On the maximal exponent of the prime power divisor of integers

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Abstract. The largest exponent of the prime powers function is investigated on the set of numbers of form one plus squares of primes.

1 Introduction

1.1. Notation. Let, as usual, $\mathcal{P}$, $\mathbb{N}$ be the set of primes, positive integers, respectively. For a prime divisor p of n let $v_p(n)$ be defined by $p^{v_p(n)} \parallel n$. Then $n = \prod_{p|n} p^{v_p(n)}$. Let

$$H(n) = \max_{p|n} v_p(n) \quad \text{and} \quad h(n) = \min_{p|n} v_p(n).$$

We denote by $\pi(x)$ the number of primes $p \leq x$ and by $\pi(x, k, \ell)$ the number of primes $p \leq x, p \equiv \ell \pmod{k}$.

1.2. Preliminaries. A. Niven proved in [7] that

$$\sum_{n \leq x} h(n) = x + \frac{\zeta(3/2)}{\zeta(3)} \sqrt{x} + o(\sqrt{x}) \quad (x \rightarrow \infty) \quad (1)$$

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and that

$$\frac{1}{x} \sum_{n \leq x} H(n) \rightarrow B \quad (x \rightarrow \infty), \quad \text{where} \quad B = 1 + \sum_{k=2}^{\infty} \left(1 - \frac{1}{\zeta(k)}\right). \quad (2)$$

W. Schwarz and J. Spilker showed in [8] that

$$\sum_{n \leq x} H(n) = \mathcal{M}(H)x + O\left(x^{3/4} \exp(-\gamma \sqrt{\log x})\right) \quad (x \rightarrow \infty), \quad (3)$$

$$\sum_{n \leq x} \frac{1}{H(n)} = \mathcal{M}\left(\frac{1}{H}\right)x + O(x^{3/4} \exp(-\gamma \sqrt{\log x})) \quad (x \rightarrow \infty), \quad (4)$$

where $\gamma > 0$ is a suitable constant, $\mathcal{M}(H) = B$, $\mathcal{M}\left(\frac{1}{H}\right)$ are suitable positive numbers.

D. Suryanayana and Sita Ramachandra Rao [9] proved that the error term in (3) and (4) can be improved to

$$O(\sqrt{x} \exp(-\gamma(\log x)^{3/5}(\log \log x)^{-1/5})).$$

They proved furthermore that

$$\sum_{n \leq x} h(n) = c_1 x + c_2 x^{1/2} + c_3 x^{1/3} + c_4 x^{1/4} + c_5 x^{1/5} + O(x^{1/6}), \quad (5)$$

$$\sum_{n \leq x} \frac{1}{h(n)} = d_1 x + d_2 x^{1/2} + d_3 x^{1/3} + d_4 x^{1/4} + d_5 x^{1/5} + O(x^{1/6}). \quad (6)$$

Gu Tongxing and Cao Huizhong announced in [4] that they can improve the error term in (3) to

$$O(\sqrt{x} \exp(-c(\log x)^{3/5}(\log \log x)^{-1/5})).$$

I. Kátai and M. V. Subbarao [5] investigated the asymptotic of

$$A_x(r) := \sharp\{n \in [x, x+Y] \mid H(n) = r\}, \quad Y = x^{\frac{1}{2r+1}} \log x,$$

and

$$B_x(r) := \sharp\{p \in \mathcal{P}, p \in [x, x+Y] \mid H(p+1) = r\}, \quad Y = x^{\frac{7}{12}+\epsilon}$$

for fixed $r \geq 1$.

Namely, they proved that

$$A_x(r) = Y\left(\eta(r+1) - \eta(r)\right) + O\left(\frac{Y}{\log x}\right), \quad \eta(s) = \frac{1}{\zeta(s)} - 1 \quad (s = 1, 2, \dots)$$

and

$$B_x(r) = e(r) \frac{Y}{\log x} + O\left(\frac{Y}{(\log x)^2}\right),$$

where

$$e(1) = \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p(p-1)}\right),$$

and for $r \geq 2$

$$e(r) = \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{(p-1)p^r}\right) - \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{(p-1)p^{r-1}}\right).$$

In [6] we can read some results on (5) assuming the Riemann conjecture.

Our main interest now is to give the asymptotic of the number of those $n \leq x, n \in \mathcal{B}$, for which $H(n) = r$ uniformly as $1 \leq r \leq \kappa(x)$, where $\kappa(x)$ is as large as it is possible. We shall investigate it when $\mathcal{B}$ = set of shifted primes.

1.3. Auxiliary results.

Lemma 1 (Brun-Titchmarsh inequality). *We have*

$$\pi(x, k, \ell) < C \frac{x}{\varphi(k) \log \frac{x}{k}}.$$

Lemma 2 (Siegel-Walfisz theorem). *We have*

$$\pi(x, k, \ell) = \frac{\text{lix}}{\varphi(k)} \left(1 + O(e^{-c\sqrt{\log x}})\right)$$

uniformly as $(k, \ell) = 1, k \leq (\log x)^A$. Here A is arbitrary, $c > 0$ is a fixed constant.

Lemma 3 ([1]) *Let q be an odd prime, $D = q^n$ ($n = 1, 2, \dots$), $\epsilon > 0$ be an arbitrary small, and M be an arbitrary large positive number. Then the asymptotic law*

$$\pi(x, D, \ell) = \frac{\text{lix}}{\varphi(D)} \left(1 + O\left((\log x)^{-M}\right)\right)$$

holds for $D \leq x^{3/8-\epsilon}$, $(\ell, D) = 1$.

Lemma 4 ([2]) *Let α be an integer, $\alpha \geq 2$. If $A > 0$, then there is a $B > 0$ for which*

$$\sum_{\substack{d \leq \frac{x^{1/2}}{q} (\log x)^{-B} \\ (d, q) = 1}} \max_{(r, qd) = 1} \max_{y \leq x} \left| \pi(y, qd, r) - \frac{\text{lix}}{\varphi(qd)} \right| \ll \frac{x}{\varphi(q)(\log x)^A}, \text{lix} = \int_2^x \frac{du}{\log u}$$

uniformly for moduli $q \leq x^{1/3} \exp(-(\log \log x)^3)$ that are powers of α .

While the implicit constant in $\ll$ may depend upon α , B is a function of A alone. $B = A + 6$ is permissible.

We shall use a special consequence of this assertion:

Corollary. *Let α be an integer, $\alpha \geq 2$, $D = \alpha^n$ ($n = 1, 2, \dots$), $D \leq x^{1/3} \exp(-(\log \log x)^3)$. Let $A > 0$ be an arbitrary constant. Then*

$$\pi(x, D, \ell) = \frac{\text{lix}}{\varphi(D)} \left(1 + O\left(\frac{1}{(\log x)^A}\right) \right), \quad (\ell, D) = 1$$

Lemma 5 ([3]) *Let $q = p^r$, p an odd prime, $qx^{\frac{3}{5}+\epsilon} \leq h \leq x$. Then*

$$\pi(x+h, q, \ell) - \pi(x, q, \ell) = (1 + o_x(1)) \frac{h}{\varphi(q) \log x}$$

as $x \rightarrow \infty$, $(\ell, q) = 1$.

2 Formulation of the theorems

Let $(0 <) U, V$ be coprime integers, and let Q be the smallest prime for which

$$U(1+2m) + V \equiv 0 \pmod{Q}$$

has a solution, that is

$$Q = \begin{cases} 2 & \text{if } 2|U+V \\ \text{smallest prime for which } (Q2U) = 1, & \text{if } 2 \nmid U+V. \end{cases}$$

Let

$$M_{U,V}(x | k) = \#\{p \leq x \mid H(Up + V) = k\}.$$

Theorem 1 Assume that $r(x) \rightarrow \infty$ arbitrarily slowly. Then, in the interval $r(x) < k < (\frac{1}{3} - \epsilon) \frac{\log x}{\log Q}$, we have

$$M_{U,V}(x | k) = \frac{\text{lix}}{\varphi(Q^k)} \left(1 - \frac{1}{Q}\right) \cdot (1 + o_x(1)).$$

Let $P(n) = n^2 + 1$. Then $4 \nmid P(n)$, $3 \nmid P(n)$, $5 \mid P(2)$, $5 \mid P(3)$. For every k there exists $1 \leq \ell_k < \frac{5^k}{2}$, such that $P(\ell_k) \equiv 0 \pmod{5^k}$. The congruence $P(n) \equiv 0 \pmod{5^k}$ has exactly two solutions: ℓ_k and $5^k - \ell_k$. It obvious that $(\ell_k, 5) = 1$.

Let

$$E(x | k) = \#\{p \leq x \mid H(p^2 + 1) = k\}.$$

Theorem 2 Assume that $r(x) \rightarrow \infty$ arbitrarily slowly. Then, in the interval $r(x) < k < (\frac{1}{3} - \epsilon) \frac{\log x}{\log 5}$, we have

$$E(x | k) = \frac{2}{5^k} \text{lix}(1 + o_x(1)).$$

3 Proof of Theorem 1.

It is obvious that

$$M_{U,V}(x | k) \leq \sum_q^* \left[Q(x, q^k, r_{q,k}) - Q(x, q^{k+1}, r_{q,k+1}) \right],$$

where q runs over all those primes for which $U(1 + 2m) + V \equiv 0 \pmod{q}$ has a solution, $r_{q,k} \equiv VU^{-1} \pmod{q^k}$, $r_{q,k+1} \equiv VU^{-1} \pmod{q^{k+1}}$.

By using Lemma 3 and Lemma 1 we obtain that

$$\begin{aligned} M_{U,V}(x | k) &\leq \frac{\text{lix}}{\varphi(Q^k)} \left(1 - \frac{1}{Q}\right) \cdot \left(1 + O\left(\frac{1}{(\log x)^M}\right)\right) + \\ &+ C \sum_{\substack{q > Q \\ q \in \mathcal{P}}} \frac{\text{lix}}{\varphi(q^k)} + C \sum_{\substack{Q < q \\ q^k \geq \sqrt{x}}} \frac{x}{q^k}. \end{aligned}$$

It is clear that

$$\sum_{\substack{q > Q \\ q \in \mathcal{P}}} \frac{1}{\varphi(q^k)} = \frac{o_x(1)}{\varphi(Q^k)}$$

and that

$$\sum_{\substack{q^k \geq \sqrt{x} \\ q \in \mathcal{P}}} \frac{1}{\varphi(q^k)} = O\left(\frac{1}{x^{1/4}}\right),$$

thus

$$M_{U,V}(x | k) \leq \left(1 + o_x(1)\right) \frac{\text{li } x}{Q^k}.$$

On the other hand

$$M_{U,V}(x | k) \geq \left[Q(x, Q^k, r_{Q,k}) - Q(x, Q^{k+1}, r_{Q,k+1}) \right] - \sum_{\substack{q > Q \\ q \in \mathcal{P}}} Q(x, Q^k q^k, r_{Qq,k}).$$

The sum on right hand side is less than

$$C \frac{\text{li } x}{Q^k} \sum_{\substack{(Q <) q}} \frac{1}{q^k} + O(x^{3/4}) \leq o_x(1) \frac{\text{li } x}{Q^k}.$$

From Lemma 3 our theorem follows.

4 Proof of Theorem 2

We have

$$E(x | k) = S + O(T),$$

where

$$S = \sharp\{p \leq x : 5^k \parallel p^2 + 1\}$$

and

$$T = \sum_{\substack{q \in \mathcal{P} \\ q > 5}} \sharp\{p \leq x : q^k \parallel p^2 + 1\}.$$

Thus, by using Lemma 1 and $k \geq \gamma(x)$,

$$T \leq \sum_{\substack{q \in \mathcal{P} \\ q > 5}} \frac{2C \text{li } x}{\varphi(q^k)} + \sum_{\substack{q^k > x \\ q \in \mathcal{P}}} \frac{x}{q^k} = o_x(1) \frac{\text{li } x}{5^k}.$$

Hence we obtain that

$$E(x | k) \leq \frac{2}{5^k} \text{li } x (1 + o_x(1)).$$

On the other hand

$$E(x | k) \geq S - \sum_{\substack{q \in \mathcal{P} \\ q > 5}} \mathfrak{h}\{p \leq x : 5^k \cdot q^k \parallel p^2 + 1\}.$$

By using Lemma 1, the sum on the right can be overestimated by

$$\frac{Cl x}{5^k} \sum_{q > 5} \frac{1}{\varphi(q^k)} + \frac{x}{5^k} \sum_{q^k > \sqrt{x}} \frac{1}{q^k},$$

which is clearly $o_x(1)S$.

This completes the proof of Theorem 2.

5 Further remarks

By using Lemma 5 we can prove short interval version of Theorem 1 and 2.

Theorem 3 *Let $5^k x^{3/5+\epsilon} \leq h \leq x$, $k \geq g(x)$. Then*

$$E(x + h | k) - E(x) = \frac{h}{5^k} \frac{1}{\log x} (1 + o_x(1)).$$

Theorem 4 *Let U, V be coprime integers, $U > 0$, $U + V = \text{odd}$, Q be the smallest prime which is not a divisor of $2U$. Let $k \geq g(x)$, $Q^k x^{3/5+\epsilon} \leq h \leq x$. Then*

$$M_{U,V}(x + h | k) - M_{U,V}(x) = (1 + o_x(1)) \frac{h}{Q^k} \frac{1}{\log x} \quad \text{as } x \rightarrow \infty.$$

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