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Some classes of analytic and multivalent functions associated with q-derivative operators

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Abstract. By applying the q-derivative operator of order m ($m \in \mathbb{N}_0$), we introduce two new subclasses of p-valently analytic functions of complex order. For these classes of functions, we obtain the coefficient inequalities and distortion properties. Some consequences of the main results are also considered.

1 Introduction and preliminaries

The theory of q-analysis in recent past has been applied in many areas of mathematics and physics, as for example, in the areas of ordinary fractional calculus, optimal control problems, q-difference and q-integral equations, and in q-transform analysis. One may refer to the books [5], [7], and the recent papers [1], [2], [3], [4], [8] and [12] on the subject. Purohit and Raina recently in [10], [11] have used the fractional q-calculus operators in investigating certain classes of functions which are analytic in the open disk. Purohit [9] also studied

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similar work and considered new classes of multivalently analytic functions in the open unit disk.

In the present paper, we aim at introducing some new subclasses of functions defined by applying the q-derivative operator of order m ($m \in \mathbb{N}_0$) which are p-valent and analytic in the open unit disk. The results derived include the coefficient inequalities and distortion theorems for the subclasses defined and introduced below. Some consequences of the main results are also pointed out in the concluding section.

To make this paper self contained, we present below the basic definitions and related details of the q-calculus, which are used in the sequel.

The q-shifted factorial (see [5]) is defined for α, $q \in \mathbb{C}$ as a product of n factors by

$$(\alpha; q)_n = \begin{cases} 1; & n = 0 \\ (1-\alpha)(1-\alpha q)\dots(1-\alpha q^{n-1}); & n \in \mathbb{N} \end{cases}, \tag{1}$$

and in terms of the basic analogue of the gamma function by

$$(q^\alpha; q)_n = \frac{(1-q)^n \Gamma_q(\alpha+n)}{\Gamma_q(\alpha)} \quad (n > 0), \tag{2}$$

where the q-gamma function is defined by [5, p. 16, eqn. (1.10.1)]

$$\Gamma_q(x) = \frac{(q;q)_\infty (1-q)^{1-x}}{(q^x;q)_\infty} \; (0 < q < 1). \tag{3}$$

If $|q| < 1$, the definition (1) remains meaningful for $n = \infty$, as a convergent infinite product given by

$$(\alpha; q)_\infty = \prod_{j=0}^{\infty} (1-\alpha q^j) \, .$$

We recall here the following q-analogue definitions given by Gasper and Rahman [5]. The recurrence relation for q-gamma function is given by

$$\Gamma_q(x+1) = \frac{(1-q^x)\,\Gamma_q(x)}{1-q}, \tag{4}$$

and the q-binomial expansion is given by

$$(x-y)_\nu = x^\nu(-y/x;q)_\nu = x^\nu \prod_{n=0}^{\infty} \left[\frac{1-(y/x)q^n}{1-(y/x)q^{\nu+n}}\right]. \tag{5}$$

Also, the Jackson's q-derivative and q-integral of a function f defined on a subset of $\mathbb{C}$ are, respectively, defined by (see Gasper and Rahman [5, pp. 19, 22])

$$D_{q,z}f(z) = \frac{f(z) - f(zq)}{z(1-q)} \quad (z \neq 0, q \neq 1) \tag{6}$$

and

$$\int_0^z f(t)\, d_q t = z(1-q) \sum_{k=0}^{\infty} q^k f(zq^k). \tag{7}$$

Following the image formula for fractional q-derivative [10, pp. 58–59], namely:

$$D^{\alpha}_{q,z} z^{\lambda} = \frac{\Gamma_q(1+\lambda)}{\Gamma_q(1+\lambda-\alpha)}\, z^{\lambda-\alpha} \quad (\alpha \geq 0,\ \lambda > -1), \tag{8}$$

we have for $\alpha = m$ $(m \in \mathbb{N})$:

$$D^{m}_{q,z} z^{\lambda} = \frac{\Gamma_q(1+\lambda)}{\Gamma_q(1+\lambda-m)}\, z^{\lambda-m} \quad (m \in \mathbb{N},\ \lambda > -1). \tag{9}$$

Further, in view of the relation that

$$\operatorname*{Lim}_{q \to 1^-} \frac{(q^{\alpha};q)_n}{(1-q)^n} = (\alpha)_n, \tag{10}$$

we observe that the q-shifted factorial (1) reduces to the familiar Pochhammer symbol $(\alpha)_n$, where $(\alpha)_0 = 1$ and $(\alpha)_n = \alpha(\alpha+1)\dots(\alpha+n-1)$ $(n \in \mathbb{N})$.

2 New classes of functions

By $\mathcal{A}_p(n)$, we denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=n+p}^{\infty} a_k\, z^k \qquad (n,\ p \in \mathbb{N}), \tag{11}$$

which are analytic and p-valent in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}$. Also, let $\mathcal{A}^-_p(n)$ denote the subclass of $\mathcal{A}_p(n)$ consisting of analytic and p-valent functions expressed in the form

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k\, z^k \quad (a_k \geq 0, n,\ p \in \mathbb{N}). \tag{12}$$

Differentiating (12) m times with respect to z and making use of (9), we get

$$D^{m}_{q,z}f(z) = \frac{\Gamma_q(1+p)}{\Gamma_q(1+p-m)}\, z^{p-m} - \sum_{k=n+p}^{\infty} a_k \frac{\Gamma_q(1+k)}{\Gamma_q(1+k-m)}\, z^{k-m} \tag{13}$$
$$(n,\ p \in \mathbb{N}, m \in \mathbb{N}_0, p > m).$$

By applying the q-derivative operator of order m to the function $f(z)$, we define here a new subclass $\mathcal{M}^{m}_{n,p}(\lambda, \delta, q)$ of the p-valent class $\mathcal{A}^{-}_{p}(n)$, which consist of functions $f(z)$ satisfying the inequality that

$$\left| \frac{1}{\delta} \left\{ \frac{z\, D^{1+m}_{q,z}f(z) + \lambda\, q\, z^2\, D^{2+m}_{q,z}f(z)}{\lambda\, z\, D^{1+m}_{q,z}f(z) + (1-\lambda)\, D^{m}_{q,z}f(z)} - [p-m]_q \right\} \right| < 1, \tag{14}$$
$$(m < p; p \in \mathbb{N}, m \in \mathbb{N}_0;\ 0 \leq \lambda \leq 1;\ \delta \in \mathbb{C} \setminus \{0\};\ 0 < q < 1;\ z \in \mathbb{U}),$$

where the q-natural number is expressed as

$$[n]_q = \frac{1-q^n}{1-q} \quad (0 < q < 1). \tag{15}$$

Also, let $\mathcal{N}^{m}_{n,p}(\lambda, \delta, q)$ denote the subclass of $\mathcal{A}^{-}_{p}(n)$ consisting of functions $f(z)$ which satisfy the inequality that

$$\left| \frac{1}{\delta} \left\{ D^{1+m}_{q,z}f(z) + \lambda\, z\, D^{2+m}_{q,z}f(z) - [p-m]_q \right\} \right| < [p-m]_q, \tag{16}$$
$$(m < p; p \in \mathbb{N}, m \in \mathbb{N}_0;\ 0 \leq \lambda \leq 1;\ \delta \in \mathbb{C} \setminus \{0\};\ 0 < q < 1;\ z \in \mathbb{U}).$$

The following results give the characterization properties for functions of the form (12) which belong to the classes defined above.

Theorem 1 *Let the function* $f(z)$ *be defined by* (12), *then* $f(z) \in \mathcal{M}^{m}_{n,p}(\lambda, \delta, q)$ *if and only if*

$$\sum_{k=n+p}^{\infty} (|\delta| - q^{k-m}[p-k]_q)\, \Delta(k, m, \lambda, q)\, a_k \leq |\delta|\, \Delta(p, m, \lambda, q), \tag{17}$$

where $\Delta(k, m, \lambda, q)$ *is given by*

$$\Delta(k, m, \lambda, q) = \frac{(1 + [k-m-1]_q\, q\, \lambda)\Gamma_q(1+k)}{\Gamma_q(1+k-m)}, \tag{18}$$

such that

$$\Delta(p, m, \lambda, q) - \sum_{k=n+p}^{\infty} \Delta(k, m, \lambda, q)\, a_k > 0. \tag{19}$$

The result is sharp.

Proof. Let $f(z) \in \mathcal{M}^{m}_{n,p}(\lambda, \delta, q)$, then on using (14), we get

$$\Re\left\{ \frac{z\ D^{1+m}_{q,z} f(z) + \lambda\ q\ z^2\ D^{2+m}_{q,z} f(z)}{\lambda\ z\ D^{1+m}_{q,z} f(z) + (1-\lambda)\ D^{m}_{q,z} f(z)} - [p-m]_q \right\} > -|\delta|. \tag{20}$$

Now, in view of (13), we have

$$\begin{aligned}
\mathcal{N} &\equiv z\ D^{1+m}_{q,z} f(z) + \lambda\ q\ z^2\ D^{2+m}_{q,z} f(z) \\
&= z \left[\frac{\Gamma_q(1+p)}{\Gamma_q(p-m)}\ z^{p-m-1} - \sum_{k=n+p}^{\infty} a_k \frac{\Gamma_q(1+k)}{\Gamma_q(k-m)}\ z^{k-m-1} \right] \\
&\quad + \lambda\ q z^2 \left[\frac{\Gamma_q(1+p)}{\Gamma_q(p-m-1)}\ z^{p-m-2} - \sum_{k=n+p}^{\infty} a_k \frac{\Gamma_q(1+k)}{\Gamma_q(k-m-1)}\ z^{k-m-2} \right] \\
&= \Gamma_q(1+p) z^{p-m} \left[\frac{1}{\Gamma_q(p-m)} + \frac{\lambda\ q}{\Gamma_q(p-m-1)} \right] \\
&\quad - \sum_{k=n+p}^{\infty} a_k\ \Gamma_q(1+k)\ z^{k-m} \left[\frac{1}{\Gamma_q(k-m)} + \frac{\lambda\ q}{\Gamma_q(k-m-1)} \right] \\
&= \frac{[p-m]_q (1 + [p-m-1]_q\ q\ \lambda) \Gamma_q(1+p)}{\Gamma_q(1+p-m)} z^{p-m} \\
&\quad - \sum_{k=n+p}^{\infty} a_k\ \frac{[k-m]_q (1 + [k-m-1]_q\ q\ \lambda) \Gamma_q(1+k)}{\Gamma_q(1+k-m)}\ z^{k-m} \\
&= [p-m]_q\ \Delta(p, m, \lambda, q)\ z^{p-m} - \sum_{k=n+p}^{\infty} a_k\ [k-m]_q\ \Delta(k, m, \lambda, q)\ z^{k-m},
\end{aligned}$$

where $\Delta(k, m, \lambda, q)$ is given by (18).
Similarly, we can obtain

$$\begin{aligned}
\mathcal{D} \equiv \lambda\ z\ D^{1+m}_{q,z} f(z) + (1-\lambda)\ D^{m}_{q,z} f(z) &= \Delta(p, m, \lambda, q) z^{p-m} \\
&\quad - \sum_{k=n+p}^{\infty} a_k\ \Delta(k, m, \lambda, q)\ z^{k-m}.
\end{aligned}$$

Hence

$$\mathcal{N} - [p-m]_q \mathcal{D} = \sum_{k=n+p}^{\infty} q^{k-m}\ [p-k]_q\ \Delta(k, m, \lambda, q)\ a_k\ z^{k-m}.$$

Therefore, from (20), we obtain the simplified form of the inequality that

$$\Re\left(\frac{\sum_{k=n+p}^{\infty} q^{k-m}\ [p-k]_q\ \Delta(k,m,\lambda,q)\ a_k\ z^{k-m}}{\Delta(p,m,\lambda,q)z^{p-m}-\sum_{k=n+p}^{\infty}\Delta(k,m,\lambda,q)\ a_k\ z^{k-m}}\right) > -|\delta|\,. \tag{21}$$

By putting $z=r$, the denominator of (21) (say $\mathrm{DN}(r)$) becomes

$$\begin{aligned}\mathrm{DN}(r) &= \Delta(p,m,\lambda,q)r^{p-m} - \sum_{k=n+p}^{\infty}\Delta(k,m,\lambda,q)\ a_k\ r^{k-m}\\ &= r^{p-m}\left(\Delta(p,m,\lambda,q) - \sum_{k=n+p}^{\infty}\Delta(k,m,\lambda,q)\ a_k\ r^{k-p}\right),\end{aligned}$$

which is positive for $r=0$, and also remains positive for $0<r<1$, with the condition (19). So that on letting $r\to 1^-$ through real values, we get the desired assertion (17) of Theorem 1.

To prove the converse of Theorem 1, first we would show that

$$\begin{aligned}&\left|\frac{z\ D_{q,z}^{1+m}f(z)+\lambda\ q\ z^2\ D_{q,z}^{2+m}f(z)}{\lambda\ z\ D_{q,z}^{1+m}f(z)+(1-\lambda)\ D_{q,z}^{m}f(z)} - [p-m]_q\right|\\ &\quad\le \frac{\sum_{k=n+p}^{\infty} q^{k-m}\ [p-k]_q\ \Delta(k,m,\lambda,q)\ a_k}{\Delta(p,m,\lambda,q)-\sum_{k=n+p}^{\infty}\ \Delta(k,m,\lambda,q)\ a_k}.\end{aligned} \tag{22}$$

We have

$$\begin{aligned}&\left|\frac{z\ D_{q,z}^{1+m}f(z)+\lambda\ q\ z^2\ D_{q,z}^{2+m}f(z)}{\lambda\ z\ D_{q,z}^{1+m}f(z)+(1-\lambda)\ D_{q,z}^{m}f(z)} - [p-m]_q\right|\\ &= \frac{\left|\sum_{k=n+p}^{\infty} q^{k-m}\ [p-k]_q\ \Delta(k,m,\lambda,q)\ a_k z^{k-m}\right|}{\left|\Delta(p,m,\lambda,q)z^{p-m}-\sum_{k=n+p}^{\infty}\ \Delta(k,m,\lambda,q)\ a_k z^{k-m}\right|}.\end{aligned} \tag{23}$$

On the other hand if $|z|=1$, then

$$\begin{aligned}&\left|\sum_{k=n+p}^{\infty} q^{k-m}\ [p-k]_q\ \Delta(k,m,\lambda,q)\ a_k z^{k-m}\right|\\ &\le \sum_{k=n+p}^{\infty}\left|q^{k-m}\ [p-k]_q\ \Delta(k,m,\lambda,q)\ a_k z^{k-m}\right|\\ &= \sum_{k=n+p}^{\infty} q^{k-m}\ [p-k]_q\ \Delta(k,m,\lambda,q)\ a_k\end{aligned} \tag{24}$$

and

$$\begin{aligned} &\left|\Delta(p,m,\lambda,q)z^{p-m} - \sum_{k=n+p}^{\infty} \Delta(k,m,\lambda,q)a_k z^{k-m}\right| \\ &\geq \left|\Delta(p,m,\lambda,q)z^{p-m}\right| - \sum_{k=n+p}^{\infty} \left|\Delta(k,m,\lambda,q)a_k z^{k-m}\right| \\ &= \Delta(p,m,\lambda,q) - \sum_{k=n+p}^{\infty} \Delta(k,m,\lambda,q)\ a_k. \end{aligned} \tag{25}$$

Now (23), (24) and (25) imply (22), and then by applying the hypothesis (17), we find that

$$\begin{aligned} &\left|\frac{z\ D_{q,z}^{1+m}f(z) + \lambda\ q\ z^2\ D_{q,z}^{2+m}f(z)}{\lambda\ z\ D_{q,z}^{1+m}f(z) + (1-\lambda)\ D_{q,z}^{m}f(z)} - [p-m]_q\right| \\ &\leq \frac{|\delta|\left\{\Delta(p,m,\lambda,q) - \sum_{k=n+p}^{\infty}\ \Delta(k,m,\lambda,q)\ a_k\right\}}{\Delta(p,m,\lambda,q) - \sum_{k=n+p}^{\infty}\ \Delta(k,m,\lambda,q)\ a_k} = |\delta|\,. \end{aligned}$$

Hence, by the maximum modulus principle, we infer that

$$f(z) \in \mathcal{M}_{n,p}^{m}(\lambda,\delta,q).$$

It is easy to verify that the equality in (17) is attained for the function $f(z)$ given by

$$f(z) = z^p - \frac{|\delta|\,\Delta(p,m,\lambda,q)}{(|\delta| + q^{p-m}[n]_q)\Delta(n+p,m,\lambda,q)}\ z^{n+p} \quad (m < p; p,n \in \mathbb{N}, m \in \mathbb{N}_0), \tag{26}$$

where $\Delta(p,m,\lambda,q)$ is given by (18). □

We now derive the following corollaries from Theorem 1.

From Theorem 1, we easily get the following corollary:

Corollary 1 *If the function* $f(z)$ *is defined by* (12) *and* $f(z) \in \mathcal{M}_{n,p}^{m}(\lambda,\delta,q)$, *then*

$$\sum_{k=n+p}^{\infty} a_k \ \leq |\delta|\ \Xi(p,n,m,\lambda,\delta,q), \tag{27}$$

where $\Xi(p,n,m,\lambda,\delta,q)$ *is defined by*

$$\Xi(p,n,m,\lambda,\delta,q)=\frac{\Delta(p,m,\lambda,q)}{(|\delta|+q^{p-m}[n]_q)\Delta(n+p,m,\lambda,q)}, \tag{28}$$

and $\Delta(k,m,\lambda,q)$ *is given by* (18).

Corollary 2 *If* $f(z)\in\mathcal{M}^m_{n,p}(\lambda,\delta,q)$*, then*

$$\sum_{k=n+p}^{\infty}[k]_q[k-1]_q\cdots[k-p+1]_q\ a_k\leq|\delta|\ \Theta(p,n,m,\lambda,\delta,q), \tag{29}$$

where $\Theta(p,n,m,\lambda,\delta,q)$ *is defined by*

$$\Theta(p,n,m,\lambda,\delta,q)=\frac{\Gamma_q(1+n+p-m)\ \Delta(p,m,\lambda,q)}{(|\delta|+q^{p-m}[n]_q)\ (1+[n+p-m-1]_q\ q\ \lambda)\Gamma_q(1+n)}, \tag{30}$$

and $\Delta(k,m,\lambda,q)$ *is given by* (18).

Proof. Since $f(z)\in\mathcal{M}^m_{n,p}(\lambda,\delta,q)$, then under the hypotheses of Theorem 1, we have

$$\begin{aligned}&\sum_{k=n+p}^{\infty}\frac{(|\delta|-q^{k-m}[p-k]_q)\ (1+[k-m-1]_q\ q\ \lambda)\ \Gamma_q(1+k)}{\Gamma_q(1+k-m)}\ a_k\\&\leq|\delta|\ \Delta(p,m,\lambda,q),\end{aligned} \tag{31}$$

where $\Delta(k,m,\lambda,q)$ is given by (18).

Using the recurrence relation (4) successively p times, we can write

$$\Gamma_q(1+k)=[k]_q[k-1]_q\ldots[k-p+1]_q\ \Gamma_q(k-p+1). \tag{32}$$

We now show here that

$$\alpha_k\leq\alpha_{k+1},$$

where

$$\begin{aligned}\alpha_k&=\frac{(|\delta|-q^{k-m}[p-k]_q)\ (1+q\ \lambda\ [k-m-1]_q)\Gamma_q(1+k-p)}{\Gamma_q(1+k-m)}\\&=(A_k)\,(B_k)\,(C_k)\,,\end{aligned} \tag{33}$$

$$\begin{aligned} A_k &= |\delta| - q^{k-m}[p-k]_q, \\ B_k &= 1 + q\ \lambda\ [k-m-1]_q \quad \text{and} \\ C_k &= \frac{\Gamma_q(1+k-p)}{\Gamma_q(1+k-m)}. \end{aligned}$$

It is sufficient to show that

$$\frac{\alpha_k}{\alpha_{k+1}} = \frac{(A_k)\ (B_k)\ (C_k)}{(A_{k+1})(B_{k+1})(C_{k+1})} \leq 1.$$

Evidently, for $k = n + p$, we have

$$\frac{A_k}{A_{k+1}} = \frac{|\delta| + q^{p-m}[n]_q}{|\delta| + q^{p-m}[n+1]_q},$$

and since $[n+1]_q > [n]_q$, hence A_k is positive and consequently

$$\frac{A_k}{A_{k+1}} < 1. \tag{34}$$

Also, it follows easily that

$$\frac{B_k}{B_{k+1}} = \frac{1 + q\ \lambda\ [n+p-m-1]_q}{1 + q\ \lambda\ [n+p-m]_q} < 1. \tag{35}$$

Further, upon using the familiar asymptotic formula ([6, pp. 311, eqn. (1.7)]) given by

$$\Gamma_q(x) \approx (1-q)^{1-x} \prod_{n=0}^{\infty}(1-q^{n+1}) \qquad (x \to \infty,\ 0 < q < 1),$$

it can be verified that

$$\begin{aligned} C_k = \frac{\Gamma_q(1+k-p)}{\Gamma_q(1+k-m)} &\approx \frac{(1-q)^{1-1-k+p} \prod_{n=0}^{\infty}(1-q^{n+1})}{(1-q)^{1-1-k+m} \prod_{n=0}^{\infty}(1-q^{n+1})} \\ &= (1-q)^{p-m} \qquad (k \to \infty,\ 0 < q < 1, m < p). \end{aligned} \tag{36}$$

Thus, for large k, we conclude that

$$\frac{\alpha_k}{\alpha_{k+1}} \leq 1.$$

We, therefore, from (31) and (32) infer that

$$\sum_{k=n+p}^{\infty} [k]_q[k-1]_q \ldots [k-p+1]_q \ a_k$$
$$\leq \frac{|\delta| \ \Delta(k,m,\lambda,q)\Gamma_q(1+n+p-m)}{(|\delta|+q^{p-m}[n]_q)(1+[n+p-m-1]_q \ q \ \lambda) \ \Gamma_q(1+n)},$$

which in view of (30) yields the desired inequality (31). □

Next, we prove the following result.

Theorem 2 *Let the function* $f(z)$ *be defined by* (12), *then* $f(z) \in \mathcal{N}_{n,p}^{m}(\lambda,\delta,q)$ *if and only if*

$$\sum_{k=n+p}^{\infty} [k-m]_q \Omega(k,m,\lambda,q) a_k \leq [p-m]_q \left[\frac{|\delta|-1}{\Gamma_q(1+m)} + \Omega(p,m,\lambda,q)\right], \quad (37)$$

where $\Omega(k,m,\lambda,q)$ *is given by*

$$\Omega(k,m,\lambda,q) = \begin{bmatrix} k \\ m \end{bmatrix}_q (1+[k-m-1]_q \ \lambda). \quad (38)$$

The result is sharp with the extremal function given by

$$f(z) = z^p - \frac{[p-m]_q \left[|\delta|-1+\Gamma_q(1+m) \ \Omega(p,m,\lambda,q)\right]}{[n+p-m]_q \ \Gamma_q(1+m) \ \Omega(n+p,m,\lambda,q)} z^{n+p}. \quad (39)$$

Proof. Let $f(z) \in \mathcal{N}_{n,p}^{m}(\lambda,\delta,q)$, then on using (16), we get

$$\Re\left\{D_{q,z}^{1+m} f(z) + \lambda \ z \ D_{q,z}^{2+m} f(z) - [p-m]_q\right\} > -|\delta| \ [p-m]_q. \quad (40)$$

Now, in view of (13), we have

$$D_{q,z}^{1+m} f(z) + \lambda \ z \ D_{q,z}^{2+m} f(z) = \frac{\Gamma_q(1+p)}{\Gamma_q(p-m)} \ z^{p-m-1} - \sum_{k=n+p}^{\infty} a_k \frac{\Gamma_q(1+k)}{\Gamma_q(k-m)} \ z^{k-m-1}$$
$$+\lambda \ z \left[\frac{\Gamma_q(1+p)}{\Gamma_q(p-m-1)} \ z^{p-m-2} - \sum_{k=n+p}^{\infty} a_k \frac{\Gamma_q(1+k)}{\Gamma_q(k-m-1)} \ z^{k-m-2}\right]$$
$$= \Gamma_q(1+p) z^{p-m-1} \left[\frac{1}{\Gamma_q(p-m)} + \frac{\lambda}{\Gamma_q(p-m-1)}\right]$$

$$- \sum_{k=n+p}^{\infty} a_k \, \Gamma_q(1+k) \, z^{k-m-1} \left[\frac{1}{\Gamma_q(k-m)} + \frac{\lambda}{\Gamma_q(k-m-1)} \right]$$
$$= \frac{[p-m]_q(1+[p-m-1]_q \, \lambda)\Gamma_q(1+p)}{\Gamma_q(1+p-m)} z^{p-m-1}$$
$$- \sum_{k=n+p}^{\infty} a_k \, \frac{[k-m]_q(1+[k-m-1]_q \, \lambda)\Gamma_q(1+k)}{\Gamma_q(1+k-m)} \, z^{k-m-1}.$$

From (40), we obtain a simplified form of the inequality which is given by

$$\Re \left\{ - \sum_{k=n+p}^{\infty} a_k \, \frac{[k-m]_q(1+[k-m-1]_q \, \lambda)\Gamma_q(1+k)}{\Gamma_q(1+k-m)} \, z^{k-m-1} \right.$$
$$\left. -[p-m]_q \left(1 - \frac{(1+[p-m-1]_q \, \lambda)\Gamma_q(1+p)}{\Gamma_q(1+p-m)} z^{p-m-1} \right) \right\} > -|\delta| \, [p-m]_q.$$

Now taking (38) into account, the above inequality yields

$$\Re \left\{ - \sum_{k=n+p}^{\infty} [k-m]_q \Omega(k,m,\lambda,q) \Gamma_q(1+m) \, a_k \, z^{k-m-1} \right.$$
$$\left. -[p-m]_q \left(1 - \Omega(k,m,\lambda,q)\Gamma_q(1+m) z^{p-m-1} \right) \right\} > -|\delta| \, [p-m]_q. \tag{41}$$

By putting $z = r$ in (41), and letting $r \to 1^-$ through real values, we get the desired assertion (37) of Theorem 2.

To prove the converse of Theorem 2, we have

$$\left| \left\{ D_{q,z}^{1+m} f(z) + \lambda \, z \, D_{q,z}^{2+m} f(z) - [p-m]_q \right\} \right|$$
$$\leq \left| \sum_{k=n+p}^{\infty} [k-m]_q \Omega(k,m,\lambda,q)\Gamma_q(1+m) \, a_k \, z^{k-m-1} \right|$$
$$+ \left| [p-m]_q \left(1 - \Omega(k,m,\lambda,q)\Gamma_q(1+m) z^{p-m-1} \right) \right|.$$

Letting $|z| = 1$, we find that

$$\left| \left\{ D_{q,z}^{1+m} f(z) + \lambda \, z \, D_{q,z}^{2+m} f(z) - [p-m]_q \right\} \right|$$
$$\leq \sum_{k=n+p}^{\infty} [k-m]_q \Omega(k,m,\lambda,q)\Gamma_q(1+m) \, a_k$$
$$+ [p-m]_q \left(1 - \Omega(k,m,\lambda,q)\Gamma_q(1+m) \right),$$

then by applying the hypothesis (37), we find that

$$\left|\left\{D_{q,z}^{1+m}f(z) + \lambda\, z\, D_{q,z}^{2+m}f(z) - [p-m]_q\right\}\right| \leq |\delta|\ [p-m]_q.$$

Hence, by the maximum modulus principle, we infer that

$$f(z) \in \mathcal{N}_{n,p}^{m}(\lambda,\delta,q).$$

□

The following corollaries follow from Theorem 2 in the same manner as Corollaries 1 and 2 from Theorem 1.

Corollary 3 *If the function* $f(z)$ *be defined by* (12) *and* $f(z) \in \mathcal{N}_{n,p}^{m}(\lambda,\delta,q)$, *then*

$$\sum_{k=n+p}^{\infty} a_k \ \leq\ X(p,n,m,\lambda,\delta,q), \tag{42}$$

where $X(p,n,m,\lambda,\delta,q)$ *is given by*

$$X(p,n,m,\lambda,\delta,q) = \frac{[p-m]_q\,[|\delta| - 1 + \Gamma_q(1+m)\ \Omega(p,m,\lambda,q)]}{\Gamma_q(1+m)[n+p-m]_q\ \Omega(n+p,m,\lambda,q)}. \tag{43}$$

Corollary 4 *If* $f(z) \in \mathcal{N}_{n,p}^{m}(\lambda,\delta,q)$, *then*

$$\sum_{k=n+p}^{\infty} [k]_q[k-1]_q \cdots [k-p+1]_q\ a_k \ \leq\ \Psi(p,n,m,\lambda,\delta,q), \tag{44}$$

where $\Psi(p,n,m,\lambda,\delta,q)$ *is given by*

$$\begin{aligned}&\Psi(p,n,m,\lambda,\delta,q)\\ &= \frac{[p-m]_q\,[|\delta| - 1 + \Gamma_q(1+m)\ \Omega(p,m,\lambda,q)]\,\Gamma_q(1+n+p-m)}{\Gamma_q(1+n)[n+p-m]_q\ (1+[n+p-m-1]_q\ \lambda)}.\end{aligned} \tag{45}$$

3 Distortion theorems

In this section, we establish certain distortion theorems for the classes of functions defined above involving the q-differential operator.

Theorem 3 *Let* $\lambda \in \mathbb{R}$ *and* $\delta \in \mathbb{C} \setminus \{0\} \in \mathbb{N}$ *satisfy the inequalities:*

$$m < p;\ m \in \mathbb{N}_0;\ p,n \in \mathbb{N};\ 0 \leq \lambda \leq 1,\ 0 < q < 1.$$

Also, let the function $f(z)$ *defined by* (12) *be in the class* $\mathcal{M}^{m}_{n,p}(\lambda,\delta,q)$, *then*

$$||f(z)|-|z|^{p}|\leq|\delta|\;\;\Xi(p,n,m,\lambda,\delta,q)\;\;|z|^{n+p}\quad(z\in\mathbb{U}),\tag{46}$$

where $\Xi(p,n,m,\lambda,\delta,q)$ *is given by* (28).

Proof. Since $f(z)\in\mathcal{M}^{m}_{n,p}(\lambda,\delta,q)$, then from the Corollary 1, we have

$$\sum_{k=n+p}^{\infty}a_k\;\leq|\delta|\;\;\Xi(p,n,m,\lambda,\delta,q),$$

where $\Xi(p,n,m,\lambda,\delta,q)$ is given by (28).

This inequality in conjunction with the following inequality (easily obtainable from (11)):

$$|z|^{p}-|z|^{n+p}\sum_{k=n+p}^{\infty}a_k\leq|f(z)|\leq|z|^{p}+|z|^{n+p}\sum_{k=n+p}^{\infty}a_k,\tag{47}$$

yields the assertion (46) of Theorem 3. □

To obtain the distortion theorem for a normalized multivalent analytic function of the form (12), we define here a q-differential operator $\mathbb{D}^{m}_{q,z}$ which is expressed in the form

$$\mathbb{D}^{m}_{q,z}f(z)=\frac{\Gamma_q(1+p-m)}{\Gamma_q(1+p)}z^{m}D^{m}_{q,z}f(z).\tag{48}$$

Theorem 4 *Let* $m<p;\;m\in\mathbb{N}_0,\;p,n\in\mathbb{N},\;0\leq\lambda\leq1,\;\delta\in\mathbb{C}\setminus\{0\}\in\mathbb{N},\;0<q<1$, *and let the function* $f(z)$ *defined by* (12) *be in the class* $\mathcal{M}^{m}_{n,p}(\lambda,\delta,q)$. *Then*

$$\left|\left|\mathbb{D}^{m}_{q,z}f(z)\right|-|z|^{p}\right|\leq|\delta|\;\;\mathbb{A}(p,n,m,\lambda,\delta,q)\;\;|z|^{n+p},\tag{49}$$

where

$$\mathbb{A}(p,n,m,\lambda,\delta,q)=\frac{1+[p-m-1]_q\;\;q\;\lambda}{(|\delta|+q^{p-m}[n]_q)\;\;(1+[n+p-m-1]_q\;\;q\;\lambda)}.\tag{50}$$

Proof. Since

$$\mathbb{D}^{m}_{q,z}f(z)=\frac{\Gamma_q(1+p-m)}{\Gamma_q(1+p)}z^{m}D^{m}_{q,z}f(z)=z^{p}-\sum_{k=n+p}^{\infty}a_k\,\frac{\Gamma_q(1+k)\Gamma_q(1+p-m)}{\Gamma_q(1+p)\Gamma_q(1+k-m)}\,z^{k},$$

therefore, on using the relation (32), we can write

$$\begin{aligned}\mathbb{D}^m_{q,z}f(z) &= z^p - \sum_{k=n+p}^{\infty} a_k \frac{[k]_q[k-1]_q\dots[k-p+1]_q\Gamma_q(1+k-p)\Gamma_q(1+p-m)}{\Gamma_q(1+p)\Gamma_q(1+k-m)} z^k \\ &= z^p - \sum_{k=n+p}^{\infty} a_k\,[k]_q[k-1]_q\dots[k-p+1]_q\ \phi(k)\ z^k,\end{aligned} \tag{51}$$

where

$$\phi(k) = \frac{\Gamma_q(1+k-p)\Gamma_q(1+p-m)}{\Gamma_q(1+p)\Gamma_q(1+k-m)}. \tag{52}$$

Now, we show that the function $\phi(k)$ ($m \in \mathbb{N}_0,\ k \geq n+p;\ p,n \in \mathbb{N}, m < p$) is a decreasing function of k for $m \in \mathbb{N}_0,\ 0 < q < 1$.

We note that

$$\frac{\phi(k+1)}{\phi(k)} = \frac{\Gamma_q(2+k-p)\Gamma_q(1+k-m)}{\Gamma_q(2+k-m)\Gamma_q(1+k-p)} \quad (k \geq n+p; n,p \in \mathbb{N}),$$

and it is sufficient here to consider the value $k = n+p$, so that on using (4), we get

$$\frac{\phi(k+1)}{\phi(k)} = \frac{1-q^{1+n}}{1-q^{1+n+p-m}} \quad (0 < q < 1).$$

The function $\phi(k)$ is a decreasing function of k if $\frac{\phi(k+1)}{\phi(k)} \leq 1$ $(n,p \in \mathbb{N})$, and this gives

$$\frac{1-q^{1+n}}{1-q^{1+n+p-m}} \leq 1 \quad (0 < q < 1).$$

Multiplying the above inequality both sides by $1-q^{1+n+p-m}$ (provided that $m < p$), we are at once lead to the inequality $m \leq p$. Thus, $\phi(k)$ $(k \geq n+p; n,p \in \mathbb{N})$ is a decreasing function of k for $m < p, m \in \mathbb{N}_0, 0 < q < 1$.

Using (51), we observe that

$$\begin{aligned}\left|\mathbb{D}^m_{q,z}f(z)\right| &\geq |z|^p - \sum_{k=n+p}^{\infty} [k]_q[k-1]_q\dots[k-p+1]_q\ \phi(k)\,|a_k|\ |z|^k \\ &\geq |z|^p - \phi(n+p)\,|z|^{n+p} \sum_{k=n+p}^{\infty} [k]_q[k-1]_q\dots[k-p+1]_q\ |a_k|,\end{aligned}$$

which in view of (29) of Corollary 2 leads to

$$\left|\mathbb{D}^{m}_{q,z}f(z)\right| \geq |z|^{p} - |\delta| \;\; \phi(n+p)\;\Theta(p,n,m,\lambda,\delta,q) \;\; |z|^{n+p}$$

$$\geq |z|^{p} - |\delta| \;\; \mathbb{A}(p,n,m,\lambda,\delta,q) \;\; |z|^{n+p}, \tag{53}$$

where $\mathbb{A}(p,n,m,\lambda,\delta,q)$ is given by (50).

Similarly, it follows that

$$\left|\mathbb{D}^{m}_{q,z}f(z)\right| \leq |z|^{p} + \;|\delta| \;\; \mathbb{A}(p,n,m,\lambda,\delta,q) \;\; |z|^{n+p}, \tag{54}$$

and hence, (53) and (54) establish the assertion (49) of Theorem 4. □

The following distortion inequalities for the function $f(z) \in \mathcal{N}^{m}_{n,p}(\lambda,\delta,q)$ can be proved in the same manner as detailed in the proof of Theorem 4 above:

Theorem 5 *Let* $\lambda \in \mathbb{R}$ *and* $\delta \in \mathbb{C}\setminus\{0\} \in \mathbb{N}$ *satisfy the inequalities:*

$$m < p; m \in \mathbb{N}_0; p,n \in \mathbb{N};\; 0 \leq \lambda \leq 1,\; 0 < q < 1.$$

Also, let the function $f(z)$ *defined by* (12) *be in the class* $\mathcal{N}^{m}_{n,p}(\lambda,\delta,q)$, *then*

$$||f(z)| - |z|^{p}| \leq |\delta| \;\; X(p,n,m,\lambda,\delta,q) \;\; |z|^{n+p} \quad (z \in \mathbb{U}), \tag{55}$$

where $X(p,n,m,\lambda,\delta,q)$ *is given by* (43).

Theorem 6 *Let* $m < p;\; m \in \mathbb{N}_0,\; p,n \in \mathbb{N},\; 0 \leq \lambda \leq 1,\; \delta \in \mathbb{C}\setminus\{0\} \in \mathbb{N},\; 0 < q < 1$ *and let the function* $f(z)$ *defined by* (12) *be in the class* $\mathcal{N}^{m}_{n,p}(\lambda,\delta,q)$. *Then*

$$\left|\left|\mathbb{D}^{m}_{q,z}f(z)\right| - |z|^{p}\right| \leq |\delta| \;\; \mathbb{B}(p,n,m,\lambda,\delta,q) \;\; |z|^{n+p}, \tag{56}$$

where

$$\mathbb{B}(p,n,m,\lambda,\delta,q) = \frac{[p-m]_q \;[|\delta| - 1 + \Gamma_q(1+m)\;\Omega(p,m,\lambda,q)]\,\Gamma_q(1+p-m)}{\Gamma_q(1+p)[n+p-m]_q \;\;(1+[n+p-m-1]_q\;\lambda)}, \tag{57}$$

$\Omega(p,m,\lambda,q)$ *is given by* (38).

4 Some consequences of the main results

In this section, we briefly consider some special cases of the results derived in the preceding sections.

When $m = 0$ and $\delta = \gamma\beta$ $(\gamma \in \mathbb{C} \setminus \{0\},\ 0 < \beta \leq 1)$, the condition (14) reduces to the inequality:

$$\left| \frac{1}{\gamma} \left\{ \frac{z\ D_{q,z}f(z) + \lambda\ qz^2\ D^2_{q,z}f(z)}{\lambda\ z\ D_{q,z}f(z) + (1-\lambda)f(z)} - [p]_q \right\} \right| < \beta, \tag{58}$$

$$(p \in \mathbb{N},\ 0 \leq \lambda \leq 1;\ 0 < \beta \leq 1;\ \gamma \in \mathbb{C} \setminus \{0\};\ 0 < q < 1;\ z \in \mathbb{U})$$

and we write

$$\mathcal{M}^0_{n,p}(\lambda, \gamma\beta, q) = \mathcal{R}_{n,p}(\lambda, \beta, \gamma, q), \tag{59}$$

where $\mathcal{R}_{n,p}(\lambda, \beta, \gamma, q)$ represents a subclass of p-valently analytic functions which satisfy the condition (58).

Similarly, the condition (16) when $m = 0$ and $\delta = \gamma\beta$ reduces to the inequality:

$$\left| \frac{1}{\gamma} \left\{ D_{q,z}f(z) + \lambda\ z\ D^2_{q,z}f(z) - [p]_q \right\} \right| < \beta\ [p]_q, \tag{60}$$

$$(p \in \mathbb{N},\ 0 \leq \lambda \leq 1;\ 0 < \beta \leq 1;\ \gamma \in \mathbb{C} \setminus \{0\};\ 0 < q < 1;\ z \in \mathbb{U})$$

and we write

$$\mathcal{N}^0_{n,p}(\lambda, \gamma\beta, q) = \mathcal{L}_{n,p}(\lambda, \beta, \gamma, q), \tag{61}$$

where $\mathcal{L}_{n,p}(\lambda, \beta, \gamma, q)$ is another subclass of p-valently analytic functions which satisfy the condition (60).

Now, by setting $m = 0$, $\delta = \gamma\beta$, and making use of the relations (59) and (61), Theorems 1 and 2 give the following coefficient inequalities for the classes $\mathcal{R}_{n,p}(\lambda, \beta, \gamma, q)$ and $\mathcal{L}_{n,p}(\lambda, \beta, \gamma, q)$, respectively.

Corollary 5 *Let the function* $f(z)$ *be defined by* (12), *then* $f(z) \in \mathcal{R}_{n,p}(\lambda, \beta, \gamma, q)$ *if and only if*

$$\sum_{k=n+p}^{\infty} (\beta\ |\gamma| - q^k[p-k]_q)\ (1 + [k-1]_q\ q\ \lambda)\ a_k\ \leq \beta\ |\gamma|\,(1 + [p-1]_q\ q\ \lambda)\,. \tag{62}$$

The result is sharp with the extremal function given by

$$f(z) = z^p - \frac{\beta\ |\gamma|\,(1 + [p-1]_q\ q\ \lambda)}{(\beta\ |\gamma| + q^p[n]_q)\ (1 + [n+p-1]_q\ q\ \lambda)}\ z^{n+p}. \tag{63}$$

Corollary 6 *Let the function* $f(z)$ *be defined by* (12), *then* $f(z) \in \mathcal{L}_{n,p}(\lambda, \beta, \gamma, q)$ *if and only if*

$$\sum_{k=n+p}^{\infty} [k]_q \ (1 + [k-1]_q \ \lambda) \ a_k \ \leq [p]_q \, [\beta \ |\gamma| + [p-1]_q \ \lambda] \ . \tag{64}$$

The result is sharp with the extremal function given by

$$f(z) = z^p - \frac{[p]_q \, [\beta \ |\gamma| + [p-1]_q \ \lambda]}{[n+p]_q \ (1 + [n+p-1]_q \ \lambda)} z^{n+p}. \tag{65}$$

Again, if we put $m = 0$, $\delta = \gamma\beta$, then Theorem 3 and Theorem 5, respectively, yield the following distortion theorems for the classes $\mathcal{R}_{n,p}(\lambda, \beta, \gamma, q)$ and $\mathcal{L}_{n,p}(\lambda, \beta, \gamma, q)$.

Corollary 7 *Let* $\lambda, \beta \in \mathbb{R}$ *and* $\gamma \in \mathbb{C} \setminus \{0\} \in \mathbb{N}$ *satisfy the inequalities:*

$$p, n \in \mathbb{N}; \ 0 \leq \lambda \leq 1, \ 0 < q < 1.$$

Also, let the function $f(z)$ *defined by* (12) *be in the class* $\mathcal{R}_{n,p}(\lambda, \beta, \gamma, q)$, *then*

$$||f(z)| - |z|^p| \leq \beta \, |\gamma| \ \mathbb{E}(p, n, \lambda, \beta, \gamma, q) \, |z|^{n+p} \quad (z \in \mathbb{U}), \tag{66}$$

where

$$\mathbb{E}(p, n, \lambda, \beta, \gamma, q) = \frac{1 + [p-1]_q \ q \ \lambda}{(\beta \, |\gamma| + q^p [n]_q)(1 + [n+p-1]_q \ q \ \lambda)}. \tag{67}$$

Corollary 8 *Let* $\lambda, \beta \in \mathbb{R}$ *and* $\gamma \in \mathbb{C} \setminus \{0\} \in \mathbb{N}$ *satisfy the inequalities:*

$$p, n \in \mathbb{N}; \ 0 \leq \lambda \leq 1, \ 0 < q < 1.$$

Also, let the function $f(z)$ *defined by* (12) *be in the class* $\mathcal{L}_{n,p}(\lambda, \beta, \gamma, q)$, *then*

$$||f(z)| - |z|^p| \leq \mathbb{F}(p, n, \lambda, \beta, \gamma, q) \, |z|^{n+p} \quad (z \in \mathbb{U}), \tag{68}$$

where

$$\mathbb{F}(p, n, \lambda, \beta, \gamma, q) = \frac{[p]_q \, [\beta \, |\gamma| + [p-1]_q \ \lambda]}{[n+p]_q (1 + [n+p-1]_q \ \lambda)}. \tag{69}$$

Further, if we set $p = 1$, then from (59) and (61), we get

$$\mathcal{M}^0_{n,1}(\lambda, \gamma\beta, q) = \mathcal{R}_{n,1}(\lambda, \beta, \gamma, q) = \mathcal{H}_n(\lambda, \gamma, \beta, q) \tag{70}$$

and

$$\mathcal{N}^0_{n,1}(\lambda, \gamma\beta, q) = \mathcal{L}_{n,1}(\lambda, \beta, \gamma, q) = \mathcal{G}_n(\lambda, \gamma, \beta, q), \tag{71}$$

where $\mathcal{H}_n(\lambda, \gamma, \beta, q)$ and $\mathcal{G}_n(\lambda, \gamma, \beta, q)$ are precisely the subclass of analytic and univalent functions studied recently by Purohit and Raina [11]. Thus, if we set $p = 1$, and taking into consideration the relations (70) and (71), Corollary 5 to Corollary 8 yield the known results obtained recently by Purohit and Raina [11].

Finally, by letting $q \to 1^-$, and making use of the limit formula (10), we observe that the function classes $\mathcal{M}^m_{n,p}(\lambda, \delta, q)$, $\mathcal{N}^m_{n,p}(\lambda, \delta, q)$ and the inequalities (17) and (37) of Theorem 1 and Theorem 2 provide, respectively, the q-extensions of the known results due to Srivastava and Orhan [13, pp. 687-688, eqn. (11) and (14)].

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