then by the second fundamental theorem we get

$$(n-2)\;T(r,h)\leq\sum_{j=1}^{n}\overline{N}(r,u_j;h)+S(r,f)\leq\overline{N}(r,\infty;f)+S(r,f).$$

Similarly we have

$$(n-2)\;T(r,h)\leq\overline{N}(r,\infty;g)+S(r,g).$$

Adding and simplifying these we get

$$2(n-2)T(r,h)\leq n(2-\Theta(\infty;f)-\Theta(\infty;g)+\varepsilon)T(r,h)+S(r,h),$$

where $0<\varepsilon<\Theta(\infty;f)+\Theta(\infty;g)$. This leads to a contradiction as $\Theta(\infty;f)+\Theta(\infty;g)>\frac{4}{n}$.

For $m\geq 2$: Then from (51) we can say that f and g satisfying the algebraic equation $R(f,g)=0$, where

$$R(\omega_1,\omega_2)=\omega_1^n(\omega_1-1)^m-\omega_2^n(\omega_2-1)^m.$$

Next we suppose that h is a constant.
Then from (51) we get

$$f^n\sum_{i=0}^{m}(-1)^i\;{}^mC_{m-i}\;f^{m-i}\equiv g^n\sum_{i=0}^{m}(-1)^i\;{}^mC_{m-i}g^{m-i}. \tag{53}$$

Now substituting $g = fh$ in (53) we get

$$\sum_{i=0}^{m} (-1)^i \, {}^mC_{m-i} \, f^{n+m-i}(h^{n+m-i} - 1) \equiv 0,$$

which implies that $h = 1$. Hence $f \equiv g$. This completes the proof. □

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References

[1] T. C. Alzahary, H. X. Yi, Weighted value sharing and a question of I. Lahiri, *Complex Var. Theory Appl.*, **49** (15) (2004), 1063–1078.

[2] A. Banerjee, Meromorphic functions sharing one value, *Int. J. Math. Math. Sci.*, **22** (2005), 3587–3598.

[3] M. L. Fang, X. H. Hua, Entire functions that share one value, *J. Nanjing Univ. Math. Biquarterly*, **13** (1) (1996), 44–48.

[4] M. L. Fang, Uniqueness and value-sharing of entire functions, *Comput. Math. Appl.*, **44** (2002), 823–831.

[5] G. Frank, Eine Vermutung Von Hayman über Nullslellen meromorphic Funktion, *Math. Z.*, **149** (1976), 29–36.

[6] W. K. Hayman, *Meromorphic Functions*, The Clarendon Press, Oxford (1964).

[7] I. Lahiri, Weighted sharing and uniqueness of meromorphic functions, *Nagoya Math. J.*, **161** (2001), 193–206.

[8] I. Lahiri, Weighted value sharing and uniqueness of meromorphic functions, *Complex Var. Theory Appl.*, **46** (2001), 241–253.

[9] I. Lahiri, S. Dewan, Value distribution of the product of a meromorphic function and its derivative, *Kodai Math. J.*, **26** (2003), 95–100.

[10] I. Lahiri, A. Sarkar, Nonlinear differential polynomials sharing 1-points with weight two, *Chinese J. Contemp. Math.*, **25** (3) (2004), 325–334.

[11] P. Li, C. C. Yang, On the characteristics of meromorphic functions that share three values CM, *J. Math. Anal. Appl.*, **220** (1998), 132–145.

[12] L. Liu, Uniqueness of meromorphic functions and differential polynomials, *Comput. Math. Appl.*, **56** (2008), 3236–3245.

[13] R. Nevanlinna, Einige Eindeutigkeitssätze in der Theorie der Meromorphen Funktionen, *Acta Math.*, **48** (1926), 367–391.

[14] P. Sahoo, Uniqueness and weighted value sharing of meromorphic functions, *Applied. Math. E-Notes.*, **11** (2011), 23–32.

[15] C. C. Yang, On deficiencies of differential polynomials II, *Math. Z.*, **125** (1972), 107–112.

[16] C. C. Yang, X. H. Hua, Uniqueness and value sharing of meromorphic functions, *Ann. Acad. Sci. Fenn. Math.*, **22** (1997), 395–406.

[17] K. Yamanoi, The second main theorem for small functions and related problems, *Acta Math.*, **192** (2004), 225–294.

[18] H. X. Yi, On characteristic function of a meromorphic function and its derivative, *Indian J. Math.*, **33** (2)(1991), 119–133.

[19] H. X. Yi, C. C. Yang, *Uniqueness Theory of meromorphic functions*, Science Press, Beijing, 1995.

[20] Q. C. Zhang, Meromorphic function that shares one small function with its derivative, *J. Inequal. Pure Appl. Math.*, **6** (4)(2005), Art.116 [ONLINE *http://jipam.vu.edu.au/*].

[21] X. Y. Zhang, W. C. Lin, Uniqueness and value sharing of entire functions, *J. Math. Anal. Appl.*, **343** (2008), 938–950.

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