Related fixed point theorem for four complete metric spaces

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Abstract. In the present paper, we obtain a new result on fixed point theorem for four metric spaces. Here we choose continuous mappings. In fact our result is the generalization of many results of fixed point theorem on two and three metric spaces. We also give some illustrative examples to justify our result.

1 Introduction

Related fixed point theorems on two metric spaces have been studied by B. Fisher [2]. Also some fixed point theorems on three metric spaces have been studied by B. Fisher et al [3], R. K. Jain et al. [4], R. K. Namdeo and B. Fisher [7], K. Kikina et al. [6], Z. Ansari et al. [1], and V. Gupta [8]. Also, the fixed point theorems on four metric spaces have been studied by L. Kikina et al. [5]. In the present paper a generalization is given for four complete metric space. Our theorem improves Theorem (2.1) of R. K. Jain et al. [4].

The following fixed point theorem was proved by R. K. Jain, H. K. Sahu, B. Fisher [4].

**Theorem 1** Let $(X, d)$, $(Y, ρ)$ and $(Z, σ)$ be complete metric spaces. If $T$ is continuous mapping of $X \mapsto Y$, $S$ is a continuous mapping of $Y \mapsto Z$ and $R$ is mapping of $Z \mapsto X$ satisfying the inequalities

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2 Main result

Theorem 2 Let \((Z_1, d_1), (Z_2, d_2), (Z_3, d_3),\) and \((Z_4, d_4)\) be complete metric spaces. If \(A_1\) is a continuous mapping of \(Z_1 \mapsto Z_2, A_2\) is continuous mapping of \(Z_2 \mapsto Z_3, A_3\) is continuous mapping of \(Z_3 \mapsto Z_4\) and \(A_4\) is a mappings of \(Z_4 \mapsto Z_1\), satisfying the inequalities

\[
d_1(A_4A_3A_2A_1z_1, A_4A_3A_2A_1z_1') \leq c \max\{d_1(z_1, z_1'), d_1(z_1, A_4A_3A_2A_1z_1),
\]
\[
d_1(z_1', A_4A_3A_2A_1z_1'), d_2(A_1z_1, A_1z'),
\]
\[
d_3(A_2A_1z_1, A_2A_1z_1'), d_4(A_3A_2A_1z_1, A_3A_2A_1z_1')\},
\]
\[
d_2(A_1A_4A_3A_2z_2, A_1A_4A_3A_2z_2') \leq c \max\{d_2(z_2, z_2'), d_2(z_2, A_1A_4A_3A_2z_2),
\]
\[
d_2(z_2', A_1A_4A_3A_2z_2'), d_3(A_2z_2, A_2z_2'),
\]
\[
d_4(A_3A_2z_2, A_3A_2z_2'), d_1(A_4A_3A_2z_2, A_4A_3A_2z_2')\},
\]
\[
d_3(A_2A_1A_4A_3z_3, A_2A_1A_4A_3z_3') \leq c \max\{d_3(z_3, z_3'), d_3(z_3, A_2A_1A_4A_3z_3),
\]
\[
d_3(z_3', A_2A_1A_4A_3z_3'), d_4(A_3z_3, A_3z_3'),
\]
\[
d_1(A_4A_3z_3, A_4A_3z_3'), d_2(A_1A_4A_3z_3, A_1A_4A_3z_3')\},
\]
\[
d_4(A_3A_2A_1A_4z_4, A_3A_2A_1A_4z_4') \leq c \max\{d_4(z_4, z_4'), d_4(z_4, A_3A_2A_1A_4z_4),
\]
\[
d_4(z_4', A_3A_2A_1A_4z_4'), d_1(A_4z_4, A_4z_4'),
\]
\[
d_2(A_1A_4z_4, A_1A_4z_4'), d_3(A_2A_1A_4z_4, A_2A_1A_4z_4')\},
\]

\(\forall x, x' \in X, y, y' \in Y\) and \(z, z' \in Z\), where \(0 \leq c < 1\), then RST has a unique fixed point \(u \in X\), TRS has a unique fixed point \(v \in Y\) and STR has a unique fixed point \(w \in Z\). Further \(Tu = v, Sv = w\) and \(Rw = u\).
\[ \forall z_1, z'_1 \in Z_1, z_2, z'_2 \in Z_2, z_3, z'_3 \in Z_3 \text{ and } z_4, z'_4 \in Z_4, \text{ where } 0 \leq c < 1, \text{ then} \]
\[ A_4A_3A_2A_1 \text{ has a unique fixed point } \alpha_1 \in Z_1, A_1A_4A_3A_2 \text{ has a unique fixed point } \alpha_2 \in Z_2, A_2A_1A_4A_3 \text{ has a unique fixed point } \alpha_3 \in Z_3 \text{ and } A_3A_2A_1A_4 \text{ has a unique fixed point } \alpha_4 \in Z_4. \]

Further \( A_1\alpha_1 = \alpha_2, A_2\alpha_2 = \alpha_3, A_3\alpha_3 = \alpha_4, A_4\alpha_4 = \alpha_1. \)

**Proof.** Let \( z_1^0 \) be an arbitrary point in \( Z_1. \)

Define the sequence \( \{z_n^1\}, \{z_n^2\}, \{z_n^3\} \) and \( \{z_n^4\} \) in \( Z_1, Z_2, Z_3 \) and \( Z_4 \) respectively by
\[
\begin{align*}
(A_4A_3A_2A_1)^n z_i^0 &= z_{n}^1 \\
A_1z_{n-1}^1 &= z_n^2 \\
A_2z_n^2 &= z_n^3 \\
A_3z_n^3 &= z_n^4 \\
A_4z_n^4 &= z_{n-1}^1
\end{align*}
\]
for \( n = 1, 2, \ldots \)

Applying inequality (5), we get,
\[
d_2(z_n^2, z_{n+1}^2) = d_2(A_1A_4A_3A_2z_{n-1}^2, A_1A_4A_3A_2z_n^2) \\
\leq c \max \{d_2(z_{n-1}^2, z_n^2), d_2(z_{n-1}^2, A_1A_4A_3A_2z_{n-1}^2), \\
d_2(z_n^2, A_1A_4A_3A_2z_n^2), d_3(A_2z_n^2, A_2z_n^2), \\
d_3(A_3z_n^3, A_3A_2z_n^3)) \}
\]
\[
d_2(z_n^2, z_{n+1}^2) \leq c \max \{d_2(z_{n-1}^2, z_n^2), d_2(z_{n-1}^2, z_n^2), d_2(z_n^2, z_n^2), \\
d_3(z_n^3, z_n^3), d_4(z_n^4, z_n^4), d_1(z_{n-1}^1, z_n^1), \}
\]
\[
d_2(z_n^2, z_{n+1}^2) \leq c \max \{d_1(z_{n-1}^1, z_n^1), d_2(z_{n-1}^2, z_n^2), d_3(z_{n-1}^3, z_n^3), d_4(z_{n-1}^4, z_n^4)\}
\]
Using inequality (6), we get,
\[
d_3(z_n^3, z_{n+1}^3) = d_3(A_2A_1A_4A_3z_{n-1}^3, A_2A_1A_4A_3z_n^3) \\
\leq \max \{d_3(z_{n-1}^3, z_n^3), d_3(z_{n-1}^3, A_2A_1A_4A_3z_{n-1}^3), \\
d_3(z_n^3, A_2A_1A_4A_3z_n^3), d_4(A_3z_n^3, A_3z_n^3), \\
d_1(A_4A_3z_n^3, A_4A_3z_n^3), d_2(A_1A_4A_3z_{n-1}^3, A_1A_4A_3z_n^3)\}
\]
\[
d_3(z_n^3, z_{n+1}^3) \leq c \max \{d_3(z_{n-1}^3, z_n^3), d_3(z_{n-1}^3, z_n^3), d_3(z_n^3, z_{n+1}^3), \\
d_4(z_{n-1}^4, z_n^4), d_1(z_{n-1}^1, z_n^1), d_2(z_n^2, z_{n+1}^2), \}
\]
\[
d_3(z_n^3, z_{n+1}^3) \leq c \max \{d_1(z_{n-1}^1, z_n^1), d_2(z_{n-1}^2, z_n^2), d_3(z_n^3, z_n^3), d_4(z_{n-1}^4, z_n^4)\}
\]
Using inequality (7), we have,

\[
d_4(z_n^4, z_{n+1}^4) = d_4(A_3 A_2 A_1 A_4 z_{n-1}^4, A_3 A_2 A_1 A_4 z_n^4)
\leq c \max \{d_4(z_{n-1}^4, z_n^4), d_4(z_n^4, A_3 A_2 A_1 A_4 z_{n-1}^4),
\]
\[d_4(z_n^4, A_3 A_2 A_1 A_4 z_n^4), d_1(A_4 z_{n-1}^4, A_4 z_n^4),
\]
\[d_2(A_1 A_4 z_{n-1}^4, A_4 z_n^4), d_3(A_2 A_1 A_4 z_{n-1}^4, A_2 A_1 A_4 z_n^4)\}
\]
\[
d_4(z_n^4, z_{n+1}^4) \leq c \max \{d_4(z_{n-1}^4, z_n^4), d_4(z_n^4, d_4(z_n^4, z_{n+1}^4),
\]
\[d_1(z_{n-1}^4, z_n^4), d_2(z_n^4, z_{n+1}^4), d_3(z_n^4, z_{n+1}^4)\} \tag{10}
\]
\[
d_4(z_n^4, z_{n+1}^4) \leq c \max \{d_1(z_{n-1}^4, z_n^4), d_2(z_n^4, z_{n+1}^4),
\]
\[d_3(z_n^4, z_{n+1}^4)\}
\]

Using inequality (4), we have,

\[
d_1(z_n^1, z_{n+1}^1) = d_1(A_4 A_3 A_2 A_1 z_{n-1}^1, A_4 A_3 A_2 A_1 z_n^1)
\leq c \max \{d_1(z_{n-1}^1, z_n^1), d_1(z_n^1, A_4 A_3 A_2 A_1 z_{n-1}^1),
\]
\[d_1(z_n^1, A_4 A_3 A_2 A_1 z_n^1), d_2(A_1 z_{n-1}^1, A_1 z_n^1),
\]
\[d_3(A_2 A_1 z_{n-1}^1, A_2 A_1 z_n^1), d_4(A_3 A_2 A_1 z_{n-1}^1, A_3 A_2 A_1 z_n^1)\} \tag{11}
\]
\[
d_1(z_n^1, z_{n+1}^1) \leq c \max \{d_1(z_{n-1}^1, z_n^1), d_1(z_n^1, z_{n+1}^1),
\]
\[d_2(z_n^1, z_{n+1}^1), d_3(z_n^1, z_{n+1}^1), d_4(z_n^1, z_{n+1}^1)\}
\]
\[
d_1(z_n^1, z_{n+1}^1) \leq c \max \{d_1(z_{n-1}^1, z_n^1), d_2(z_n^1, z_{n+1}^1),
\]
\[d_3(z_n^1, z_{n+1}^1), d_4(z_n^1, z_{n+1}^1)\}
\]

By induction on using inequalities (8), (9), (10) and (11), we have,

\[
d_1(z_n^1, z_{n+1}^1) \leq c^{n-1} \max \{d_1(z_1^1, z_2^1), d_2(z_1^2, z_2^2),
\]
\[d_3(z_1^3, z_2^3), d_4(z_1^4, z_2^4)\}
\]
\[
d_2(z_n^2, z_{n+1}^2) \leq c^{n-1} \max \{d_1(z_1^1, z_2^1), d_2(z_1^2, z_2^2),
\]
\[d_3(z_1^3, z_2^3), d_4(z_1^4, z_2^4)\}
\]
\[
d_3(z_n^3, z_{n+1}^3) \leq c^{n-1} \max \{d_1(z_1^1, z_2^1), d_2(z_1^2, z_2^2),
\]
\[d_3(z_1^3, z_2^3), d_4(z_1^4, z_2^4)\}
\]
\[
d_4(z_n^4, z_{n+1}^4) \leq c^{n-1} \max \{d_1(z_1^1, z_2^1), d_2(z_1^2, z_2^2),
\]
\[d_3(z_1^3, z_2^3), d_4(z_1^4, z_2^4)\}
Since $c < 1$, it follows that $\{z^1_n\}, \{z^2_n\}, \{z^3_n\}$ and $\{z^4_n\}$ are Cauchy sequences with limit $\alpha_1$, $\alpha_2$, $\alpha_3$ and $\alpha_4$ in $Z_1$, $Z_2$, $Z_3$ and $Z_4$ respectively.

Since $A_1$, $A_2$ and $A_3$ are continuous, we have,
\[
\lim_{n \to \infty} z^2_n = \lim_{n \to \infty} A_1 z^1_n = A_1 \alpha_1 = \alpha_2
\]
\[
\lim_{n \to \infty} z^3_n = \lim_{n \to \infty} A_2 z^2_n = A_2 \alpha_2 = \alpha_3
\]
\[
\lim_{n \to \infty} z^4_n = \lim_{n \to \infty} A_3 z^3_n = A_3 \alpha_3 = \alpha_4
\]

Using inequality (4), again, we get,
\[
d_1(A_4A_3A_2A_1\alpha_1, z^1_n) = d_1(A_4A_3A_2A_1\alpha_1, A_4A_3A_2A_1z^1_{n-1})
\leq c \max\{d_1(\alpha_1, z^1_{n-1}), d_1(\alpha_1, A_4A_3A_2A_1\alpha_1),
\]
\[d_1(z^1_{n-1}, A_4A_3A_2A_1z^1_{n-1}), d_2(A_1\alpha_1, A_1z^1_{n-1}),
\]
\[d_3(A_2A_1\alpha_1, A_2A_1z^1_{n-1}), d_4(A_3A_2A_1\alpha_1, A_3A_2A_1z^1_{n-1})\}
\]

Since $A_1$, $A_2$ and $A_3$ are continuous, it follows on letting $n \to \infty$ that
\[
d_1(A_4A_3A_2A_1\alpha_1, \alpha_1) \leq c \max\{d_1(\alpha_1, A_4A_3A_2A_1\alpha_1)\}
\]

Thus, we have, $A_4A_3A_2A_1\alpha_1 = \alpha_1$. Since $c < 1$ and $\alpha_1$ is the fixed point of $A_4A_3A_2A_1$

\[
A_1A_4A_3A_2\alpha_2 = A_1A_4A_3A_2A_1\alpha_1 = A_1\alpha_1 = \alpha_2
\]

and

\[
A_2A_1A_4A_3\alpha_3 = A_2A_1A_4A_3A_2\alpha_2 = A_2\alpha_2 = \alpha_3
\]

and

\[
A_3A_2A_1A_4\alpha_4 = A_3A_2A_1A_4A_3\alpha_3 = A_3\alpha_3 = \alpha_4
\]

Hence $\alpha_2$, $\alpha_3$ and $\alpha_4$ are fixed points of $A_1A_4A_3A_2$, $A_2A_1A_4A_3$ and $A_3A_2A_1A_4$ respectively.

### 2.1 Uniqueness

Suppose that $A_4A_3A_2A_1$ has a second fixed point $\alpha'_1$. Then, using inequality (4), we have,
\[
d_1(\alpha_1, \alpha'_1) = d_1(A_4A_3A_2A_1\alpha_1, A_4A_3A_2A_1\alpha'_1)
\leq c \max\{d_1(\alpha_1, \alpha'_1), d_1(\alpha_1, A_4A_3A_2A_1\alpha_1),
\]
\[d_1(\alpha'_1, A_4A_3A_2A_1\alpha'_1), d_2(A_1\alpha_1, A_1\alpha'_1),
\]
\[d_3(A_2A_1\alpha_1, A_2A_1\alpha'_1),
\]
\[d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha'_1)\}
\[ d_1(\alpha_1, \alpha'_1) \leq c \max\{d_1(\alpha_1, \alpha'_1), d_1(\alpha_1, \alpha_1), d_1(\alpha'_1, \alpha'_1), \\
\quad d_2(A_1\alpha_1, A_1\alpha'_1), d_3(A_2A_1\alpha_1, A_2A_1\alpha'_1), \\
\quad d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha'_1)\} \tag{12} \]

Using inequality (5), we have

\[ d_2(A_1\alpha_1, A_1\alpha'_1) = d_2(A_1A_4A_3A_2A_1\alpha_1, A_1A_4A_3A_2A_1\alpha'_1) \]
\[ \leq c \max\{d_2(A_1\alpha_1, A_1\alpha'_1), \\
\quad d_2(A_1\alpha_1, A_1A_4A_3A_2A_1\alpha_1), \\
\quad d_2(A_1\alpha'_1, A_1A_4A_3A_2A_1\alpha'_1), \\
\quad d_3(A_2A_1\alpha_1, A_2A_1\alpha'_1), \\
\quad d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha'_1), \\
\quad d_4(A_4A_3A_2A_1\alpha_1, A_4A_3A_2A_1\alpha'_1)\} \]

\[ d_2(A_1\alpha_1, A_1\alpha'_1) \leq c \max\{d_2(A_1\alpha_1, A_1\alpha'_1), d_2(A_1\alpha_1, A_1\alpha'_1), \\
\quad d_1(A_1\alpha'_1, A_1\alpha'_1), \\
\quad d_3(A_2A_1\alpha_1, A_2A_1\alpha'_1), \\
\quad d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha'_1), \\
\quad d_1(A_4A_3A_2A_1\alpha_1, A_4A_3A_2A_1\alpha'_1)\} \]

\[ d_2(A_1\alpha_1, A_1\alpha'_1) \leq c \max\{d_2(A_1\alpha_1, A_1\alpha'_1), d_3(A_2A_1\alpha_1, A_2A_1\alpha'_1), \\
\quad d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha'_1), d_1(\alpha_1, \alpha'_1)\} \]

\[ d_2(A_1\alpha_1, A_1\alpha'_1) \leq c \max\{d_3(A_2A_1\alpha_1, A_2A_1\alpha'_1), \\
\quad d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha'_1), \} \tag{13} \]

Now, we have

\[ d_2(A_1\alpha_1, A_1\alpha'_1) \leq c \max\{d_3(A_2A_1\alpha_1, A_2A_1\alpha'_1), \\
\quad d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha'_1), \\
\quad cd_2(A_1\alpha_1, A_1\alpha'_1), \}
\quad cd_3(A_2A_1\alpha_1, A_2A_1\alpha'_1), \\
\quad cd_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha'_1)\}] \]

\[ d_2(A_1\alpha_1, A_1\alpha'_1) \leq c \max\{d_3(A_2A_1\alpha_1, A_2A_1\alpha'_1), \\
\quad d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha'_1)\}]\]
Similarly on using inequality (6), we get
\[
d_3(A_2A_1\alpha_1, A_2A_1\alpha_1') \leq c \max\{d_3(A_2A_1\alpha_1, A_2A_1\alpha_1'),
\]
\[
d_3(A_2A_1\alpha_1, A_2A_1\alpha_1'),
\]
\[
d_3(A_2A_1\alpha_1', A_2A_1\alpha_1'),
\]
\[
d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha_1'),
\]
\[
d_1(A_4A_3A_2A_1\alpha_1, A_4A_3A_2A_1\alpha_1'),
\]
\[
d_2(A_1A_4A_3A_2A_1\alpha_1, A_1A_4A_3A_2A_1\alpha_1')\}
\]
\[
d_3(A_2A_1\alpha_1, A_2A_1\alpha_1') \leq c \max\{d_3(A_2A_1\alpha_1, A_2A_1\alpha_1'),
\]
\[
d_3(A_2A_1\alpha_1, A_2A_1\alpha_1'),
\]
\[
d_3(A_2A_1\alpha_1', A_2A_1\alpha_1'),
\]
\[
d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha_1'),
\]
\[
d_1(A_1, \alpha_1'), d_2(A_1\alpha_1, A_1\alpha_1')\}
\]
\[
d_3(A_2A_1\alpha_1, A_2A_1\alpha_1') \leq c \max\{d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha_1'),
\]
\[
d_1(A_1, \alpha_1'), d_2(A_1\alpha_1, A_1\alpha_1')\}
\]
\[
d_3(A_2A_1\alpha_1, A_2A_1\alpha_1') \leq c \max\{d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha_1'),
\]
\[
c d_2(A_1\alpha_1, A_1\alpha_1'), c d_3(A_2A_1\alpha_1, A_2A_1\alpha_1')
\]
\[
c d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha_1')\}
\]

Using inequality (13) and (14), we have
\[
d_3(A_2A_1\alpha_1, A_2A_1\alpha_1') \leq c \max\{d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha_1')\}\]
\]
\[
(15)
\]

Similarly on using inequality (7), (13) and (15), we have,
\[
d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha_1') \leq c d_1(\alpha_1, \alpha_1')\]
\]
\[
(16)
\]

Using inequality (12), (13), (15) and (16), we have
\[
d_1(\alpha_1, \alpha_1') \leq c d_2(A_1\alpha_1, A_1\alpha_1')
\]
\[
\leq c^2 d_3(A_2A_1\alpha_1, A_2A_1\alpha_1')
\]
\[
\leq c^3 d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha_1')
\]
\[
\leq c^4 d_1(\alpha_1, \alpha_1')
\]
\]

Now we have
\[
d_1(\alpha_1, \alpha_1') \leq c^4 d_1(\alpha_1, \alpha_1')\]
Since $0 \leq c < 1$, we have
\[ d_1(\alpha_1, \alpha'_1) = 0 \]
\[ \Rightarrow \alpha_1 = \alpha'_1, \] proving the uniqueness of $\alpha_1$.

We can similarly prove that $A_1A_4A_3A_2$ has a unique fixed point $d_2 \in Z_2$ and $A_2A_1A_3A_4$ has a unique fixed point $\alpha_3 \in Z_3$ and $A_3A_2A_1A_4$ has unique fixed point $\alpha_4 \in Z_4$.

Now, in support of our result, we give some examples.

**Example 1** Let suppose $X = [0, 1], Y = [1, 2], Z = [2, 3]$ and $L = [3, 4]$ be complete metric spaces with usual metric. If $T : [0, 1] \to [1, 2], S : [1, 2] \to [2, 3]$ and $R : [2, 3] \to [3, 4]$ are continuous mappings and $U : [3, 4] \to [0, 1]$ is a mapping satisfying given conditions (in Theorem 2.1), where

\[ T(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq \frac{3}{4} \\ \frac{4}{3}x, & \text{if } \frac{3}{4} < x \leq 1 \end{cases}, \quad S(y) = \begin{cases} 2, & \text{if } 1 \leq y \leq \frac{3}{2} \\ \frac{4}{3}y, & \text{if } \frac{3}{2} < y \leq 2 \end{cases} \]

\[ R(z) = \begin{cases} 3, & \text{if } 2 \leq z \leq \frac{5}{2} \\ \frac{6}{5}z, & \text{if } \frac{5}{2} < z \leq 3 \end{cases}, \quad U(u) = \begin{cases} 1, & \text{if } 3 \leq u \leq \frac{7}{2} \\ \frac{3}{5}, & \text{if } \frac{7}{2} < u \leq 4 \end{cases} \]

then $URST$ has fixed point 1 such that $URST(1) = 1$, $TURS$ has fixed point $4/3$ such that $TURS(4/3) = 4/3$, $STUR$ has fixed point 2 such that $STUR(2) = 2$ and $RSTU$ has fixed point 3 such that $RSTU(3) = 3$. Also $T(1) = 4/3, S(4/3) = 2, R(2) = 3$ and $U(3) = 1$.

**Remark 1** Below we give an example which satisfies all the condition of Theorem 2.1 but does not satisfies the condition of Theorem 1.1.

**Example 2** Let $X = [0, 1], Y = [1, 2], Z = [2, 3]$ and $L = [3, 4]$ be complete metric space with usual metric. If $T : [0, 1] \to [1, 2], S : [1, 2] \to [2, 3]$ and $R : [2, 3] \to [3, 4]$ are continuous mappings and $U : [3, 4] \to [0, 1]$ is a mapping satisfying given conditions (in Theorem 2.1), where

\[ T(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq \frac{1}{4} \\ 2x + \frac{1}{2}, & \text{if } \frac{1}{4} \leq x \leq \frac{3}{4} \\ 2, & \text{if } \frac{3}{4} \leq x \leq 1 \end{cases}, \quad S(y) = \begin{cases} 2, & \text{if } 1 \leq y \leq \frac{5}{4} \\ \frac{4}{3}y + 1, & \text{if } \frac{5}{4} \leq y \leq \frac{7}{4} \\ 2.4, & \text{if } \frac{7}{4} \leq y \leq 2 \end{cases} \]

\[ R(z) = \begin{cases} 3, & \text{if } 2 \leq z \leq \frac{9}{4} \\ z + \frac{3}{4}, & \text{if } \frac{9}{4} \leq z \leq \frac{11}{4} \\ 3.5, & \text{if } \frac{11}{4} \leq z \leq 3 \end{cases}, \quad U(u) = \begin{cases} 0, & \text{if } 3 \leq u \leq \frac{7}{2} \\ \frac{7}{2}u - 1, & \text{if } \frac{7}{2} \leq u \leq 4 \end{cases} \]
then URST has fixed point 0 such that URST(0) = 0, TURS has fixed point 1 such that TURS(1) = 1, STUR has fixed point 2 such that STUR(2) = 2 and RSTU has fixed point 3 such that RSTU(3) = 3. Also T(0)=1, S(1)=2, R(2)=3 and U(3)=0.

Example 3 Let suppose $X = [0, 2], Y = [1, 5], Z = [0, 10]$ and $L = [1, 12]$ be complete metric spaces with usual metric. If $T : [0, 2] \to [1, 5], S : [1, 5] \to [0, 10]$ and $R : [0, 10] \to [1, 12]$ are continuous mappings and $U : [1, 12] \to [0, 2]$ is a mapping satisfying given conditions (in Theorem 2.1), where

$$T(x) = [1 + x, 2], \quad S(y) = [2y + 1, 5]$$

$$R(z) = [1 + z, 10], \quad U(u) = \begin{cases} \left[\frac{u}{6}, 5\right] & \text{if} \quad 1 \leq z \leq 6 \\ \left[1, \frac{u}{6}\right] & \text{if} \quad 6 < z \leq 12 \end{cases}$$

then URST has fixed point 1 such that URST(1) = 1, TURS has fixed point 2 such that TURS(2) = 2, STUR has fixed point 5 such that STUR(5) = 5 and RSTU has fixed point 6 such that RSTU(6) = 6. Also T(1) = 2, S(2) = 5, R(5) = 6 and U(6) = 1.

Example 4 Let suppose $X = [0, 3], Y = [1, 4], Z = [4, 7]$ and $L = [3, 10]$ be complete metric spaces with usual metric. If $T : [0, 3] \to [1, 4], S : [1, 4] \to [4, 7]$ and $R : [4, 7] \to [3, 10]$ be a continuous mappings and $U : [3, 10] \to [0, 3]$ be a mapping satisfying given conditions (in Theorem 2.1), where

$$T(x) = 1 + x, \quad S(y) = y + 2,$$

$$R(z) = z + 3, \quad U(u) = \begin{cases} \frac{u}{7} & \text{if} \quad 3 \leq z \leq 5 \\ \frac{u+1}{8} & \text{if} \quad 5 < z \leq 10 \end{cases}$$

then URST has fixed point 1 such that URST(1) = 1, TURS has fixed point 2 such that TURS(2) = 2, STUR has fixed point 4 such that STUR(4) = 4 and RSTU has fixed point 7 such that RSTU(7) = 7. Also T(1) = 2, S(2) = 4, R(5) = 7 and U(7) = 1.

References


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