All maximal idempotent submonoids of $\text{Hyp}_G(2)$

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Abstract. The purpose of this paper is to determine all maximal idempotent submonoids and some maximal compatible idempotent submonoids of the monoid of all generalized hypersubstitutions of type $\tau = (2)$.

1 Introduction

In Universal Algebra, identities are used to classify algebras into collections, called varieties and hyperidentities are used to classify varieties into collections, called hypervarieties. The concept of a hypersubstitution is a tool to study hyperidentities and hypervarieties. The notion of a hypersubstitution originated by K. Denecke, D. Lau, R. Pöschel and D. Schweigert [3]. In 2000, S. Leeratanavalee and K. Denecke generalized the concepts of a hypersubstitution and a hyperidentity to the concepts of a generalized hypersubstitution and a strong hyperidentity, respectively [4]. The set of all generalized hypersubstitutions together with a binary operation and the identity hypersubstitution forms a monoid. There are several published papers on algebraic properties of this monoid and its submonoids.

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The concept of regular subsemigroup plays an important role in the theory of semigroup. The concept of an idempotent submonoid is an example of a regular subsemigroup. In 2013, W. Puninagool and S. Leeratanavalee studied the natural partial order on the set $E(\text{Hyp}_G(2))$ of all idempotent elements of $\text{Hyp}_G(2)$, see [6]. In 2012, the authors studied the natural partial order on $\text{Hyp}_G(2)$, see [7]. In this paper we determine all maximal idempotent submonoids and give some maximal compatible idempotent submonoids of $\text{Hyp}_G(2)$ under this partial order.

2 Generalized hypersubstitutions

Let $n \in \mathbb{N}$ be a natural number and $X_n := \{x_1, x_2, \dots, x_n\}$ be an n -element set. Let $\{f_i \mid i \in I\}$ be a set of n_i -ary operation symbols indexed by the set I . We call the sequence $\tau = (n_i)_{i \in I}$ of arities of f_i , the *type*. An n -ary term of type τ is defined inductively by the following.

- (i) Every $x_i \in X_n$ is an n -ary term of type τ .
- (ii) If $t_1, t_2, \dots, t_{n_i}$ are n -ary terms of type τ , then $f_i(t_1, t_2, \dots, t_{n_i})$ is an n -ary term of type τ .

We denote the smallest set which contains $x_1, \dots, x_n$ and is closed under finite number of applications of (ii) by $W_\tau(X_n)$ and let $W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n)$ be the set of all terms of type τ .

A mapping σ from $\{f_i \mid i \in I\}$ into $W_\tau(X)$ which does not necessarily preserve the arity is called a *generalized hypersubstitution of type τ* . The set of all generalized hypersubstitutions of type τ is denoted by $\text{Hyp}_G(\tau)$. In general, to combine two mappings together we use a composition of mappings. But in this case to combine two generalized hypersubstitutions we need the concept of a generalized superposition of terms and the extension of a generalized hypersubstitution which are defined by the following.

Definition 1 A generalized superposition of terms is a mapping $S^m : W_\tau(X)^{m+1} \longrightarrow W_\tau(X)$ where

- (i) $S^m(x_j, t_1, \dots, t_m) := t_j, 1 \leq j \leq m,$
- (ii) $S^m(x_j, t_1, \dots, t_m) := x_j, m < j \in \mathbb{N},$
- (iii) $S^m(f_i(s_1, \dots, s_{n_i}), t_1, \dots, t_m) := f_i(S^m(s_1, t_1, \dots, t_m), \dots, S^m(s_{n_i}, t_1, \dots, t_m)).$

Definition 2 Let $\sigma \in \text{Hyp}_G(\tau)$. The extension of σ is a mapping $\hat{\sigma} : W_\tau(X) \longrightarrow W_\tau(X)$ where

- (i) $\hat{\sigma}[x] := x \in X$,
- (ii) $\hat{\sigma}[f_i(t_1, \dots, t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_{n_i}])$, for any n_i -ary operation symbol f_i where $\hat{\sigma}[t_j]$, $1 \leq j \leq n_i$ are already defined.

Proposition 1 ([4]) For arbitrary $t, t_1, t_2, \dots, t_n \in W_\tau(X)$ and for arbitrary generalized hypersubstitution $\sigma, \sigma_1, \sigma_2$ we have

- (i) $S^n(\hat{\sigma}[t], \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_n]) = \hat{\sigma}[S^n(t, t_1, \dots, t_n)]$,
- (ii) $(\hat{\sigma}_1 \circ \hat{\sigma}_2)^\wedge = \hat{\sigma}_1 \circ \hat{\sigma}_2$.

The binary operation of two generalized hypersubstitutions σ_1, σ_2 is defined by $\sigma_1 \circ_G \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ where $\circ$ denotes the usual composition of mappings. It turns out that $\text{Hyp}_G(\tau)$ together with the identity element σ_{id} where $\sigma_{\text{id}}(f_i) = f_i(x_1, \dots, x_{n_i})$ is a monoid under $\circ_G$, see [4].

3 All Maximal idempotent submonoids of $\text{Hyp}_G(2)$

We recall first the definition of an idempotent element of a semigroup. Let S be a semigroup. An element $a \in S$ is called *idempotent* if $aa = a$. We denote the set of all idempotent elements of a semigroup S by $E(S)$. Let $E(S) \neq \emptyset$. Define $a \leq b$ ($a, b \in E(S)$) iff $a = ab = ba$. Then $\leq$ is a partial order on $E(S)$. We call $\leq$ a natural partial order on $E(S)$. A natural partial order $\leq$ on a semigroup S is said to be a *compatible* if $a \leq b$ implies $ac \leq bc$ and $ca \leq cb$ for all $a, b, c \in S$. Throughout this paper, let f be a binary operation symbol of type $\tau = (2)$. By σ_t we denote a generalized hypersubstitution which maps f to the term $t \in W_{(2)}(X)$. For $t \in W_{(2)}(X)$ we introduce the following notation:

- (i) $\text{leftmost}(t) :=$ the first variable (from the left) occurring in t ,
- (ii) $\text{rightmost}(t) :=$ the last variable occurring in t ,
- (iii) $\text{var}(t) :=$ the set of all variables occurring in t .

Let $\sigma_t \in \text{Hyp}_G(2)$, we denote $R_1 := \{\sigma_t \mid t = f(x_1, t') \text{ where } t' \in W_{(2)}(X) \text{ and } x_2 \notin \text{var}(t')\}$, $R_2 := \{\sigma_t \mid t = f(t', x_2) \text{ where } t' \in W_{(2)}(X) \text{ and } x_1 \notin \text{var}(t')\}$, $R_3 := \{\sigma_t \mid t \in \{x_1, x_2, f(x_1, x_2)\}\}$ and $R_4 := \{\sigma_t \mid \text{var}(t) \cap \{x_1, x_2\} = \emptyset\}$.

In 2008, W. Puninagool and S. Leeratanavalee [5] proved that: $\bigcup_{i=1}^4 R_i = E(\text{Hyp}_G(2))$.

Example 1 Let $\sigma_s \in R_1$ and $\sigma_t \in R_2$ such that $s = f(x_1, s')$ and $t = f(t', x_2)$ where $s' = f(x_4, x_1)$ and $t' = f(x_2, x_6)$. Consider

$$\begin{aligned} (\sigma_s \circ_G \sigma_t)(f) &= \widehat{\sigma}_s[f(f(x_2, x_6), x_2)] \\ &= S^2(f(x_1, f(x_4, x_1)), \widehat{\sigma}_s[f(x_2, x_6)], \widehat{\sigma}_s[x_2]) \\ &= S^2(f(x_1, f(x_4, x_1)), f(x_2, f(x_4, x_2)), x_2) \\ &= f(f(x_2, f(x_4, x_2)), f(x_4, f(x_2, f(x_4, x_2)))). \end{aligned}$$

So $\sigma_s \circ_G \sigma_t \notin \bigcup_{i=1}^4 R_i$.

By the previous example, we have $\bigcup_{i=1}^4 R_i$ is not a subsemigroup of $\text{Hyp}_G(2)$.

Let $\sigma_t \in \text{Hyp}_G(2)$, we denote $R'_1 := \{\sigma_t \mid t = f(x_1, t') \text{ where } t' \in W_{(2)}(X), x_2 \notin \text{var}(t') \text{ and } \text{rightmost}(t') \neq x_1\}$ and $R'_2 := \{\sigma_t \mid t = f(t', x_2) \text{ where } t' \in W_{(2)}(X), x_1 \notin \text{var}(t') \text{ and } \text{leftmost}(t') \neq x_2\}$.

We denote $(\text{MI})_{\text{Hyp}_G(2)} = R'_1 \cup R'_2 \cup R_3 \cup R_4$, $(\text{MI}_1)_{\text{Hyp}_G(2)} = R_1 \cup R_3 \cup R_4$ and $(\text{MI}_2)_{\text{Hyp}_G(2)} = R_2 \cup R_3 \cup R_4$.

Proposition 2 $(\text{MI})_{\text{Hyp}_G(2)}$ is an idempotent submonoid of $\text{Hyp}_G(2)$.

Proof. It is clear that $(\text{MI})_{\text{Hyp}_G(2)} \subseteq \text{Hyp}_G(2)$ and every element in $(\text{MI})_{\text{Hyp}_G(2)}$ is idempotent. Next, we show that $(\text{MI})_{\text{Hyp}_G(2)}$ is a submonoid of $\text{Hyp}_G(2)$.

Case 1: $\sigma_t \in R'_1$. Then $t = f(x_1, t')$ where $t' \in W_{(2)}(X)$ such that $x_2 \notin \text{var}(t')$ and $\text{rightmost}(t') \neq x_1$. Let $\sigma_s \in (\text{MI})_{\text{Hyp}_G(2)}$.

Case 1.1: $\sigma_s \in R'_1$. Then $s = f(x_1, s')$ where $x_2 \notin \text{var}(s')$ and $\text{rightmost}(s') \neq x_1$. Consider

$$\begin{aligned} (\sigma_t \circ_G \sigma_s)(f) &= \widehat{\sigma}_t[f(x_1, s')] \\ &= S^2(f(x_1, t'), x_1, \widehat{\sigma}_t[s']) \\ &= f(x_1, t') \quad \text{since } x_2 \notin \text{var}(t'). \end{aligned}$$

Then $\sigma_t \circ_G \sigma_s \in R'_1 \subseteq (\text{MI})_{\text{Hyp}_G(2)}$.

Case 1.2: $\sigma_s \in R'_2$. Then $s = f(s', x_2)$ where $x_1 \notin \text{var}(s')$ and $\text{leftmost}(s') \neq x_2$. Consider $(\sigma_s \circ_G \sigma_t)(f) = \widehat{\sigma}_s[f(x_1, t')] = S^2(f(s', x_2), x_1, \widehat{\sigma}_s[t'])$

$= f(S^2(s', x_1, w), S^2(x_2, x_1, w))$, where $w = \widehat{\sigma}_s[t']$. Since $x_2 \notin \text{var}(t')$ and $\text{rightmost}(t') \neq x_1$, then $x_1, x_2 \notin \text{var}(w)$. Since $x_1 \notin \text{var}(s')$ and $x_1, x_2 \notin \text{var}(w)$, then $x_1, x_2 \notin \text{var}(S^2(s', x_1, w))$. Consider $(\sigma_t \circ_G \sigma_s)(f) = \widehat{\sigma}_t[f(s', x_2)] = S^2(f(x_1, t'), \widehat{\sigma}_t[s'], x_2) = f(S^2(x_1, u, x_2), S^2(t', u, x_2))$, where $u = \widehat{\sigma}_t[s']$. Since $x_1 \notin \text{var}(s')$ and $\text{leftmost}(s') \neq x_2$, we have $x_1, x_2 \notin \text{var}(u)$. Since $x_2 \notin \text{var}(t')$ and $x_1, x_2 \notin \text{var}(u)$, we have $x_1, x_2 \notin \text{var}(S^2(t', u, x_2))$. Then $\sigma_s \circ_G \sigma_t, \sigma_t \circ_G \sigma_s \in R'_4 \subseteq (\text{MI})_{\text{Hyp}_G(2)}$.

Case 1.3: $\sigma_s \in R_3$. Then $s = x_1$ or $s = x_2$ or $s = f(x_1, x_2)$.

If $s = x_1$, then $(\sigma_t \circ_G \sigma_s)(f) = \widehat{\sigma}_t[x_1] = x_1$ and $(\sigma_s \circ_G \sigma_t)(f) = \widehat{\sigma}_{x_1}[f(x_2, t')] = S^2(x_1, x_2, \widehat{\sigma}_{x_1}[t']) = x_2$.

If $s = x_2$, then $(\sigma_t \circ_G \sigma_s)(f) = \widehat{\sigma}_t[x_2] = x_2$ and $(\sigma_s \circ_G \sigma_t)(f) = \widehat{\sigma}_{x_2}[f(x_2, t')] = S^2(x_2, x_2, \widehat{\sigma}_{x_2}[t'])$.

Since $x_1 \notin \text{var}(t')$ and $\text{rightmost}(t') \neq x_2$, then $S^2(x_2, x_2, \widehat{\sigma}_{x_2}[t']) = x_i \notin \{x_1, x_2\}$.

If $s = f(x_1, x_2)$, then $\sigma_s = \sigma_{\text{id}}$ such that $\sigma_t \circ_G \sigma_{\text{id}} = \sigma_t = \sigma_{\text{id}} \circ_G \sigma_t$.

Therefore $\sigma_s \circ_G \sigma_t, \sigma_s \circ_G \sigma_t \in (\text{MI})_{\text{Hyp}_G(2)}$.

Case 1.4: $\sigma_s \in R_4$. Then $s = f(s_1, s_2)$ where $x_1, x_2 \notin \text{var}(s)$. Consider $(\sigma_t \circ_G \sigma_s)(f) = \widehat{\sigma}_t[f(s_1, s_2)] = S^2(f(x_2, t'), \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2]) = f(S^2(x_2, w_1, w_2), S^2(t', w_1, w_2))$, where $w_1 = \widehat{\sigma}_t[s_1]$ and $w_2 = \widehat{\sigma}_t[s_2]$. Then $x_1, x_2 \notin \text{var}(w_1) \cup \text{var}(w_2)$. The consequence is $x_1, x_2 \notin \text{var}(S^2(t', w_1, w_2))$.

Since $x_1, x_2 \notin \text{var}(w_2) \cup \text{var}(S^2(t', w_1, w_2))$, so that $\sigma_t \circ_G \sigma_s \in R'_4 \subseteq (\text{MI})_{\text{Hyp}_G(2)}$. Consider $(\sigma_s \circ_G \sigma_t)(f) = \widehat{\sigma}_s[f(x_2, t')] = S^2(f(s_1, s_2), x_2, \widehat{\sigma}_s[t']) = f(s_1, s_2)$ since $x_1, x_2 \notin \text{var}(s)$. So that $\sigma_s \circ_G \sigma_t \in R_4 \subseteq (\text{MI})_{\text{Hyp}_G(2)}$.

Case 2: $\sigma_t \in R'_2$ and $\sigma_s \in R'_2 \cup R_3 \cup R_4$. It can be proved similarly as in Case 1. Then we have $\sigma_t \circ_G \sigma_s, \sigma_s \circ_G \sigma_t \in (\text{MI})_{\text{Hyp}_G(2)}$.

Case 3: $\sigma_t \in R_3$ and $\sigma_s \in R_3 \cup R_4$. It can be proved similarly as in Case 1.3. Then we have $\sigma_t \circ_G \sigma_s, \sigma_s \circ_G \sigma_t \in (\text{MI})_{\text{Hyp}_G(2)}$.

Case 4: $\sigma_t \in R_4$ and $\sigma_s \in R_4$. Then $\sigma_t \circ_G \sigma_s = \sigma_t \in R_4 \subseteq (\text{MI})_{\text{Hyp}_G(2)}$.

Therefore $(\text{MI})_{\text{Hyp}_G(2)}$ is a submonoid of $\text{Hyp}_G(2)$. $\square$

Corollary 1 $(\text{MI}_1)_{\text{Hyp}_G(2)}$ and $(\text{MI}_2)_{\text{Hyp}_G(2)}$ are idempotent submonoids of $\text{Hyp}_G(2)$.

Proposition 3 $(\text{MI})_{\text{Hyp}_G(2)}$ is a maximal idempotent submonoid of $\text{Hyp}_G(2)$.

Proof. Let K be a proper idempotent submonoid of $\text{Hyp}_G(2)$ such that $(\text{MI})_{\text{Hyp}_G(2)} \subseteq K \subset \text{Hyp}_G(2)$. Let $\sigma_t \in K$. Then σ_t is an idempotent element.

Case 1: $\sigma_t \in R_1 \setminus R'_1$. Then $t = f(x_1, t')$ where $x_2 \notin \text{var}(t')$ and $\text{rightmost}(t') =$

x_1 . Choose $\sigma_s \in R'_2 \subseteq K$, then $s = f(s', x_2)$ such that $x_1 \notin \text{var}(s')$ and $\text{leftmost}(s') \neq x_2$. Consider $(\sigma_s \circ_G \sigma_t)(f) = \widehat{\sigma}_s[f(x_1, t')] = S^2(f(s', x_2), x_1, \widehat{\sigma}_s[t']) = f(S^2(s', x_1, w), S^2(x_2, x_1, w))$ where $w = \widehat{\sigma}_s[t']$. Since $x_2 \in \text{var}(s)$ and $\text{rightmost}(t') = x_1$, we have $x_1 \in \text{var}(w)$ and $S^2(s', x_1, w) \in W_{(2)}(X) \setminus X$. Since $x_1 \in \text{var}(w)$, $\sigma_s \circ_G \sigma_t$ is not idempotent. So $\sigma_t \in R'_1$.

Case 2: $\sigma_t \in R_2 \setminus R'_2$. Then $t = f(t', x_2)$ where $x_1 \notin \text{var}(t')$ and $\text{leftmost}(t') = x_2$. Choose $\sigma_s \in R'_1 \subseteq K$, then $s = f(x_1, s')$ such that $x_2 \notin \text{var}(s')$ and $\text{rightmost}(s') \neq x_1$. Consider $(\sigma_s \circ_G \sigma_t)(f) = \widehat{\sigma}_s[f(t', x_2)] = S^2(f(x_1, s'), \widehat{\sigma}_s[t'], x_2) = f(S^2(x_1, w, x_2), S^2(s', w, x_2))$, where $w = \widehat{\sigma}_s[t']$. Since $x_1 \in \text{var}(s)$ and $\text{leftmost}(t') = x_2$, we have $x_2 \in \text{var}(w)$ and $S^2(s', w, x_2) \in W_{(2)}(X) \setminus X$. Since $x_2 \in \text{var}(w)$, $\sigma_s \circ_G \sigma_t$ is not idempotent. So $\sigma_t \in R'_2$. Then $\sigma_t \in (\text{MI})_{\text{Hyp}_G(2)}$. Therefore $K \subseteq (\text{MI})_{\text{Hyp}_G(2)}$ and thus $K = (\text{MI})_{\text{Hyp}_G(2)}$. $\square$

Proposition 4 $(\text{MI}_1)_{\text{Hyp}_G(2)}$ is a maximal idempotent submonoid of $\text{Hyp}_G(2)$.

Proof. Let K be a proper idempotent submonoid of $\text{Hyp}_G(2)$ such that $(\text{MI}_1)_{\text{Hyp}_G(2)} \subseteq K \subset \text{Hyp}_G(2)$. Let $\sigma_t \in K$. Then σ_t is an idempotent element. If $\sigma_t \in R_2$. Then $t = f(t', x_2)$ where $x_1 \notin \text{var}(t')$. Choose $\sigma_s \in R_1$ such that $s = f(x_1, s')$ where $x_2 \notin \text{var}(s')$, $s' \in W_{(2)}(X) \setminus X$ and $\text{rightmost}(s') = x_1$. Consider $(\sigma_t \circ_G \sigma_s)(f) = \widehat{\sigma}_t[f(x_1, s')] = S^2(f(t', x_2), x_1, \widehat{\sigma}_t[s']) = f(S^2(t', x_1, w), S^2(x_2, x_1, w))$, where $w = \widehat{\sigma}_t[s']$. Since $x_2 \in \text{var}(t)$, we have $x_1 \in \text{var}(w)$ and $S^2(t', x_1, w) \in W_{(2)}(X) \setminus X$. Since $x_1 \in \text{var}(w)$, $\sigma_t \circ_G \sigma_s$ is not idempotent, so $\sigma_t \in (\text{MI}_1)_{\text{Hyp}_G(2)}$. Therefore $K = (\text{MI}_1)_{\text{Hyp}_G(2)}$. $\square$

Proposition 5 $(\text{MI}_2)_{\text{Hyp}_G(2)}$ is a maximal idempotent submonoid of $\text{Hyp}_G(2)$.

Proof. The proof is similar to the proof of Proposition 4. $\square$

Corollary 2 $\{(\text{MI})_{\text{Hyp}_G(2)}, (\text{MI}_1)_{\text{Hyp}_G(2)}, (\text{MI}_2)_{\text{Hyp}_G(2)}\}$ is the set of all maximal idempotent submonoids of $\text{Hyp}_G(2)$.

Proposition 6 ([6]) Let σ_t be an idempotent element. Then $\sigma_{x_1} \leq \sigma_t$ if and only if $\text{leftmost}(t) = x_1$.

Proposition 7 ([6]) Let σ_t be an idempotent element. Then $\sigma_{x_2} \leq \sigma_t$ if and only if $\text{rightmost}(t) = x_2$.

Proposition 8 For each $t \in W_{(2)}(X)$ where $x_2 \notin \text{var}(t)$, $\{\sigma_{x_1}, \sigma_{\text{id}}, \sigma_{f(x_1, t)}\}$ is a maximal compatible idempotent submonoid of $\text{Hyp}_G(2)$.

Proof. By using Proposition 6, $\sigma_{x_1} \leq \sigma_{f(x_1,t)}$. Then $\sigma_{x_1} = \sigma_{x_1} \circ_G \sigma_{f(x_1,t)} = \sigma_{f(x_1,t)} \circ_G \sigma_{x_1}$ and σ_{id} is the identity element. We have $\{\sigma_{x_1}, \sigma_{id}, \sigma_{f(x_1,t)}\}$ is an idempotent submonoid of $\text{Hyp}_G(2)$. Since

$$\sigma_{f(x_1,t)} \circ_G \sigma_{x_1} = \sigma_{x_1} \circ_G \sigma_{f(x_1,t)} = \sigma_{x_1} \circ_G \sigma_{x_1} = \sigma_{x_1} \leq \sigma_{f(x_1,t)} = \sigma_{f(x_1,t)} \circ_G \sigma_{f(x_1,t)}.$$

We have $\{\sigma_{x_1}, \sigma_{id}, \sigma_{f(x_1,t)}\}$ is a compatible idempotent submonoid of $\text{Hyp}_G(2)$.

Let K be a proper compatible idempotent submonoid of $\text{Hyp}_G(2)$ such that $\{\sigma_{x_1}, \sigma_{id}, \sigma_{f(x_1,t)}\} \subseteq K \subset \text{Hyp}_G(2)$. Let $\sigma_s \in K$. Then σ_s is an idempotent element.

Case 1: $\sigma_s \in R_1 \setminus \{\sigma_{x_1}, \sigma_{id}, \sigma_{f(x_1,t)}\}$. Then $s = f(x_1, s')$ where $x_2 \notin \text{var}(s')$. Since K is a compatible idempotent submonoid and $\sigma_{f(x_1,t)} \leq \sigma_{id}$, we have $\sigma_{f(x_1,t)} \circ_G \sigma_{f(x_1,s')} = \sigma_{f(x_1,t)} \leq \sigma_{f(x_1,s')} = \sigma_{id} \circ_G \sigma_{f(x_1,s')}$ which is a contradiction.

Case 2: $\sigma_s \in R_2$. Then $s = f(s', x_2)$ where $x_1 \notin \text{var}(s')$. Since K is a compatible idempotent submonoid and $\sigma_{f(s',x_2)} \leq \sigma_{id}$, we have $\sigma_{x_1} \circ_G \sigma_{f(s',x_2)} = \sigma_{\text{leftmost}(s')} \leq \sigma_{x_1} = \sigma_{x_1} \circ_G \sigma_{f(x_1,s')}$. So $\text{leftmost}(s') = x_1$ which is a contradiction.

Case 3: $\sigma_s = \sigma_{x_2}$. Since K is a compatible idempotent submonoid and $\sigma_{x_1} \leq \sigma_{id}$, we have $\sigma_{x_2} \circ_G \sigma_{x_1} = \sigma_{x_1} \leq \sigma_{x_2} = \sigma_{x_2} \circ_G \sigma_{id}$ which is a contradiction.

Case 4: $\sigma_s \in R_4$. Then $s = f(s_1, s_2) \in W_{(2)}X \setminus X$ where $x_1, x_2 \notin \text{var}(s)$. Since K is a compatible idempotent submonoid and $\sigma_{x_1} \leq \sigma_{id}$, we have $\sigma_s \circ_G \sigma_{x_1} = \sigma_{x_1} \leq \sigma_s = \sigma_s \circ_G \sigma_{id}$ which is a contradiction.

Therefore $K = \{\sigma_{x_1}, \sigma_{id}, \sigma_{f(x_1,t)}\}$ is a maximal compatible idempotent submonoid of $\text{Hyp}_G(2)$. $\square$

Proposition 9 *For each $t \in W_{(2)}X$ where $x_1 \notin \text{var}(t)$, $\{\sigma_{x_2}, \sigma_{id}, \sigma_{f(t,x_2)}\}$ is a maximal compatible idempotent submonoid of $\text{Hyp}_G(2)$.*

Proof. The proof is similar to the proof of Proposition 8. $\square$

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