Some inequalities in bicentric quadrilateral

Mihály Bencze

505600 Săcele-Négyfalu,

Jud. Braşov, Romania

email: benczemihaly@gmail.com

Marius Drăgan

061311 bd. Timișoara nr. 35,

bl. OD6, sc. E, et. 7 ap. 176, Sect. 6.,

Bucureşti, Romania

email: marius.dragan2005@yahoo.com

Dedicated to the memory of Professor Antal Bege

Abstract. In this paper we prove some results concerning bicentric quadrilaterals. We offer a new proof of the Blundon-Eddy inequality, which we use to obtain other inequalities in bicentric quadrilaterals.

1 Introduction

Let $ABCD$ be a bicentric quadrilateral with $a = AB, b = BC, c = CD, d = AD, d_1 = AC, d_2 = BD, s = \frac{a+b+c+d}{2}$, R the radius of the circumscribed circle of the quadrilateral $ABCD$ and r the radius of the inscribed circle, F the area.

In [1] W. J. Blundon and R. H. Eddy proved that:

$$8r \left(\sqrt{4R^2 + r^2} - r \right) \leq s^2 \leq \left(r + \sqrt{4R^2 + r^2} \right)^2.$$

In the following we give a simple proof to this double inequality using the product

$$(a-b)^2 (a-c)^2 (a-d)^2 (b-c)^2 (b-d)^2 (c-d)^2,$$

then we deduce many other important new inequalities. We mention that the result concerning the above product is new.

We denote:

$$\sigma_1 = \sum a, \sigma_2 = \sum ab, \sigma_3 = \sum abc, x_1 = bc+ad, x_2 = ab+cd, x_3 = ac+bd.$$

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2 Main results

Lemma 1 *In every bicentric quadrilateral ABCD the following equalities are true:*

- 1) $F^2 = (s - a)(s - b)(s - c)(s - d) = abcd$;
- 2) $x_1 x_2 x_3 = 16R^2 r^2 s^2$;
- 3) $x_1 + x_2 = s^2$;
- 4) $x_1 + x_2 + x_3 = s^2 + 2r^2 + 2r\sqrt{r^2 + 4R^2}$;
- 5) $x_3 = 2r \left(r + \sqrt{4R^2 + r^2} \right)$;
- 6) $(a - b)^2 (a - c)^2 (a - d)^2 (b - c)^2 (b - d)^2 (c - d)^2 = (x_1 - x_2)^2 (x_2 - x_3)^2 (x_3 - x_1)^2$.

Proof.

- 1) We have $a + c = b + d$. It results that $s - b = d$ and three similar equalities which imply

$$(s - a)(s - b)(s - c)(s - d) = abcd.$$

- 2) From Ptolemy's theorem it results that $x_3 = d_1 d_2$. We have the equalities:

$$ad \sin A + bc \sin C = 2F, \quad ab \sin B + dc \sin D = 2F.$$

We obtain $(ad + bc) d_1 = 4RF$, $(ab + dc) d_2 = 4RF$ which implies

$$(ad + bc) (ab + dc) d_1 d_2 = 16R^2 F^2 \text{ or } x_1 x_2 x_3 = 16R^2 r^2 s^2. \quad (1)$$

$$3) \text{ We have } x_1 + x_2 = ad + bc + ab + cd = (a + c)(d + b) = (a + c)^2 = \left(\frac{a+b+c+d}{2}\right) = s^2.$$

4) From (1) it results that

$$\begin{aligned} (ab + bc) (ad + dc) (ac + bd) &= 16R^2 F^2 \text{ or} \\ abcd \sum a^2 + \sigma_3^2 - 2abcd\sigma_2 &= 16R^2 F^2 \text{ or} \\ \sigma_3^2 - 4s^2 r^2 \sigma_2 + 4s^4 r^2 &= 16R^2 r^2 s^2 v. \end{aligned} \quad (2)$$

But $(s - a)(s - b)(s - c)(s - d) = s^2 r^2$ or $-s^3 + \sigma_2 s - \sigma_3 = 0$ which implies

$$\sigma_3 = s \left(\sigma_2 - s^2 \right). \quad (3)$$

From (2) and (3) we have:

$$\begin{aligned} s^2 \left(\sigma_2 - s^2 \right)^2 - 4s^2 r^2 \sigma_2 + 4s^4 r^2 &= 16R^2 r^2 s^2 \text{ or} \\ \sigma_2^2 - \left(2s^2 + 4r^2 \right) \sigma_2 + s^4 + 2s^2 r^2 - 16r^2 R^2 &= 0. \end{aligned}$$

It results that: $\sigma_2 = s^2 + 2r^2 + 2r\sqrt{r^2 + 4R^2}$. But $\sigma_2 = x_1 + x_2 + x_3$, so it follows that

$$x_1 + x_2 + x_3 = s^2 + 2r^2 + 2r\sqrt{r^2 + 4R^2}. \quad (4)$$

5) From 4) since $x_1 + x_2 = s^2$ it follows that $x_3 = 2r^2 + 2r\sqrt{4R^2 + r^2}$.

$$\begin{aligned} 6) \text{ We have } (a - b)^2 (a - c)^2 (a - d)^2 (b - c)^2 (b - d)^2 (c - d)^2 &= \\ [(a - b) (c - d)]^2 [(a - c) (b - d)]^2 [(a - d) (b - c)]^2 &= \\ (x_1 - x_2)^2 (x_2 - x_3)^2 (x_2 - x_1)^2. \end{aligned}$$

□

Theorem 1 *In every bicentric quadrilateral ABCD the following equality is true:*

$$\begin{aligned} &(a - b)^2 (a - c)^2 (a - d)^2 (b - c)^2 (b - d)^2 (c - d)^2 \\ &= 16r^4 s^2 \left[s^2 - 8r \left(\sqrt{4R^2 + r^2} - r \right) \right] \left[s^2 - \left(r + \sqrt{4R^2 + r^2} \right)^2 \right]^2. \end{aligned}$$

Proof. We denote $\triangle = (a-b)^2(a-c)^2(a-d)^2(b-c)^2(b-d)^2(c-d)^2$. From Lemma 1 6) we have:

$$\begin{aligned}\triangle &= (x_1 - x_2)^2 (x_3 - x_1)^2 (x_3 - x_2)^2 \\ &= \left[(x_1 + x_2)^2 - 4x_1x_2 \right] \left[x_3^2 - x_3(x_1 + x_2) + x_1x_2 \right]^2.\end{aligned}\quad (5)$$

From Lemma 1 2) and 5) it results that:

$$x_1x_2 = \frac{8R^2r^2s^2}{r(r + \sqrt{4R^2 + r^2})} = 2r \left(\sqrt{4R^2 + r^2} - r \right) s^2. \quad (6)$$

From Lemma 1 3), 5) and equalities (5), (6) we obtain:

$$\begin{aligned}\triangle &= \left[s^4 - 8r \left(\sqrt{4R^2 + r^2} - r \right) s^2 \right] \left[4r^2 \left(r + \sqrt{4R^2 + r^2} \right)^2 \right. \\ &\quad \left. - 2s^2r \left(r + \sqrt{4R^2 + r^2} \right) + 2r \left(\sqrt{4R^2 + r^2} - r \right) s^2 \right]^2 \\ &= s^2 \left[s^2 - 8r \left(\sqrt{4R^2 + r^2} - r \right) \right] \left[4r^2 \left(r + \sqrt{4R^2 + r^2} \right)^2 - 4r^2s^2 \right]^2 \\ &= 16r^4s^2 \left[s^2 - 8r \left(\sqrt{4R^2 + r^2} - r \right) \right] \left[s^2 - \left(r + \sqrt{4R^2 + r^2} \right)^2 \right].\end{aligned}$$

□

Theorem 2 *In every bicentric quadrilateral ABCD the following double inequality is true: $8r \left(\sqrt{4R^2 + r^2} - r \right) \leq s^2 \leq \left(r + \sqrt{4R^2 + r^2} \right)^2$. The equality holds in the case of two bicentric quadrilaterals $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ with the sides*

$$\begin{aligned}a_1 &= c_1 = \sqrt{2r\sqrt{4R^2 + r^2} - 2r^2} \\ b_1 &= \sqrt{2r\sqrt{4R^2 + r^2} - 2r^2} - \sqrt{2r\sqrt{4R^2 + r^2} - 6r^2} \\ d_1 &= \sqrt{2r\sqrt{4R^2 + r^2} - 2r^2} + \sqrt{2r\sqrt{4R^2 + r^2} - 6r^2} \\ a_2 &= d_2 = \frac{r + \sqrt{r^2 + 4R^2} - \sqrt{4R^2 - 2r^2 - 2r\sqrt{4R^2 + r^2}}}{2} \\ b_2 &= c_2 = \frac{r + \sqrt{r^2 + 4R^2} + \sqrt{4R^2 - 2r^2 - 2r\sqrt{4R^2 + r^2}}}{2}.\end{aligned}$$

Proof. We have $(x_3 - x_1)(x_3 - x_2) = (a - b)(b - c)(c - d)(d - a)$ and because $a + c = b + d$ it results that $(a - b)(b - c)(c - d)(d - a) = (a - b)^2(b - c)^2 \geq 0$, which implies $(x_3 - x_1)(x_3 - x_2) \geq 0$ or

$$s^2 \leq \left(r + \sqrt{4R^2 + r^2}\right)^2.$$

But, from Theorem 1 since $\Delta \geq 0$, it results that

$$8r \left(\sqrt{4R^2 + r^2} - r\right) \leq s^2.$$

It remain to study the equality cases for $s_1 \leq s \leq s_2$ where

$$s_1 = \sqrt{8r \left(\sqrt{4R^2 + r^2} - r\right)}, \quad s_2 = r + \sqrt{4R^2 + r^2}.$$

From Theorem 1 it results that we may have the cases:

Case 1.

$$a = c.$$

We denote $a = x$. Then

$$a = x, b = y, c = x, d = 2x - y.$$

From Lemma 1 we have:

$$x_3 = 2r \left(r + \sqrt{4R^2 + r^2}\right) \text{ or } x^2 + y(2x - y) = 2r \left(r + \sqrt{4R^2 + r^2}\right).$$

But $F^2 = abcd$ or $(2x - y)y = 4r^2$. It results that $x^2 = 2r\sqrt{4R^2 + r^2} - 2r^2$. Since $s_1^2 = 4x^2 = 8r \left(\sqrt{4R^2 + r^2} - r\right)$ represents the left side of the inequality from the statement, so:

$$x = \sqrt{2r\sqrt{4R^2 + r^2} - 2r^2}$$

$$(y - x)^2 = 2r\sqrt{4R^2 + r^2} - 6r^2 \text{ or } |y - x| = \sqrt{2r\sqrt{4R^2 + r^2} - 6r^2}.$$

We denote $u_1 = 2r\sqrt{4R^2 + r^2} - 2r^2$, $u_2 = 2r\sqrt{4R^2 + r^2} - 6r^2$.

If $x \leq y$ we have

$$a = x = \sqrt{u_1}, \quad b = y = \sqrt{u_1} + \sqrt{u_2}, \quad c = \sqrt{u_1}, \quad d = 2x - y = \sqrt{u_1} - \sqrt{u_2}.$$

If $x > y$ we have

$$a = x = \sqrt{u_1}, b = y = x - \sqrt{u_2} = \sqrt{u_1} - \sqrt{u_2}, c = \sqrt{u_1}, d = 2x - y = \sqrt{u_1} + \sqrt{u_2}.$$

It results that the equality from the left side of the inequality of the statement holds in the case of bicentric quadrilateral $A_1B_1C_1D_1$ with the sides

$$\sqrt{u_1}, \sqrt{u_1} - \sqrt{u_2}, \sqrt{u_1}, \sqrt{u_1} + \sqrt{u_2}.$$

Case 2.

$$a = d = x, b = c = y.$$

In this case $m(\angle D) = m(\angle B) = 90^\circ$, $AC = 2R$. It results that $F = sr = 2\frac{xy}{2}$ or $xy = (x + y)r$.

We denote $\alpha = x + y$, $\beta = xy$.

We have $\beta = \alpha r$. But $x^2 + y^2 = 4R^2$ which implies $\alpha^2 - 2\beta = 4R^2$ so we have $\alpha^2 - 2\alpha r - 4R^2 = 0$.

It results that $\alpha = r + \sqrt{r^2 + 4R^2}$.

But $s_1 = x + y = \alpha = r + \sqrt{r^2 + 4R^2}$ which represents the right side of the inequality from the statement. We have $\begin{cases} x + y = \alpha \\ xy = r\alpha \end{cases}$, so x, y are the solutions of the equation $u^2 - \alpha u + r\alpha = 0$ which implies:

$$x = \frac{\alpha - \sqrt{\alpha^2 - 4r\alpha}}{2} = \frac{r + \sqrt{r^2 + 4R^2} - \sqrt{4R^2 - 2r^2 - 2r\sqrt{4R^2 + r^2}}}{2},$$

$$y = \frac{r + \sqrt{r^2 + 4R^2} + \sqrt{4R^2 - 2r^2 - 2r\sqrt{4R^2 + r^2}}}{2}.$$

So, the equality for the right side of the inequality from the statement is true in the case of bicentric quadrilateral $A_2B_2C_2D_2$ with the sides

$$a_2 = x, b_2 = x, c_2 = y, d_2 = y.$$

□

Theorem 3 *In every bicentric quadrilateral ABCD the following inequalities are true:*

$$2r \left(r + \sqrt{4R^2 + r^2} \right) \leq \min\{ab + cd, bc + ad\} \leq 4r \left(\sqrt{4R^2 + r^2} - r \right) \\ \leq \max\{ab + cd + bc + ad\} \leq 4R^2.$$

Proof. We suppose that $x_1 \leq x_2$, $x_1 + x_2 = s^2$, $x_1 x_2 = \alpha s^2$ where

$$\alpha = \frac{8R^2 r}{\sqrt{4R^2 + r^2} + r} = 2r \left(\sqrt{4R^2 + r^2} - r \right).$$

It results that: $x_1 = \frac{s^2 - \sqrt{s^4 - 4\alpha s^2}}{2}$, $x_2 = \frac{s^2 + \sqrt{s^4 - 4\alpha s^2}}{2}$. We consider the functions $f, g : (0, +\infty) \rightarrow \mathbb{R}$.

$$f(s) = \frac{s^2 - \sqrt{s^4 - 4\alpha s^2}}{2}, g(s) = \frac{s^2 + \sqrt{s^4 - 4\alpha s^2}}{2}.$$

After differentiation we obtain:

$$f'(s) = \frac{s \left(\sqrt{s^4 - 4\alpha s^2} - s^2 + 2\alpha \right)}{\sqrt{s^4 - 4\alpha s^2}} \leq 0, g'(s) = \frac{s \left(\sqrt{s^4 - 4\alpha s^2} + s^2 - 4\alpha \right)}{\sqrt{s^4 - 4\alpha s^2}} \geq 0.$$

From Theorem 2 it results that: $s^2 \geq 8r \left(\sqrt{4R^2 + r^2} - r \right) = 4\alpha$.

It results that f is a decreasing and g is an increasing function. Because $s \leq r + \sqrt{4R^2 + r^2}$ we have $f \left(r + \sqrt{4R^2 + r^2} \right) \leq f(s) = x_1$. It follows that

$$\begin{aligned} x_1 &\geq \frac{1}{2} \left[\left(r + \sqrt{4R^2 + r^2} \right)^2 \right. \\ &\quad \left. - \left(r + \sqrt{4R^2 + r^2} \right) \sqrt{\left(r + \sqrt{4R^2 + r^2} \right)^2 - 8r \left(\sqrt{4R^2 + r^2} - r \right)} \right] \\ &= \frac{\left(r + \sqrt{4R^2 + r^2} \right)}{2} \left[r + \sqrt{4R^2 + r^2} \right. \\ &\quad \left. - \sqrt{r^2 + 4R^2 + r^2 + 2r\sqrt{4R^2 + r^2} - 8r\sqrt{4R^2 + r^2} + 8r^2} \right] \\ &= \frac{\left(r + \sqrt{4R^2 + r^2} \right)}{2} \left[r + \sqrt{4R^2 + r^2} - \sqrt{\left(\sqrt{4R^2 + r^2} \right)^2 + 9r^2 - 6r\sqrt{4R^2 + r^2}} \right] \\ &= 2r \left(r + \sqrt{4R^2 + r^2} \right). \end{aligned}$$

It follows that

$$x_1 \geq 2r \left(r + \sqrt{4R^2 + r^2} \right). \quad (7)$$

From $s \leq r + \sqrt{4R^2 + r^2}$ it results also that

$$\begin{aligned} x_2 = g(s) &\leq g\left(r + \sqrt{4R^2 + r^2}\right) \\ &= \frac{1}{2} \left[\left(r + \sqrt{4R^2 + r^2}\right)^2 \right. \\ &\quad \left. + \left(r + \sqrt{4R^2 + r^2}\right) \sqrt{\left(r + \sqrt{4R^2 + r^2}\right)^2 - 8r\left(\sqrt{4R^2 + r^2} - r\right)} \right] \\ &= \left(\sqrt{4R^2 + r^2} + r\right) \left(\sqrt{4R^2 + r^2} - r\right) = 4R^2. \end{aligned}$$

Thus we get the following inequality

$$x_2 \leq 4R^2. \quad (8)$$

Since $8r\left(\sqrt{4R^2 + r^2} - r\right) \leq s^2$ we have $x_1 = f(s) \leq f\left(\sqrt{8r\left(\sqrt{4R^2 + r^2} - r\right)}\right)$ or in an equivalent form

$$\begin{aligned} x_1 &\leq \frac{1}{2} \left[8r\left(\sqrt{4R^2 + r^2} - r\right) \right. \\ &\quad \left. - \sqrt{8r\left(\sqrt{4R^2 + r^2} - r\right)} \sqrt{8r\left(\sqrt{4R^2 + r^2} - r\right) - 8r\left(\sqrt{4R^2 + r^2} - r\right)} \right] \\ &= 4r\left(\sqrt{4R^2 + r^2} - r\right). \end{aligned}$$

It follows that

$$x_1 \leq 4r\left(\sqrt{4R^2 + r^2} - r\right). \quad (9)$$

Because $8r\left(\sqrt{4R^2 + r^2} - r\right) \leq s^2$ and g is an increasing function it follows that:

$$g\left(\sqrt{8r\left(\sqrt{4R^2 + r^2} - r\right)}\right) \leq g(s) = x_2 \text{ or } x_2 \geq 4r\left(\sqrt{4R^2 + r^2} - r\right). \quad (10)$$

From (7) (8) (9) and (10) it results that:

$$x_3 = 2r\left(r + \sqrt{4R^2 + r^2}\right) \leq x_1 \leq 4r\left(\sqrt{4R^2 + r^2} - r\right) \leq x_2 \leq 4R^2.$$

□

Remark 1 From Theorem 3 it results that $2r(r + \sqrt{4R^2 + r^2}) \leq 4r(\sqrt{4R^2 + r^2} - r)$ which, after performing some calculation, represent the well-known Fejes inequality $R \geq \sqrt{2}r$.

Theorem 4 In every bicentric quadrilateral ABCD the following inequalities are true:

$$\begin{aligned} \frac{r(\sqrt{4R^2 + r^2} + r)}{R} &\leq \min\{d_1, d_2\} \leq \frac{\sqrt{4R^2 + r^2} + r}{R} \sqrt{\frac{(\sqrt{4R^2 + r^2} - r)r}{2}} \\ &\leq \max\{d_1, d_2\} \leq 2R. \end{aligned}$$

Proof. We suppose that $x_1 \leq x_2$.

From Ptolemy's theorem it results that $\frac{x_1}{x_2} = \frac{d_1}{d_2}$ which implies $d_1 \leq d_2$.

Because $d_1 d_2 = x_3$ we have

$$\begin{aligned} d_1^2 &= \frac{x_1}{x_2} x_3 = \frac{s^2 - \sqrt{s^4 - 4\alpha s^2}}{s^2 + \sqrt{s^4 - 4\alpha s^2}} x_3 = x_3 \frac{(s^2 - \sqrt{s^4 - 4\alpha s^2})^2}{4\alpha s^2} \\ &= x_3 \frac{2s^4 - 4\alpha s^2 - 2s^2 \sqrt{s^4 - 4\alpha s^2}}{4\alpha s^2} = \frac{x_3 (s^2 - 2\alpha - \sqrt{s^4 - 4\alpha s^2})}{2\alpha} \\ &= \frac{2r(r + \sqrt{4R^2 + r^2})}{4r(\sqrt{4R^2 + r^2} - r)} [s^2 - \sqrt{s^4 - 4\alpha s^2} - 2\alpha] \\ &= \frac{(\sqrt{4R^2 + r^2} + r)^2}{8R^2} [s^2 - \sqrt{s^4 - 4\alpha s^2} - 2\alpha] = B(2x_1 - 2\alpha), \end{aligned}$$

where we denote $B = \frac{(\sqrt{4R^2 + r^2} + r)^2}{8R^2}$.

But from Theorem 3 we have

$$4r(r + \sqrt{4R^2 + r^2}) \leq 2x_1 \leq 8r(\sqrt{4R^2 + r^2} - r).$$

We obtain

$$\begin{aligned} 4r(r + \sqrt{4R^2 + r^2}) - 2\alpha &\leq 2x_1 - 2\alpha \leq 8r(\sqrt{4R^2 + r^2} - r) - 2\alpha \text{ or} \\ 8r^2 &\leq 2x_1 - 2\alpha \leq 4r(\sqrt{4R^2 + r^2} - r) \text{ or} \\ 8r^2 B &\leq B(2x_1 - 2\alpha) \leq 4r(\sqrt{4R^2 + r^2} - r) B \text{ or} \\ \frac{8r^2 (\sqrt{4R^2 + r^2} + r)^2}{8R^2} &\leq d_1^2 \leq \frac{4r(\sqrt{4R^2 + r^2} - r)(\sqrt{4R^2 + r^2} + r)^2}{8R^2}. \end{aligned}$$

It results that:

$$\frac{r \left(\sqrt{4R^2 + r^2} + r \right)}{r} < d_1 \leq \frac{\sqrt{4R^2 + r^2} + r}{R} \sqrt{\frac{\left(\sqrt{4R^2 + r^2} - r \right) r}{2}}. \quad (11)$$

Also:

$$\begin{aligned} d_2^2 &= \frac{x_2}{x_1} x_3 = \frac{s^2 + \sqrt{s^4 - 4\alpha s^2}}{s^2 - \sqrt{s^4 - 4\alpha s^2}} x_3 = \frac{\left(s^2 + \sqrt{s^4 - 4\alpha s^2} \right)^2}{4\alpha s^2} x_3 \\ &= \frac{x_3}{4\alpha s^2} \left(2s^4 - 4\alpha s^2 + 2s^2 \sqrt{s^4 - 4\alpha s^2} \right) = \frac{x_3}{2\alpha} \left(s^2 + \sqrt{s^4 - 4\alpha s^2} - 2\alpha \right) \\ &= \frac{x_3}{2\alpha} (2x_2 - 2\alpha) = \frac{\left(\sqrt{4R^2 + r^2} + r \right)^2 (2x_2 - 2\alpha)}{8R^2}. \end{aligned}$$

But we have proved that $4r \left(\sqrt{4R^2 + r^2} - r \right) \leq x_2 \leq 4R$. It results that:

$$\begin{aligned} 4r \left(\sqrt{4R^2 + r^2} - r \right) &\leq 2x_2 - 2\alpha \leq 2 \left(4R^2 + 2r^2 - 2r\sqrt{4R^2 + r^2} \right) \text{ or} \\ 4r \left(\sqrt{4R^2 + r^2} - r \right) &\left(\frac{\sqrt{4R^2 + r^2} + r}{2\sqrt{2}R} \right)^2 \leq d_2^2 \\ &\leq 2 \left(\sqrt{4R^2 + r^2} - r \right)^2 \frac{\left(\sqrt{4R^2 + r^2} + r \right)^2}{8R^2} \text{ or} \\ \frac{\sqrt{4R^2 + r^2} + r}{R} &\sqrt{\frac{\left(\sqrt{4R^2 + r^2} - r \right) r}{2}} \leq d_2 \leq 2R. \end{aligned} \quad (12)$$

From (11) and (12) it results the inequalities from the statement. $\square$

Theorem 5 Let be $\alpha, \beta \in \mathbb{R}$ so that $s \leq \alpha R + \beta r$ is true in every bicentric quadrilateral ABCD. Then $2R + (4 - 2\sqrt{2})r \leq \alpha R + \beta r$ is true in every bicentric quadrilateral ABCD.

Proof. We consider the case of the square with the sides $a = b = c = d = 1$. We have $2 \leq \alpha \frac{1}{\sqrt{2}} + \beta \frac{1}{2}$. It results that

$$4 \leq \sqrt{2}\alpha + \beta. \quad (13)$$

If $a = b = 1, c = d = 0$ it results that $R = \frac{1}{2}, r = 0$.

It follows that

$$1 \leq \frac{\alpha}{2} \text{ or } \alpha \geq 2. \quad (14)$$

We know that

$$R \geq \sqrt{2}r. \quad (15)$$

From (13), (14) and (15) it results that

$$\begin{aligned} (\alpha - 2)R + (\beta - 4 + 2\sqrt{2})r &\geq (\alpha - 2)\sqrt{2}r + (\beta - 4 + 2\sqrt{2})r \\ &= (\alpha\sqrt{2} + \beta - 4)r \geq 0, \end{aligned}$$

therefore

$$\alpha R + \beta r \geq 2R + (4 - 2\sqrt{2})r.$$

□

Theorem 6 *In every bicentric quadrilateral the following inequality is true:*

$$s \leq 2R + (4 - 2\sqrt{2})r.$$

Proof. From the Theorem 1 we have $s \leq r + \sqrt{4R^2 + r^2}$. We denote $x = \frac{R}{r}$.

We prove that

$$r + \sqrt{4R^2 + r^2} \leq 2R + (4 - 2\sqrt{2})r,$$

or in an equivalent form

$$\begin{aligned} 1 + \sqrt{4x^2 + 1} &\leq 2x + 4 - 2\sqrt{2} \text{ or } \sqrt{4x^2 + 1} \leq 2x + 3 - 2\sqrt{2} \text{ or} \\ 1 &\leq 4(3 - 2\sqrt{2})x + (3 - 2\sqrt{2})^2 \text{ or } x \geq \frac{(-2 + 2\sqrt{2})(4 - 2\sqrt{2})}{4(3 - 2\sqrt{2})}. \end{aligned}$$

After performing some calculation it results that $x \geq \sqrt{2}$ which represents just the Fejes's inequality [2]. □

Theorem 7 *In every bicentric quadrilateral ABCD the following inequalities are true:*

$$1) \ 4r(3\sqrt{4R^2 + r^2} - 5r) \leq a^2 + b^2 + c^2 + d^2 \leq 8R^2;$$

- 2) $2r\sqrt{8r(\sqrt{4R^2 + r^2} - r)}(7\sqrt{4R^2 + r^2} - 9r) \leq \sum a^2b \leq 8R^2 + 2r^2;$
- 3) $2r(5\sqrt{4R^2 + r^2} - 3r) \leq \sum ab \leq 4(R^2 + r^2 + r\sqrt{4R^2 + r^2});$
- 4) $32r^2\sqrt{4R^2 + r^2}(\sqrt{4R^2 + r^2} - r) \leq \sum a^2bc$
 $\leq 4r\sqrt{4R^2 + r^2}(r + \sqrt{4R^2 + r^2})^2;$
- 5) $(2r^2 + 2r\sqrt{4R^2 + r^2})\sqrt{8r(\sqrt{4R^2 + r^2} - r)} \leq \sum abc$
 $\leq 2r(r + \sqrt{4R^2 + r^2})^2.$

Proof. We have $\sigma_2 = s^2 + \alpha$, $\sigma_3 = s\alpha$ where $\alpha = 2r^2 + 2r\sqrt{r^2 + 4R^2}$.

$$1) \sum a^2 = (2s)^2 - 2\sigma_2 = 4s^2 - 2\sigma_2 = 4s^2 - 2s^2 - 4r^2 - 4r\sqrt{4R^2 + r^2}.$$

It results that: $\sum a^2 = 2s^2 - 4r^2 - 4r\sqrt{4R^2 + r^2}$.

From Theorem 2 we obtain

$$4r(3\sqrt{4R^2 + r^2} - 5r) \leq a^2 + b^2 + c^2 + d^2 \leq 8R^2.$$

$$2) a^2b = ab(2s - b - c - d) = 2sab - ab^2 - abc - abd \text{ or } a^2b + ab^2 = 2sab - abc - abd.$$

It results that $\sum a^2b = 2s\sigma_2 - 3\sigma_3 = 2s^3 - s\alpha = s(2s^2 - \alpha)$ which implies $\sum a^2b = s(2s^2 - \alpha)$. We consider the increasing function

$$f : (0, +\infty) \rightarrow \mathbb{R}, f(s) = 2s^3 - s\alpha, \text{ with } f'(s) = 6s^2 - \alpha \geq 0 \text{ as}$$

$$s^2 \geq 8r(\sqrt{4R^2 + r^2} - r) \geq \frac{\alpha}{6} = \frac{2r^2 + 2r\sqrt{r^2 + 4R^2}}{6}.$$

The last inequality may be written as:

$$24\sqrt{4R^2 + r^2} - 24r \geq r + \sqrt{4R^2 + r^2} \text{ or } 23\sqrt{4R^2 + r^2} \geq 25r.$$

But from inequality of Fejes it results that

$$23\sqrt{4R^2 + r^2} \geq 25\sqrt{9r^2} = 75r > 25r.$$

From Theorem 2 it results that:

$$\begin{aligned} & \sqrt{8r \left(\sqrt{4R^2 + r^2} - r \right)} \left(16 \left(\sqrt{4R^2 + r^2} - r \right) - 2r^2 - 2r\sqrt{4R^2 + r^2} \right) \\ & \leq \sum a^2b \leq \left(r + \sqrt{4R^2 + r^2} \right) \\ & \left(2r^2 + 8R^2 + 2r^2 + 2r\sqrt{4R^2 + r^2} - 2r^2 - 2r\sqrt{4R^2 + r^2} \right) \end{aligned}$$

which is equivalent with the inequality from the statement.

$$3) \sigma_2 = \sum ab = s^2 + \alpha \text{ or } 8r \left(\sqrt{4R^2 + r^2} - r \right) + 2r^2 + 2r\sqrt{4R^2 + r^2} \leq \sum ab \leq r^2 + 4R^2 + r^2 + 2r\sqrt{4R^2 + r^2} + 2r^2 + 2r\sqrt{4R^2 + r^2}$$

which is equivalent with the inequality from the statement.

$$\begin{aligned} 4) a^2bc = a abc = (2s - b - c - d) abc = 2sabc - ab^2c - abc^2 - abcd \text{ or } \\ a^2bc + ab^2c + abc^2 = 2sabc - abcd \text{ or } \sum a^2bc = 2s\sigma_3 - 4abcd = 2s \\ s\alpha - 4s^2r^2 \text{ or } \sum a^2bc = s^2 (2\alpha - 4r^2) = s^2 \left(4r^2 + 4r\sqrt{4R^2 + r^2} - 4r^2 \right) = \\ 4r\sqrt{4R^2 + r^2}s^2. \end{aligned}$$

From Theorem 2 it results the inequality from the statement.

$$5) \sum abc = s\alpha.$$

According to Theorem 2 it results the inequality from the statement.

□

Theorem 8 Let be $\alpha, \beta, \gamma \in \mathbb{R}$, $\beta \geq 4$ so that $s^2 \leq \alpha R^2 + \beta Rr + \gamma r^2$ is true in all bicentric quadrilateral. Then

$$4R^2 + 4Rr + \left(8 - 4\sqrt{2} \right) r^2 \leq \alpha R^2 + \beta Rr + \gamma r^2$$

is true in all bicentric quadrilateral.

Proof. We consider the case of the bicentric quadrilateral with $a = b = c = d = 1$. It results that $4 \leq \frac{\alpha}{2} + \frac{\beta}{2\sqrt{2}} + \frac{\gamma}{4}$ or $16 \leq 2\alpha + \sqrt{2}\beta + \gamma$.

In the case of $a = b = 1$, $c = d = 0$ it results that $R = \frac{1}{2}$, $r = 0$ and $\alpha \geq 4$. But from inequality $R \geq \sqrt{2}r$ we have:

$$\begin{aligned}
& (\alpha - 4) R^2 + (\beta - 4) Rr + (\gamma - 8 + 4\sqrt{2}) r^2 \\
& \geq (\alpha - 4) 2r^2 + \sqrt{2} (\beta - 4) r^2 + (\gamma - 8 + 4\sqrt{2}) r^2 \\
& \geq (\alpha - 4) 2r^2 + \sqrt{2} (\beta - 4) r^2 + (\gamma - 8 + 4\sqrt{2}) r^2 \\
& = (2\alpha + \sqrt{2}\beta + \gamma - 16) r^2 \geq 0.
\end{aligned}$$

□

Theorem 9 *In every bicentric quadrilateral ABCD the following inequality is true:*

$$s^2 \leq 4R^2 + 4Rr + (8 - 4\sqrt{2}) r^2.$$

Proof. Since $s^2 \leq (r + \sqrt{4R^2 + r^2})^2$ it is sufficient to prove that:

$$\begin{aligned}
& (\sqrt{4x^2 + 1} + 1)^2 \leq 4x^2 + 4x + 8 - 4\sqrt{2} \text{ or} \\
& 4x^2 + 1 + 1 + 2\sqrt{4x^2 + 1} \leq 4x^2 + 4x + 8 - 4\sqrt{2} \text{ or} \\
& 2\sqrt{4x^2 + 1} \leq 4x + 6 - 4\sqrt{2} \text{ or} \\
& \sqrt{4x^2 + 1} \leq 2x + 3 - 2\sqrt{2} \text{ or } 4x^2 + 1 \leq 4x^2 + (12 - 8\sqrt{2})x + (3 - 2\sqrt{2})^2 \text{ or} \\
& x \geq \frac{(1 - 3 + 2\sqrt{2})(1 + 3 - 2\sqrt{2})}{4(3 - 2\sqrt{2})} = \frac{(\sqrt{2} - 1)(2 - \sqrt{2})}{3 - 2\sqrt{2}} = \sqrt{2}.
\end{aligned}$$

□

Theorem 10 *In every bicentric quadrilateral ABCD the following inequalities are true:*

- 1) $\sum abc \leq 8R^2r + 8Rr^2 + (16 - 8\sqrt{2}) r^3;$
- 2) $\sum ab \leq 4 [R^2 + 2Rr + (4 - 2\sqrt{2}) r^2];$
- 3) $\sum a^2bc \leq 32R^3r + 16Rr^3 + (80 - 32\sqrt{2}) R^2r^2 + (32 - 16\sqrt{2}) r^4.$

Proof.

- 1) We proved that $\sum abc \leq 2r \left(r + \sqrt{4R^2 + r^2} \right)^2$, and

$$\left(r + \sqrt{4R^2 + r^2} \right)^2 \leq 4R^2 + 4Rr + \left(8 - 4\sqrt{2} \right) r^2.$$

It results that

$$\sum abc \leq 2r \left(4R^2 + 4Rr + \left(8 - 4\sqrt{2} \right) r^2 \right).$$

- 2) Since $\sqrt{4R^2 + r^2} \leq 2R + \left(3 - 2\sqrt{2} \right) r$, from Theorem 7 3) it results that:

$$\begin{aligned} \sum ab &\leq 4 \left(R^2 + r^2 + r\sqrt{4R^2 + r^2} \right) \\ &\leq 4 \left[R^2 + r^2 + r \left(2R + \left(3 - 2\sqrt{2} \right) r \right) \right] \\ &= 4 \left[R^2 + r^2 + 2Rr + \left(3 - 2\sqrt{2} \right) r^2 \right] \text{ or} \\ \sum ab &\leq 4 \left[R^2 + 2Rr + \left(4 - 2\sqrt{2} \right) r^2 \right]. \end{aligned}$$

- 3) From Theorem 7 4) it results that:

$$\begin{aligned} \sum a^2bc &\leq 4r\sqrt{4R^2 + r^2} \left(r + \sqrt{4R^2 + r^2} \right)^2 \\ &= 4r\sqrt{4R^2 + r^2} \left(r^2 + 4R^2 + r^2 + 2r\sqrt{4R^2 + r^2} \right) \\ &= 8r\sqrt{4R^2 + r^2} \left(2R^2 + r^2 + r\sqrt{4R^2 + r^2} \right) \\ &= \left(16R^2r + 8r^3 \right) \sqrt{4R^2 + r^2} + 8r^2 \left(4R^2 + r^2 \right) \\ &\leq \left(16R^2r + 8r^3 \right) \left[2R + \left(3 - 2\sqrt{2} \right) r \right] + 32R^2r^2 + 8r^4 \\ &= 32R^3r + \left(48 - 32\sqrt{2} \right) R^2r^2 + 16Rr^3 + \left(24 - 16\sqrt{2} \right) r^4 \\ &\quad + 32R^2r^2 + 8r^4, \end{aligned}$$

which is equivalent with the inequality from the statement.

□

Theorem 11 *In every bicentric quadrilateral ABCD the following inequalities are true:*

- 1) $2r\sqrt{8r(\sqrt{4R^2+r^2}-r)}(5\sqrt{4R^2+r^2}-11r) \leq \sum a^3$
 $\leq 2\left(r+\sqrt{4R^2+r^2}\right)\left(4R^2-r^2-r\sqrt{4R^2+r^2}\right);$
- 2) $352R^2r^2+208r^4-240r^3\sqrt{4R^2+r^2}$
 $\leq \sum a^3b \leq \left(r+\sqrt{4R^2+r^2}\right)^2(8R^2-4r^2).$

Proof.

- 1) $a^3 = a^2(2s-b-c-d) = 2a^2s - a^2b - a^2c - a^2d$ or $\sum a^3 = 2s\sum a^2 - \sum a^2b = 2s(2s^2-2\alpha) - 2s^3 + s\alpha.$

It results that $\sum a^3 = 2s^3 - 3\alpha s.$

We consider the function $f : (0, +\infty) \rightarrow \mathbb{R}, f(s) = 2s^3 - 3\alpha s$, with the derivate $f'(s) = 6s^2 - 3\alpha$. We prove that $f'(s) \geq 0$ or $s^2 \geq \frac{\alpha}{2}.$

But $s^2 \geq 8r(\sqrt{4R^2+r^2}-r)$. It will be sufficient to prove that:

$$8r(\sqrt{4R^2+r^2}-r) \geq r^2 + r\sqrt{4R^2+r^2} \text{ or } \\ 8\sqrt{4x^2+1}-8 \geq 1+\sqrt{4x^2+1} \text{ or } \sqrt{4x^2+1} \geq \frac{9}{7},$$

which is true because $\sqrt{4x^2+1} \geq 2$ according to Fejes inequality.

Since f is an increasing function it results from Theorem 2 that:

$$\sqrt{8r(\sqrt{4R^2+r^2}-r)}\left[16(\sqrt{4R^2+r^2}-r)-6r^2-6r\sqrt{4R^2+r^2}\right] \\ \leq \sum a^3 \leq \left(r+\sqrt{4R^2+r^2}\right)\left[2r^2+8R^2+2r^2\right. \\ \left.+4r\sqrt{4R^2+r^2}-6r^2-6r\sqrt{4R^2+r^2}\right],$$

which is equivalent with the inequality from the statement.

- 2) $a^3b = ab(\sum a^2 - b^2 - c^2 - d^2) = ab\sum a^2 - ab^3 - abc^2 - abd^2$ or $a^3b + ab^3 = ab\sum a^2 - abc^2 - abd^2$ or $\sum a^3b = \sum ab\sum a^2 - \sum a^2bc = (s^2 + \alpha)(2s^2 - 2\alpha) - (2\alpha - 4r^2)s^2$ or $\sum a^3b = 2s^4 - (2\alpha - 4r^2)s^2 - 2\alpha^2.$

We denote $s^2 = t$ and consider the function: $f : (0, +\infty) \rightarrow \mathbb{R},$

$$f(t) = 2t^2 - (2\alpha - 4r^2)t - 2\alpha^2$$

and

$$t_v = \frac{2a - 4r^2}{4} = \frac{a - 2r^2}{2} = r\sqrt{4R^2 + r^2}.$$

We prove that $t \geq t_v$.

$s^2 \geq r\sqrt{4R^2 + r^2}$. But $s^2 \geq 8r(\sqrt{4R^2 + r^2} - r)$. It will be sufficient to prove that

$$8r(\sqrt{4R^2 + r^2} - r) \geq r\sqrt{4R^2 + r^2} \text{ or } \sqrt{4R^2 + r^2} \geq \frac{8}{7}$$

which is true because $\sqrt{4R^2 + r^2} \geq 3$.

It results that f is an increasing function which implies:

$$\begin{aligned} & 128r^2(4R^2 + 2r^2 - 2r\sqrt{4R^2 + r^2}) - 4r\sqrt{4R^2 + r^2}8r(\sqrt{4R^2 + r^2} - r) \\ & - 2(2r^2 + 2r\sqrt{4R^2 + r^2})^2 \leq \sum a^3b \leq 2(r + \sqrt{4R^2 + r^2})^4 \\ & - 4r\sqrt{4R^2 + r^2}(r + \sqrt{4R^2 + r^2})^2 - 2(2r^2 + 2r\sqrt{4R^2 + r^2})^2 \end{aligned}$$

or

$$\begin{aligned} & 512R^2r^2 + 256r^4 - 256r^3\sqrt{4R^2 + r^2} - 32r^2(4R^2 + r^2) + 32r^3\sqrt{4R^2 + r^2} \\ & - 8r^4 - 8r^2(4R^2 + r^2) - 16r^3\sqrt{4R^2 + r^2} \leq \sum a^3b \leq 2(r + \sqrt{4R^2 + r^2})^2 \\ & (r^2 + 4R^2 + r^2 + 2r\sqrt{4R^2 + r^2} - 2r\sqrt{4R^2 + r^2}) - 8r^2(r + \sqrt{4R^2 + r^2})^2 \end{aligned}$$

or

$$\begin{aligned} 352R^2r^2 + 208r^4 - 240r^3\sqrt{4R^2 + r^2} & \leq \sum a^3b \leq (r + \sqrt{4R^2 + r^2})^2 \\ & (4r^2 + 8R^2 - 8r^2). \end{aligned}$$

□

Theorem 12 *In every bicentric quadrilateral ABCD the following inequalities are true:*

- 1) $\sum a^3 \leq 16R^3 + (24 - 16\sqrt{2})R^2r - 8Rr^2 - (16 - 8\sqrt{2})r^3;$
- 2) $\sum a^3b \leq 32R^4 - 16R^2r^2 + 32R^3r + 16Rr^3 + (64 - 32\sqrt{2})R^2r^2 - (32 - 16\sqrt{2})r^4;$

$$3) \sum a^3 b \geq 352R^2 r^2 + (480\sqrt{2} - 512) r^4 - 480Rr^3.$$

Proof.

1) From Theorem 11 it results that:

$$\begin{aligned} \sum a^3 &\leq \left(r + \sqrt{4R^2 + r^2} \right) \left(8R^2 - 2r^2 - 2r\sqrt{4R^2 + r^2} \right) \\ &= 8R^2 r - 2r^3 - 2r^2 \sqrt{4R^2 + r^2} + 8R^2 \sqrt{4R^2 + r^2} \\ &\quad - 2r^2 \sqrt{4R^2 + r^2} - 8R^2 r - 2r^3 \\ &= (8R^2 - 4r^2) \sqrt{4R^2 + r^2} - 4r^3 \\ &\leq (8R^2 - 4r^2) \left[2R + (3 - 2\sqrt{2}) r \right] - 4r^3 \\ &= 16r^3 + (24 - 16\sqrt{2}) R^2 r - 8Rr^2 - (12 - 8\sqrt{2}) r^3 - 4r^3, \end{aligned}$$

which is equivalent with inequality from the statement.

2) From Theorem 11 it results that

$$\sum a^3 b \leq \left(r + \sqrt{4R^2 + r^2} \right)^2 (8R^2 - 4r^2)$$

and

$$\left(r + \sqrt{4R^2 + r^2} \right)^2 \leq 4R^2 + 4Rr + (8 - 4\sqrt{2}) r^2.$$

It results that:

$$\begin{aligned} \sum a^3 b &\leq \left[4R^2 + 4Rr + (8 - 4\sqrt{2}) r^2 \right] (8R^2 - 4r^2) \\ &= 32R^4 - 16R^2 r^2 + 32R^3 r - 16Rr^3 + (64 - 32\sqrt{2}) R^2 r^2 \\ &\quad - (32 - 16\sqrt{2}) r^4, \end{aligned}$$

which is equivalent with the inequality from the statement.

3) We prove that:

$$\begin{aligned} \sum a^3 b &\geq 352R^2 r^2 + 208r^4 - 240r^3 \sqrt{4R^2 + r^2} \\ &\geq 352R^2 r^2 + 208r^4 - 240r^3 \left[2R + (3 - 2\sqrt{2}) r \right] \\ &= 352R^2 r^2 + 208r^4 - 480Rr^3 - (720 - 480\sqrt{2}) r^4, \end{aligned}$$

which is equivalent with the inequality from the statement.

□

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