Trapezoid type inequalities for generalized Riemann-Liouville fractional integrals of functions with bounded variation

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Abstract. In this paper we establish some trapezoid type inequalities for the Riemann-Liouville fractional integrals of functions of bounded variation and of Hölder continuous functions. Applications for the *g-mean of two numbers* are provided as well. Some particular cases for Hadamard fractional integrals are also provided.

1 Introduction

Let (a, b) with $-\infty \leq a < b \leq \infty$ be a finite or infinite interval of the real line $\mathbb{R}$ and α a complex number with $\operatorname{Re}(\alpha) > 0$. Also let g be a strictly increasing function on (a, b) , having a continuous derivative g' on (a, b) . Following [18,

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p. 100], we introduce the *generalized left- and right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a, b]$ by

$$I_{a+,g}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t) f(t) dt}{[g(x) - g(t)]^{1-\alpha}}, \quad a < x \leq b$$

and

$$I_{b-,g}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \frac{g'(t) f(t) dt}{[g(t) - g(x)]^{1-\alpha}}, \quad a \leq x < b.$$

For $g(t) = t$ we have the classical *Riemann-Liouville fractional integrals*

$$J_{a+}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t) dt}{(x-t)^{1-\alpha}}, \quad a < x \leq b$$

and

$$J_{b-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t) dt}{(t-x)^{1-\alpha}}, \quad a \leq x < b,$$

while for the logarithmic function $g(t) = \ln t$ we have the *Hadamard fractional integrals* [18, p. 111]

$$H_{a+}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \left[\ln \left(\frac{x}{t} \right) \right]^{\alpha-1} \frac{f(t) dt}{t}, \quad 0 \leq a < x \leq b$$

and

$$H_{b-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \left[\ln \left(\frac{t}{x} \right) \right]^{\alpha-1} \frac{f(t) dt}{t}, \quad 0 \leq a < x < b.$$

One can consider the function $g(t) = -t^{-1}$ and define the “*Harmonic fractional integrals*” by

$$R_{a+}^{\alpha} f(x) := \frac{x^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{f(t) dt}{(x-t)^{1-\alpha} t^{\alpha+1}}, \quad 0 \leq a < x \leq b$$

and

$$R_{b-}^{\alpha} f(x) := \frac{x^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{f(t) dt}{(t-x)^{1-\alpha} t^{\alpha+1}}, \quad 0 \leq a < x < b.$$

Also, for $g(t) = \exp(\beta t)$, $\beta > 0$, we can consider the “ *β -Exponential fractional integrals*”

$$E_{a+,\beta}^{\alpha} f(x) := \frac{\beta}{\Gamma(\alpha)} \int_a^x \frac{\exp(\beta t) f(t) dt}{[\exp(\beta x) - \exp(\beta t)]^{1-\alpha}}, \quad a < x \leq b$$

and

$$\mathbb{E}_{b-, \beta}^{\alpha} f(x) := \frac{\beta}{\Gamma(\alpha)} \int_x^b \frac{\exp(\beta t) f(t) dt}{[\exp(\beta t) - \exp(\beta x)]^{1-\alpha}}, \quad a \leq x < b.$$

In the recent paper [14] we obtained the following Ostrowski type inequalities for functions of bounded variation:

Theorem 1 *Let $f : [a, b] \rightarrow \mathbb{C}$ be a function of bounded variation on $[a, b]$ and g be a strictly increasing function on (a, b) , having a continuous derivative g' on (a, b) . For any $x \in (a, b)$ we have the inequalities*

$$\begin{aligned} & \left| \mathbb{I}_{a+, g}^{\alpha} f(x) + \mathbb{I}_{b-, g}^{\alpha} f(x) - \frac{1}{\Gamma(\alpha+1)} ([g(x) - g(a)]^{\alpha} + [g(b) - g(x)]^{\alpha}) f(x) \right| \\ & \leq \frac{1}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t) V_t^x(f) dt}{[g(x) - g(t)]^{1-\alpha}} + \int_x^b \frac{g'(t) V_x^t(f) dt}{[g(t) - g(x)]^{1-\alpha}} \right] \\ & \leq \frac{1}{\Gamma(\alpha+1)} \left[[g(x) - g(a)]^{\alpha} \bigvee_a^x(f) + [g(b) - g(x)]^{\alpha} \bigvee_x^b(f) \right] \\ & \leq \frac{1}{\Gamma(\alpha+1)} \left\{ \begin{aligned} & \left[\frac{1}{2} (g(b) - g(a)) + \left| g(x) - \frac{g(a)+g(b)}{2} \right| \right]^{\alpha} V_a^b(f); \\ & ((g(x) - g(a))^{\alpha p} + (g(b) - g(x))^{\alpha p})^{1/p} \left((V_a^x(f))^q + (V_x^b(f))^q \right)^{1/q} \\ & \text{with } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ & ((g(x) - g(a))^{\alpha} + (g(b) - g(x))^{\alpha}) \left[\frac{1}{2} V_a^b(f) + \frac{1}{2} |V_a^x(f) - V_x^b(f)| \right], \end{aligned} \right. \end{aligned}$$

and

$$\begin{aligned} & \left| \mathbb{I}_{x-, g}^{\alpha} f(a) + \mathbb{I}_{x+, g}^{\alpha} f(b) - \frac{1}{\Gamma(\alpha+1)} ([g(x) - g(a)]^{\alpha} + [g(b) - g(x)]^{\alpha}) f(x) \right| \\ & \leq \frac{1}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t) V_t^x(f) dt}{[g(t) - g(a)]^{1-\alpha}} + \int_x^b \frac{g'(t) V_x^t(f) dt}{[g(b) - g(t)]^{1-\alpha}} \right] \\ & \leq \frac{1}{\Gamma(\alpha+1)} \left[[g(x) - g(a)]^{\alpha} \bigvee_a^x(f) + [g(b) - g(x)]^{\alpha} \bigvee_x^b(f) \right] \\ & \leq \frac{1}{\Gamma(\alpha+1)} \left\{ \begin{aligned} & \left[\frac{1}{2} (g(b) - g(a)) + \left| g(x) - \frac{g(a)+g(b)}{2} \right| \right]^{\alpha} V_a^b(f); \\ & ((g(x) - g(a))^{\alpha p} + (g(b) - g(x))^{\alpha p})^{1/p} \left((V_a^x(f))^q + (V_x^b(f))^q \right)^{1/q} \\ & \text{with } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ & ((g(x) - g(a))^{\alpha} + (g(b) - g(x))^{\alpha}) \left[\frac{1}{2} V_a^b(f) + \frac{1}{2} |V_a^x(f) - V_x^b(f)| \right]. \end{aligned} \right. \end{aligned}$$

If g is a function which maps an interval I of the real line to the real numbers, and is both continuous and injective then we can define the g -mean of two numbers $a, b \in I$ as

$$M_g(a, b) := g^{-1} \left(\frac{g(a) + g(b)}{2} \right).$$

If $I = \mathbb{R}$ and $g(t) = t$ is the *identity function*, then $M_g(a, b) = A(a, b) := \frac{a+b}{2}$, the *arithmetic mean*. If $I = (0, \infty)$ and $g(t) = \ln t$, then $M_g(a, b) = G(a, b) := \sqrt{ab}$, the *geometric mean*. If $I = (0, \infty)$ and $g(t) = \frac{1}{t}$, then $M_g(a, b) = H(a, b) := \frac{2ab}{a+b}$, the *harmonic mean*. If $I = (0, \infty)$ and $g(t) = t^p$, $p \neq 0$, then $M_g(a, b) = M_p(a, b) := \left(\frac{a^p + b^p}{2} \right)^{1/p}$, the *power mean with exponent* p . Finally, if $I = \mathbb{R}$ and $g(t) = \exp t$, then

$$M_g(a, b) = \text{LME}(a, b) := \ln \left(\frac{\exp a + \exp b}{2} \right),$$

the *LogMeanExp function*.

The following particular case for g -mean is of interest [14].

Corollary 1 *With the assumptions of Theorem 1 we have*

$$\begin{aligned} & \left| I_{a+,g}^\alpha f(M_g(a, b)) + I_{b-,g}^\alpha f(M_g(a, b)) - \frac{[g(b) - g(a)]^\alpha}{2^{\alpha-1} \Gamma(\alpha+1)} f(M_g(a, b)) \right| \\ & \leq \frac{1}{\Gamma(\alpha)} \left[\int_a^{M_g(a,b)} \frac{g'(t) V_t^{M_g(a,b)}(f) dt}{[g(M_g(a, b)) - g(t)]^{1-\alpha}} + \int_{M_g(a,b)}^b \frac{g'(t) V_{M_g(a,b)}^t(f) dt}{[g(t) - g(M_g(a, b))]^{1-\alpha}} \right] \\ & \leq \frac{1}{2^\alpha \Gamma(\alpha+1)} (g(b) - g(a))^\alpha \bigvee_a^b(f); \end{aligned}$$

and

$$\begin{aligned} & \left| I_{M_g(a,b)-,g}^\alpha f(a) + I_{M_g(a,b)+,g}^\alpha f(b) - \frac{[g(b) - g(a)]^\alpha}{2^{\alpha-1} \Gamma(\alpha+1)} f(M_g(a, b)) \right| \\ & \leq \frac{1}{\Gamma(\alpha)} \left[\int_a^{M_g(a,b)} \frac{g'(t) V_t^{M_g(a,b)}(f) dt}{[g(t) - g(a)]^{1-\alpha}} + \int_{M_g(a,b)}^b \frac{g'(t) V_x^t(f) dt}{[g(b) - g(t)]^{1-\alpha}} \right] \\ & \leq \frac{1}{2^\alpha \Gamma(\alpha+1)} (g(b) - g(a))^\alpha \bigvee_a^b(f). \end{aligned}$$

Remark 1 If we take in Theorem 1 $x = \frac{a+b}{2}$, then we obtain similar mid-point inequalities, however the details are not presented here. Some applications for the Hadamard fractional integrals are also provided in [14].

For several Ostrowski type inequalities for Riemann-Liouville fractional integrals see [1]-[5], [16]-[27] and the references therein.

Motivated by the above results, in this paper we establish some trapezoid type inequalities for the generalized Riemann-Liouville fractional integrals of functions of bounded variation and of Hölder continuous functions. Applications for the *g-mean of two numbers* are provided as well. Some particular cases for Hadamard fractional integrals are also provided.

2 Some identities

We have:

Lemma 1 Let $f : [a, b] \rightarrow \mathbb{C}$ be Lebesgue integrable on $[a, b]$, g be a strictly increasing function on (a, b) , having a continuous derivative g' on (a, b) and λ, μ some complex parameters:

(i) For any $x \in (a, b)$ we have the representation

$$\begin{aligned} I_{a+,g}^\alpha f(x) + I_{b-,g}^\alpha f(x) &= \frac{1}{\Gamma(\alpha+1)} (\lambda [g(x) - g(a)]^\alpha + \mu [g(b) - g(x)]^\alpha) \\ &+ \frac{1}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t) [f(t) - \lambda] dt}{[g(x) - g(t)]^{1-\alpha}} + \int_x^b \frac{g'(t) [f(t) - \mu] dt}{[g(t) - g(x)]^{1-\alpha}} \right] \end{aligned} \quad (1)$$

and

$$\begin{aligned} I_{x-,g}^\alpha f(a) + I_{x+,g}^\alpha f(b) &= \frac{1}{\Gamma(\alpha+1)} (\lambda [g(x) - g(a)]^\alpha + \mu [g(b) - g(x)]^\alpha) \\ &+ \frac{1}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t) [f(t) - \lambda] dt}{[g(t) - g(a)]^{1-\alpha}} + \int_x^b \frac{g'(t) [f(t) - \mu] dt}{[g(b) - g(t)]^{1-\alpha}} \right]. \end{aligned} \quad (2)$$

(ii) We have

$$\begin{aligned} \frac{I_{b-,g}^\alpha f(a) + I_{a+,g}^\alpha f(b)}{2} &= \frac{1}{\Gamma(\alpha+1)} [g(b) - g(a)]^\alpha \frac{\lambda + \mu}{2} \\ &+ \frac{1}{2\Gamma(\alpha)} \left[\int_a^b \frac{g'(t) [f(t) - \lambda] dt}{[g(b) - g(t)]^{1-\alpha}} + \int_a^b \frac{g'(t) [f(t) - \mu] dt}{[g(t) - g(a)]^{1-\alpha}} \right]. \end{aligned} \quad (3)$$

Proof. (i) We observe that

$$\begin{aligned}
 & \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t) [f(t) - \lambda] dt}{[g(x) - g(t)]^{1-\alpha}} \\
 &= I_{a+,g}^\alpha f(x) - \lambda \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t) dt}{[g(x) - g(t)]^{1-\alpha}} \\
 &= I_{a+,g}^\alpha f(x) - \frac{[g(x) - g(a)]^\alpha}{\alpha \Gamma(\alpha)} \lambda = I_{a+,g}^\alpha f(x) - \frac{[g(x) - g(a)]^\alpha}{\Gamma(\alpha + 1)} \lambda
 \end{aligned} \tag{4}$$

for $a < x \leq b$ and, similarly,

$$\frac{1}{\Gamma(\alpha)} \int_x^b \frac{g'(t) [f(t) - \mu] dt}{[g(t) - g(x)]^{1-\alpha}} = I_{b-,g}^\alpha f(x) - \frac{[g(b) - g(x)]^\alpha}{\Gamma(\alpha + 1)} \mu \tag{5}$$

for $a \leq x < b$.

If $x \in (a, b)$, then by adding the equalities (4) and (5) we get the representation (1).

By the definition of fractional integrals we have

$$I_{x+,g}^\alpha f(b) := \frac{1}{\Gamma(\alpha)} \int_x^b \frac{g'(t) f(t) dt}{[g(b) - g(t)]^{1-\alpha}}, \quad a \leq x < b$$

and

$$I_{x-,g}^\alpha f(a) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t) f(t) dt}{[g(t) - g(a)]^{1-\alpha}}, \quad a < x \leq b.$$

Then

$$\frac{1}{\Gamma(\alpha)} \int_x^b \frac{g'(t) [f(t) - \lambda] dt}{[g(b) - g(t)]^{1-\alpha}} = I_{x+,g}^\alpha f(b) - \frac{[g(b) - g(x)]^\alpha}{\Gamma(\alpha + 1)} \lambda \tag{6}$$

for $a \leq x < b$ and

$$\frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t) [f(t) - \mu] dt}{[g(t) - g(a)]^{1-\alpha}} = I_{x-,g}^\alpha f(a) - \frac{[g(x) - g(a)]^\alpha}{\Gamma(\alpha + 1)} \mu \tag{7}$$

for $a < x \leq b$.

If $x \in (a, b)$, then by adding the equalities (6) and (7) we get the representation (1).

If we take $x = b$ in (4) we get

$$\frac{1}{\Gamma(\alpha)} \int_a^b \frac{g'(t) [f(t) - \lambda] dt}{[g(b) - g(t)]^{1-\alpha}} = I_{a+,g}^\alpha f(b) - \frac{[g(b) - g(a)]^\alpha}{\Gamma(\alpha + 1)} \lambda \tag{8}$$

while from $x = a$ in (5) we get

$$\frac{1}{\Gamma(\alpha)} \int_a^b \frac{g'(t) [f(t) - \mu] dt}{[g(t) - g(a)]^{1-\alpha}} = I_{b-,g}^\alpha f(a) - \frac{[g(b) - g(a)]^\alpha}{\Gamma(\alpha+1)} \mu. \quad (9)$$

If we add (8) with (9) and divide by 2 we get (3). $\square$

Remark 2 If we take in (1) and (2) $x = M_g(a, b) = g^{-1}\left(\frac{g(a)+g(b)}{2}\right)$, then we get

$$\begin{aligned} & I_{a+,g}^\alpha f(M_g(a, b)) + I_{b-,g}^\alpha f(M_g(a, b)) \\ &= \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)} [g(b) - g(a)]^\alpha \left(\frac{\lambda + \mu}{2}\right) \\ &+ \frac{1}{\Gamma(\alpha)} \left[\int_a^{M_g(a,b)} \frac{g'(t) [f(t) - \lambda] dt}{[g(M_g(a, b)) - g(t)]^{1-\alpha}} + \int_{M_g(a,b)}^b \frac{g'(t) [f(t) - \mu] dt}{[g(t) - g(M_g(a, b))]^{1-\alpha}} \right] \end{aligned}$$

and

$$\begin{aligned} & I_{M_g(a,b)-,g}^\alpha f(a) + I_{M_g(a,b)+,g}^\alpha f(b) = \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)} [g(b) - g(a)]^\alpha \left(\frac{\lambda + \mu}{2}\right) \\ &+ \frac{1}{\Gamma(\alpha)} \left[\int_a^{M_g(a,b)} \frac{g'(t) [f(t) - \lambda] dt}{[g(t) - g(a)]^{1-\alpha}} + \int_{M_g(a,b)}^b \frac{g'(t) [f(t) - \mu] dt}{[g(b) - g(t)]^{1-\alpha}} \right]. \end{aligned}$$

The above lemma provides various identities of interest by taking particular values for the parameters λ and μ , out of which we give only a few:

Corollary 2 With the assumptions of Lemma 1 we have:

(i) For any $x \in (a, b)$,

$$\begin{aligned} & I_{a+,g}^\alpha f(x) + I_{b-,g}^\alpha f(x) = \frac{1}{\Gamma(\alpha+1)} ([g(x) - g(a)]^\alpha + [g(b) - g(x)]^\alpha) f(x) \\ &+ \frac{1}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t) [f(t) - f(x)] dt}{[g(x) - g(t)]^{1-\alpha}} + \int_x^b \frac{g'(t) [f(t) - f(x)] dt}{[g(t) - g(x)]^{1-\alpha}} \right] \quad (10) \end{aligned}$$

and

$$I_{x-,g}^\alpha f(a) + I_{x+,g}^\alpha f(b) = \frac{1}{\Gamma(\alpha+1)} ([g(x) - g(a)]^\alpha + [g(b) - g(x)]^\alpha) f(x)$$

$$+ \frac{1}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t) [f(t) - f(x)] dt}{[g(t) - g(a)]^{1-\alpha}} + \int_x^b \frac{g'(t) [f(t) - f(x)] dt}{[g(b) - g(t)]^{1-\alpha}} \right]. \quad (11)$$

(ii) For any $x \in [a, b]$,

$$\begin{aligned} \frac{I_{b-,g}^\alpha f(a) + I_{a+,g}^\alpha f(b)}{2} &= \frac{1}{\Gamma(\alpha+1)} [g(b) - g(a)]^\alpha f(x) \\ &+ \frac{1}{2\Gamma(\alpha)} \left[\int_a^b \frac{g'(t) [f(t) - f(x)] dt}{[g(b) - g(t)]^{1-\alpha}} + \int_a^b \frac{g'(t) [f(t) - f(x)] dt}{[g(t) - g(a)]^{1-\alpha}} \right]. \end{aligned} \quad (12)$$

The proof is obvious by taking $\lambda = \mu = f(x)$ in Lemma 1. These identities were obtained in [14]. If we take in (10)-(12) $x = M_g(a, b) = g^{-1}\left(\frac{g(a)+g(b)}{2}\right)$, then we get the corresponding identities were obtained in [14].

Corollary 3 *With the assumptions of Lemma 1 we have:*

$$\begin{aligned} I_{a+,g}^\alpha f(x) + I_{b-,g}^\alpha f(x) &= \frac{1}{\Gamma(\alpha+1)} ([g(x) - g(a)]^\alpha f(a) + [g(b) - g(x)]^\alpha f(b)) \\ &+ \frac{1}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t) [f(t) - f(a)] dt}{[g(x) - g(t)]^{1-\alpha}} + \int_x^b \frac{g'(t) [f(t) - f(b)] dt}{[g(t) - g(x)]^{1-\alpha}} \right] \end{aligned} \quad (13)$$

and

$$\begin{aligned} I_{x-,g}^\alpha f(a) + I_{x+,g}^\alpha f(b) &= \frac{1}{\Gamma(\alpha+1)} ([g(x) - g(a)]^\alpha f(a) + [g(b) - g(x)]^\alpha f(b)) \\ &+ \frac{1}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t) [f(t) - f(a)] dt}{[g(t) - g(a)]^{1-\alpha}} + \int_x^b \frac{g'(t) [f(t) - f(b)] dt}{[g(b) - g(t)]^{1-\alpha}} \right], \end{aligned} \quad (14)$$

for any $x \in (a, b)$

(ii) We also have

$$\begin{aligned} \frac{I_{b-,g}^\alpha f(a) + I_{a+,g}^\alpha f(b)}{2} &= \frac{1}{\Gamma(\alpha+1)} [g(b) - g(a)]^\alpha \frac{f(b) + f(a)}{2} \\ &+ \frac{1}{2\Gamma(\alpha)} \left[\int_a^b \frac{g'(t) [f(t) - f(b)] dt}{[g(b) - g(t)]^{1-\alpha}} + \int_a^b \frac{g'(t) [f(t) - f(a)] dt}{[g(t) - g(a)]^{1-\alpha}} \right]. \end{aligned} \quad (15)$$

The proof of (13) and (14) are obvious by taking $\lambda = f(a)$, $\mu = f(b)$ in Lemma 1. The proof of (15) follows by Lemma 1 on taking $\lambda = f(b)$ and $\mu = f(a)$.

Remark 3 If we take in (13) and (14) $x = M_g(a, b) = g^{-1}\left(\frac{g(a)+g(b)}{2}\right)$, then we get

$$\begin{aligned} & I_{a+,g}^\alpha f(M_g(a, b)) + I_{b-,g}^\alpha f(M_g(a, b)) \\ &= \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)} [g(b) - g(a)]^\alpha \left(\frac{f(a) + f(b)}{2} \right) \\ &+ \frac{1}{\Gamma(\alpha)} \left[\int_a^{M_g(a,b)} \frac{g'(t) [f(t) - f(a)] dt}{[g(M_g(a, b)) - g(t)]^{1-\alpha}} + \int_{M_g(a,b)}^b \frac{g'(t) [f(t) - f(b)] dt}{[g(t) - g(M_g(a, b))]^{1-\alpha}} \right] \end{aligned}$$

and

$$\begin{aligned} & I_{M_g(a,b)-,g}^\alpha f(a) + I_{M_g(a,b)+,g}^\alpha f(b) \\ &= \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)} [g(b) - g(a)]^\alpha \left(\frac{f(a) + f(b)}{2} \right) \\ &+ \frac{1}{\Gamma(\alpha)} \left[\int_a^{M_g(a,b)} \frac{g'(t) [f(t) - f(a)] dt}{[g(t) - g(a)]^{1-\alpha}} + \int_{M_g(a,b)}^b \frac{g'(t) [f(t) - f(b)] dt}{[g(b) - g(t)]^{1-\alpha}} \right]. \end{aligned}$$

3 Inequalities for bounded functions

Now, for $\phi, \Phi \in \mathbb{C}$ and $[a, b]$ an interval of real numbers, define the sets of complex-valued functions, see for instance [15]

$$\begin{aligned} & \bar{U}_{[a,b]}(\phi, \Phi) \\ &:= \left\{ f : [a, b] \rightarrow \mathbb{C} \mid \operatorname{Re} \left[(\Phi - f(t)) \left(\overline{f(t)} - \bar{\Phi} \right) \right] \geq 0 \text{ for almost every } t \in [a, b] \right\} \end{aligned}$$

and

$$\bar{\Delta}_{[a,b]}(\phi, \Phi) := \left\{ f : [a, b] \rightarrow \mathbb{C} \mid \left| f(t) - \frac{\phi + \Phi}{2} \right| \leq \frac{1}{2} |\Phi - \phi| \text{ for a.e. } t \in [a, b] \right\}.$$

The following representation result may be stated.

Proposition 1 For any $\phi, \Phi \in \mathbb{C}$, $\phi \neq \Phi$, we have that $\bar{U}_{[a,b]}(\phi, \Phi)$ and $\bar{\Delta}_{[a,b]}(\phi, \Phi)$ are nonempty, convex and closed sets and

$$\bar{U}_{[a,b]}(\phi, \Phi) = \bar{\Delta}_{[a,b]}(\phi, \Phi). \quad (16)$$

Proof. We observe that for any $z \in \mathbb{C}$ we have the equivalence

$$\left| z - \frac{\phi + \Phi}{2} \right| \leq \frac{1}{2} |\Phi - \phi|$$

if and only if

$$\operatorname{Re}[(\Phi - z)(\bar{z} - \phi)] \geq 0.$$

This follows by the equality

$$\frac{1}{4} |\Phi - \phi|^2 - \left| z - \frac{\phi + \Phi}{2} \right|^2 = \operatorname{Re}[(\Phi - z)(\bar{z} - \phi)]$$

that holds for any $z \in \mathbb{C}$.

The equality (16) is thus a simple consequence of this fact. $\square$

On making use of the complex numbers field properties we can also state that:

Corollary 4 *For any $\phi, \Phi \in \mathbb{C}$, $\phi \neq \Phi$, we have that*

$$\begin{aligned} \bar{U}_{[a,b]}(\phi, \Phi) = \{f : [a, b] \rightarrow \mathbb{C} \mid (\operatorname{Re} \Phi - \operatorname{Re} f(t))(\operatorname{Re} f(t) - \operatorname{Re} \phi) \\ + (\operatorname{Im} \Phi - \operatorname{Im} f(t))(\operatorname{Im} f(t) - \operatorname{Im} \phi) \geq 0 \text{ for a.e. } t \in [a, b]\}. \end{aligned}$$

Now, if we assume that $\operatorname{Re}(\Phi) \geq \operatorname{Re}(\phi)$ and $\operatorname{Im}(\Phi) \geq \operatorname{Im}(\phi)$, then we can define the following set of functions as well:

$$\begin{aligned} \bar{S}_{[a,b]}(\phi, \Phi) := \{f : [a, b] \rightarrow \mathbb{C} \mid \operatorname{Re}(\Phi) \geq \operatorname{Re} f(t) \geq \operatorname{Re}(\phi) \\ \text{and } \operatorname{Im}(\Phi) \geq \operatorname{Im} f(t) \geq \operatorname{Im}(\phi) \text{ for a.e. } t \in [a, b]\}. \end{aligned}$$

One can easily observe that $\bar{S}_{[a,b]}(\phi, \Phi)$ is closed, convex and

$$\emptyset \neq \bar{S}_{[a,b]}(\phi, \Phi) \subseteq \bar{U}_{[a,b]}(\phi, \Phi).$$

We have:

Theorem 2 *Let $f : [a, b] \rightarrow \mathbb{C}$ be a complex valued Lebesgue integrable function on the real interval $[a, b]$, g be a strictly increasing function on (a, b) , having a continuous derivative g' on (a, b) and $\phi, \Phi \in \mathbb{C}$, $\phi \neq \Phi$ such that $f \in \bar{\Delta}_{[a,b]}(\phi, \Phi)$.*

(i) *For any $x \in (a, b)$,*

$$\left| I_{a+,g}^\alpha f(x) + I_{b-,g}^\alpha f(x) - \frac{\phi + \Phi}{2\Gamma(\alpha + 1)} ([g(x) - g(a)]^\alpha + [g(b) - g(x)]^\alpha) \right| \quad (17)$$

$$\leq \frac{1}{2} |\Phi - \phi| \frac{1}{\Gamma(\alpha + 1)} ([g(x) - g(a)]^\alpha + [g(b) - g(x)]^\alpha)$$

and

$$\left| I_{x-,g}^\alpha f(a) + I_{x+,g}^\alpha f(b) - \frac{\phi + \Phi}{2\Gamma(\alpha + 1)} ([g(x) - g(a)]^\alpha + [g(b) - g(x)]^\alpha) \right| \quad (18)$$

$$\leq \frac{1}{2} |\Phi - \phi| \frac{1}{\Gamma(\alpha + 1)} ([g(x) - g(a)]^\alpha + [g(b) - g(x)]^\alpha).$$

(ii) We have

$$\left| \frac{I_{b-,g}^\alpha f(a) + I_{a+,g}^\alpha f(b)}{2} - \frac{1}{\Gamma(\alpha + 1)} [g(b) - g(a)]^\alpha \frac{\phi + \Phi}{2} \right| \quad (19)$$

$$\leq \frac{1}{2} |\Phi - \phi| \frac{1}{\Gamma(\alpha + 1)} |\Phi - \phi| [g(b) - g(a)]^\alpha.$$

Proof. Using the identity (1) for $\lambda = \mu = \frac{\phi + \Phi}{2}$, we have

$$\begin{aligned} & I_{a+,g}^\alpha f(x) + I_{b-,g}^\alpha f(x) \\ & - \frac{1}{\Gamma(\alpha + 1)} ([g(x) - g(a)]^\alpha + [g(b) - g(x)]^\alpha) \frac{\phi + \Phi}{2} \\ & = \frac{1}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t) \left[f(t) - \frac{\phi + \Phi}{2} \right] dt}{[g(x) - g(t)]^{1-\alpha}} + \int_x^b \frac{g'(t) \left[f(t) - \frac{\phi + \Phi}{2} \right] dt}{[g(t) - g(x)]^{1-\alpha}} \right] \quad (20) \end{aligned}$$

for any $x \in (a, b)$.

Taking the modulus in (20), then we get

$$\begin{aligned} & \left| I_{a+,g}^\alpha f(x) + I_{b-,g}^\alpha f(x) - \frac{1}{\Gamma(\alpha + 1)} ([g(x) - g(a)]^\alpha + [g(b) - g(x)]^\alpha) \frac{\phi + \Phi}{2} \right| \\ & \leq \frac{1}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t) \left| f(t) - \frac{\phi + \Phi}{2} \right| dt}{[g(x) - g(t)]^{1-\alpha}} + \int_x^b \frac{g'(t) \left| f(t) - \frac{\phi + \Phi}{2} \right| dt}{[g(t) - g(x)]^{1-\alpha}} \right] \\ & \leq \frac{1}{2} |\Phi - \phi| \frac{1}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t) dt}{[g(x) - g(t)]^{1-\alpha}} + \int_x^b \frac{g'(t) dt}{[g(t) - g(x)]^{1-\alpha}} \right] \\ & = \frac{1}{2} |\Phi - \phi| \frac{1}{\Gamma(\alpha + 1)} ([g(x) - g(a)]^\alpha + [g(b) - g(x)]^\alpha) \end{aligned}$$

for any $x \in (a, b)$, which proves (17).

The inequality (18) follows in a similar manner from the identity (2).

The inequality (19) follows by (3) for $\lambda = \mu = \frac{\Phi + \Phi}{2}$. $\square$

Corollary 5 *With the assumptions of Theorem 2 we have*

$$\begin{aligned} & \left| I_{a+,g}^\alpha f(M_g(a,b)) + I_{b-,g}^\alpha f(M_g(a,b)) - \frac{\Phi + \Phi}{2^\alpha \Gamma(\alpha + 1)} [g(b) - g(a)]^\alpha \right| \\ & \leq \frac{1}{2^\alpha} |\Phi - \phi| \frac{1}{\Gamma(\alpha + 1)} [g(b) - g(a)]^\alpha \end{aligned}$$

and

$$\begin{aligned} & \left| I_{M_g(a,b)-,g}^\alpha f(a) + I_{M_g(a,b)+,g}^\alpha f(b) - \frac{\Phi + \Phi}{2^\alpha \Gamma(\alpha + 1)} [g(b) - g(a)]^\alpha \right| \\ & \leq \frac{1}{2^\alpha} |\Phi - \phi| \frac{1}{\Gamma(\alpha + 1)} [g(b) - g(a)]^\alpha. \end{aligned}$$

Remark 4 *If the function $f : [a, b] \rightarrow \mathbb{R}$ is measurable and there exists the constants m, M such that $m \leq f(t) \leq M$ for a.e. $t \in [a, b]$, then for any $x \in (a, b)$ we have by (17) and (18) that*

$$\begin{aligned} & \left| I_{a+,g}^\alpha f(x) + I_{b-,g}^\alpha f(x) - \frac{m + M}{2^\alpha \Gamma(\alpha + 1)} ([g(x) - g(a)]^\alpha + [g(b) - g(x)]^\alpha) \right| \\ & \leq \frac{1}{2} (M - m) \frac{1}{\Gamma(\alpha + 1)} ([g(x) - g(a)]^\alpha + [g(b) - g(x)]^\alpha) \end{aligned}$$

and

$$\begin{aligned} & \left| I_{x-,g}^\alpha f(a) + I_{x+,g}^\alpha f(b) - \frac{m + M}{2^\alpha \Gamma(\alpha + 1)} ([g(x) - g(a)]^\alpha + [g(b) - g(x)]^\alpha) \right| \\ & \leq \frac{1}{2} (M - m) \frac{1}{\Gamma(\alpha + 1)} ([g(x) - g(a)]^\alpha + [g(b) - g(x)]^\alpha). \end{aligned}$$

In particular,

$$\begin{aligned} & \left| I_{a+,g}^\alpha f(M_g(a,b)) + I_{b-,g}^\alpha f(M_g(a,b)) - \frac{m + M}{2^\alpha \Gamma(\alpha + 1)} [g(b) - g(a)]^\alpha \right| \\ & \leq \frac{1}{2^\alpha} (M - m) \frac{1}{\Gamma(\alpha + 1)} [g(b) - g(a)]^\alpha \end{aligned}$$

and

$$\begin{aligned} & \left| I_{M_g(a,b)-,g}^\alpha f(a) + I_{M_g(a,b)+,g}^\alpha f(b) - \frac{m + M}{2^\alpha \Gamma(\alpha + 1)} [g(b) - g(a)]^\alpha \right| \\ & \leq \frac{1}{2^\alpha} (M - m) \frac{1}{\Gamma(\alpha + 1)} [g(b) - g(a)]^\alpha. \end{aligned}$$

4 Trapezoid inequalities for functions of bounded variation

We have:

Theorem 3 *Let $f : [a, b] \rightarrow \mathbb{C}$ be a complex valued function of bounded variation on the real interval $[a, b]$, and g be a strictly increasing function on (a, b) , having a continuous derivative g' on (a, b) . Then we have the inequalities*

$$\begin{aligned}
 & \left| I_{a+,g}^\alpha f(x) + I_{b-,g}^\alpha f(x) - \frac{[g(x) - g(a)]^\alpha f(a) + [g(b) - g(x)]^\alpha f(b)}{\Gamma(\alpha + 1)} \right| \\
 & \leq \frac{1}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t) V_a^t(f) dt}{[g(x) - g(t)]^{1-\alpha}} + \int_x^b \frac{g'(t) V_t^b(f) dt}{[g(t) - g(x)]^{1-\alpha}} \right] \\
 & \leq \frac{1}{\Gamma(\alpha + 1)} \left[(g(x) - g(a))^\alpha \bigvee_a^x(f) + (g(b) - g(x))^\alpha \bigvee_x^b(f) \right] \\
 & \leq \frac{1}{\Gamma(\alpha + 1)} \begin{cases} \left[\frac{1}{2} (g(b) - g(a)) + \left| g(x) - \frac{g(a) + g(b)}{2} \right| \right]^\alpha V_a^b(f); \\ ((g(x) - g(a))^{\alpha p} + (g(b) - g(x))^{\alpha p})^{1/p} \left((V_a^x(f))^q + (V_x^b(f))^q \right)^{1/q} \\ \text{with } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ ((g(x) - g(a))^\alpha + (g(b) - g(x))^\alpha) \left[\frac{1}{2} V_a^b(f) + \frac{1}{2} |V_a^x(f) - V_x^b(f)| \right] \end{cases} \quad (21)
 \end{aligned}$$

and

$$\begin{aligned}
 & \left| I_{x-,g}^\alpha f(a) + I_{x+,g}^\alpha f(b) - \frac{[g(x) - g(a)]^\alpha f(a) + [g(b) - g(x)]^\alpha f(b)}{\Gamma(\alpha + 1)} \right| \\
 & \leq \frac{1}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t) V_a^t(f) dt}{[g(t) - g(a)]^{1-\alpha}} + \int_x^b \frac{g'(t) V_t^b(f) dt}{[g(b) - g(t)]^{1-\alpha}} \right] \\
 & \leq \frac{1}{\Gamma(\alpha + 1)} \left[(g(x) - g(a))^\alpha \bigvee_a^x(f) + (g(b) - g(x))^\alpha \bigvee_x^b(f) \right] \\
 & \leq \frac{1}{\Gamma(\alpha + 1)} \begin{cases} \left[\frac{1}{2} (g(b) - g(a)) + \left| g(x) - \frac{g(a) + g(b)}{2} \right| \right]^\alpha V_a^b(f); \\ ((g(x) - g(a))^{\alpha p} + (g(b) - g(x))^{\alpha p})^{1/p} \left((V_a^x(f))^q + (V_x^b(f))^q \right)^{1/q} \\ \text{with } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ ((g(x) - g(a))^\alpha + (g(b) - g(x))^\alpha) \left[\frac{1}{2} V_a^b(f) + \frac{1}{2} |V_a^x(f) - V_x^b(f)| \right] \end{cases} \quad (22)
 \end{aligned}$$

for any $x \in (a, b)$

(ii) We also have

$$\begin{aligned}
 & \left| \frac{I_{b-,g}^\alpha f(a) + I_{a+,g}^\alpha f(b)}{2} - \frac{1}{\Gamma(\alpha+1)} [g(b) - g(a)]^\alpha \frac{f(b) + f(a)}{2} \right| \\
 & \leq \frac{1}{2\Gamma(\alpha)} \left[\int_a^b \frac{g'(t) V_t^b(f) dt}{[g(b) - g(t)]^{1-\alpha}} + \int_a^b \frac{g'(t) V_a^t(f) dt}{[g(t) - g(a)]^{1-\alpha}} \right] \\
 & \leq \frac{1}{\Gamma(\alpha+1)} [g(b) - g(a)]^\alpha \bigvee_a^b(f).
 \end{aligned} \tag{23}$$

Proof. Using the identity (13) and the properties of the modulus, we have

$$\begin{aligned}
 & \left| I_{a+,g}^\alpha f(x) + I_{b-,g}^\alpha f(x) - \frac{[g(x) - g(a)]^\alpha f(a) + [g(b) - g(x)]^\alpha f(b)}{\Gamma(\alpha+1)} \right| \\
 & \leq \frac{1}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t) |f(t) - f(a)| dt}{[g(x) - g(t)]^{1-\alpha}} + \int_x^b \frac{g'(t) |f(t) - f(b)| dt}{[g(t) - g(x)]^{1-\alpha}} \right] =: B(x)
 \end{aligned}$$

for any $x \in (a, b)$.

Since f is of bounded variation on $[a, b]$, then we have

$$|f(t) - f(a)| \leq \bigvee_a^t(f) \leq \bigvee_a^x(f) \text{ for } a \leq t \leq x$$

and

$$|f(t) - f(b)| \leq \bigvee_t^b(f) \leq \bigvee_x^b(f) \text{ for } x \leq t \leq b.$$

Therefore

$$\begin{aligned}
 B(x) & \leq \frac{1}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t) V_a^t(f) dt}{[g(x) - g(t)]^{1-\alpha}} + \int_x^b \frac{g'(t) V_t^b(f) dt}{[g(t) - g(x)]^{1-\alpha}} \right] \\
 & \leq \frac{1}{\Gamma(\alpha)} \left[\bigvee_a^x(f) \int_a^x \frac{g'(t) dt}{[g(x) - g(t)]^{1-\alpha}} + \bigvee_x^b(f) \int_x^b \frac{g'(t) dt}{[g(t) - g(x)]^{1-\alpha}} \right] \\
 & = \frac{1}{\Gamma(\alpha)} \left[\frac{(g(x) - g(a))^\alpha}{\alpha} \bigvee_a^x(f) + \frac{(g(b) - g(x))^\alpha}{\alpha} \bigvee_x^b(f) \right] \\
 & = \frac{1}{\Gamma(\alpha+1)} \left[(g(x) - g(a))^\alpha \bigvee_a^x(f) + (g(b) - g(x))^\alpha \bigvee_x^b(f) \right],
 \end{aligned}$$

which proves the first two inequalities in (21).

The last part of (21) is obvious by making use of the elementary Hölder type inequalities for positive real numbers $c, d, m, n \geq 0$

$$mc + nd \leq \begin{cases} \max\{m, n\}(c + d); \\ (m^p + n^p)^{1/p} (c^q + d^q)^{1/q} \text{ with } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1. \end{cases}$$

The inequality (22) follows in a similar way by utilising the equality (14).

From the equality (15) we have

$$\begin{aligned} & \left| \frac{I_{b-,g}^\alpha f(a) + I_{a+,g}^\alpha f(b)}{2} - \frac{1}{\Gamma(\alpha+1)} [g(b) - g(a)]^\alpha \frac{f(b) + f(a)}{2} \right| \\ & \leq \frac{1}{2\Gamma(\alpha)} \left[\int_a^b \frac{g'(t) |f(t) - f(b)| dt}{[g(b) - g(t)]^{1-\alpha}} + \int_a^b \frac{g'(t) |f(t) - f(a)| dt}{[g(t) - g(a)]^{1-\alpha}} \right] \\ & \leq \frac{1}{2\Gamma(\alpha)} \left[\int_a^b \frac{g'(t) V_t^b(f) dt}{[g(b) - g(t)]^{1-\alpha}} + \int_a^b \frac{g'(t) V_a^t(f) dt}{[g(t) - g(a)]^{1-\alpha}} \right] \\ & \leq \frac{1}{2\Gamma(\alpha)} \left[\bigvee_a^b(f) \int_a^b \frac{g'(t) dt}{[g(b) - g(t)]^{1-\alpha}} + \bigvee_a^b(f) \int_a^b \frac{g'(t) dt}{[g(t) - g(a)]^{1-\alpha}} \right] \\ & = \frac{1}{2\Gamma(\alpha)} \left[\bigvee_a^b(f) \frac{[g(b) - g(a)]^\alpha}{\alpha} + \bigvee_a^b(f) \frac{[g(b) - g(a)]^\alpha}{\alpha} \right] \\ & = \frac{1}{\Gamma(\alpha+1)} [g(b) - g(a)]^\alpha \bigvee_a^b(f), \end{aligned}$$

which proves (23). □

Corollary 6 *With the assumptions of Theorem 3 we have*

$$\begin{aligned} & \left| I_{a+,g}^\alpha f(M_g(a, b)) + I_{b-,g}^\alpha f(M_g(a, b)) - \frac{f(a) + f(b)}{2^\alpha \Gamma(\alpha+1)} [g(b) - g(a)]^\alpha \right| \\ & \leq \frac{1}{\Gamma(\alpha)} \left[\int_a^{M_g(a,b)} \frac{g'(t) V_a^t(f) dt}{[g(M_g(a, b)) - g(t)]^{1-\alpha}} + \int_{M_g(a,b)}^b \frac{g'(t) V_t^b(f) dt}{[g(t) - g(M_g(a, b))]^{1-\alpha}} \right] \\ & \leq \frac{1}{2^\alpha \Gamma(\alpha+1)} (g(b) - g(a))^\alpha \bigvee_a^b(f) \end{aligned}$$

and

$$\begin{aligned}
& \left| I_{M_g(a,b)-,g}^\alpha f(a) + I_{M_g(a,b)+,g}^\alpha f(b) - \frac{f(a) + f(b)}{2^\alpha \Gamma(\alpha + 1)} [g(b) - g(a)]^\alpha \right| \\
& \leq \frac{1}{\Gamma(\alpha)} \left[\int_a^{M_g(a,b)} \frac{g'(t) \bigvee_a^t(f) dt}{[g(t) - g(a)]^{1-\alpha}} + \int_{M_g(a,b)}^b \frac{g'(t) \bigvee_t^b(f) dt}{[g(b) - g(t)]^{1-\alpha}} \right] \\
& \leq \frac{1}{2^\alpha \Gamma(\alpha + 1)} (g(b) - g(a))^\alpha \bigvee_a^b(f).
\end{aligned}$$

5 Inequalities for Hölder's continuous functions

We say that the function $f : [a, b] \rightarrow \mathbb{C}$ is r -H-Hölder continuous on $[a, b]$ with $r \in (0, 1]$ and $H > 0$ if

$$|f(t) - f(s)| \leq H |t - s|^r \quad (24)$$

for any $t, s \in [a, b]$. If $r = 1$ and $H = L$ we call the function L -Lipschitzian on $[a, b]$.

Theorem 4 Assume that $f : [a, b] \rightarrow \mathbb{C}$ is r -H-Hölder continuous on $[a, b]$ with $r \in (0, 1]$ and $H > 0$, and g be a strictly increasing function on (a, b) , having a continuous derivative g' on (a, b) . Then

$$\begin{aligned}
& \left| I_{a+,g}^\alpha f(x) + I_{b-,g}^\alpha f(x) - \frac{[g(x) - g(a)]^\alpha f(a) + [g(b) - g(x)]^\alpha f(b)}{\Gamma(\alpha + 1)} \right| \\
& \leq \frac{H}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t) (t - a)^r dt}{[g(x) - g(t)]^{1-\alpha}} + \int_x^b \frac{g'(t) (b - t)^r dt}{[g(t) - g(x)]^{1-\alpha}} \right] \\
& \leq \frac{H}{\Gamma(\alpha + 1)} [(g(x) - g(a))^\alpha (x - a)^r + (g(b) - g(x))^\alpha (b - x)^r] \\
& \leq \frac{H}{\Gamma(\alpha + 1)} \begin{cases} \left[\frac{1}{2} (g(b) - g(a)) + \left| g(x) - \frac{g(a) + g(b)}{2} \right| \right]^\alpha [(x - a)^r + (b - x)^r]; \\ ((g(x) - g(a))^{\alpha p} + (g(b) - g(x))^{\alpha p})^{1/p} ((x - a)^{rq} + (b - x)^{rq})^{1/q} \\ \text{with } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ ((g(x) - g(a))^\alpha + (g(b) - g(x))^\alpha) \left[\frac{1}{2} (b - a) + \left| x - \frac{a+b}{2} \right| \right]^r \end{cases} \quad (25)
\end{aligned}$$

and

$$\begin{aligned}
& \left| I_{x-,g}^\alpha f(a) + I_{x+,g}^\alpha f(b) - \frac{[g(x) - g(a)]^\alpha f(a) + [g(b) - g(x)]^\alpha f(b)}{\Gamma(\alpha + 1)} \right| \\
& \leq \frac{H}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t) (t-a)^r dt}{[g(t) - g(a)]^{1-\alpha}} + \int_x^b \frac{g'(t) (b-t)^r dt}{[g(b) - g(t)]^{1-\alpha}} \right] \\
& \leq \frac{H}{\Gamma(\alpha + 1)} [(g(x) - g(a))^\alpha (x-a)^r + (g(b) - g(x))^\alpha (b-x)^r] \\
& \leq \frac{H}{\Gamma(\alpha + 1)} \begin{cases} \left[\frac{1}{2} (g(b) - g(a)) + \left| g(x) - \frac{g(a)+g(b)}{2} \right| \right]^\alpha [(x-a)^r + (b-x)^r]; \\ ((g(x)-g(a))^{\alpha p} + (g(b)-g(x))^{\alpha p})^{1/p} ((x-a)^{rq} + (b-x)^{rq})^{1/q} \\ \text{with } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ ((g(x)-g(a))^{\alpha p} + (g(b)-g(x))^{\alpha p}) \left[\frac{1}{2} (b-a) + \left| x - \frac{a+b}{2} \right| \right]^r \end{cases} \quad (26)
\end{aligned}$$

for any $x \in (a, b)$

(ii) We also have

$$\begin{aligned}
& \left| \frac{I_{b-,g}^\alpha f(a) + I_{a+,g}^\alpha f(b)}{2} - \frac{1}{\Gamma(\alpha + 1)} [g(b) - g(a)]^\alpha \frac{f(b) + f(a)}{2} \right| \\
& \leq \frac{H}{2\Gamma(\alpha)} \left[\int_a^b \frac{g'(t) (b-t)^r dt}{[g(b) - g(t)]^{1-\alpha}} + \int_a^b \frac{g'(t) (t-a)^r dt}{[g(t) - g(a)]^{1-\alpha}} \right] \quad (27) \\
& \leq \frac{H}{\Gamma(\alpha + 1)} [g(b) - g(a)]^\alpha (b-a)^r.
\end{aligned}$$

Proof. Using the identity (13) and the properties of the modulus, we have

$$\begin{aligned}
& \left| I_{a+,g}^\alpha f(x) + I_{b-,g}^\alpha f(x) - \frac{[g(x) - g(a)]^\alpha f(a) + [g(b) - g(x)]^\alpha f(b)}{\Gamma(\alpha + 1)} \right| \\
& \leq \frac{1}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t) |f(t) - f(a)| dt}{[g(x) - g(t)]^{1-\alpha}} + \int_x^b \frac{g'(t) |f(t) - f(b)| dt}{[g(t) - g(x)]^{1-\alpha}} \right] =: C(x)
\end{aligned}$$

for any $x \in (a, b)$.

Since $f : [a, b] \rightarrow \mathbb{C}$ is r -H-*Hölder continuous* on $[a, b]$ with $r \in (0, 1]$ and $H > 0$, hence

$$C(x) \leq \frac{H}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t) (t-a)^r dt}{[g(x) - g(t)]^{1-\alpha}} + \int_x^b \frac{g'(t) (b-t)^r dt}{[g(t) - g(x)]^{1-\alpha}} \right]$$

$$\begin{aligned}
&\leq \frac{H}{\Gamma(\alpha)} \left[(x-a)^r \int_a^x \frac{g'(t) dt}{[g(x)-g(t)]^{1-\alpha}} + (b-x)^r \int_x^b \frac{g'(t) dt}{[g(t)-g(x)]^{1-\alpha}} \right] \\
&= \frac{H}{\Gamma(\alpha)} \left[(x-a)^r \frac{(g(x)-g(a))^\alpha}{\alpha} + (b-x)^r \frac{(g(b)-g(x))^\alpha}{\alpha} \right] \\
&= \frac{H}{\Gamma(\alpha+1)} [(x-a)^r (g(x)-g(a))^\alpha + (b-x)^r (g(b)-g(x))^\alpha],
\end{aligned}$$

for any $x \in (a, b)$, which proves the first two inequalities in (25). The rest is obvious.

The inequality (26) follows in a similar way by utilising the equality (14).

The inequality (27) follows by utilising the equality (15). $\square$

Corollary 7 *With the assumptions of Theorem 4 we have*

$$\begin{aligned}
&\left| I_{a+,g}^\alpha f(M_g(a,b)) + I_{b-,g}^\alpha f(M_g(a,b)) - \frac{f(a)+f(b)}{2^\alpha \Gamma(\alpha+1)} [g(b)-g(a)]^\alpha \right| \\
&\leq \frac{H}{\Gamma(\alpha)} \left[\int_a^{M_g(a,b)} \frac{g'(t) (t-a)^r dt}{[g(M_g(a,b))-g(t)]^{1-\alpha}} + \int_{M_g(a,b)}^b \frac{g'(t) (b-t)^r dt}{[g(t)-g(M_g(a,b))]^{1-\alpha}} \right] \\
&\leq \frac{H}{2^\alpha \Gamma(\alpha+1)} (g(b)-g(a))^\alpha [(M_g(a,b)-a)^r + (b-M_g(a,b))^r]
\end{aligned}$$

and

$$\begin{aligned}
&\left| I_{M_g(a,b)-,g}^\alpha f(a) + I_{M_g(a,b)+,g}^\alpha f(b) - \frac{f(a)+f(b)}{2^\alpha \Gamma(\alpha+1)} [g(b)-g(a)]^\alpha \right| \\
&\leq \frac{H}{\Gamma(\alpha)} \left[\int_a^{M_g(a,b)} \frac{g'(t) (t-a)^r dt}{[g(t)-g(a)]^{1-\alpha}} + \int_{M_g(a,b)}^b \frac{g'(t) (b-t)^r dt}{[g(b)-g(t)]^{1-\alpha}} \right] \\
&\leq \frac{H}{2^\alpha \Gamma(\alpha+1)} (g(b)-g(a))^\alpha [(M_g(a,b)-a)^r + (b-M_g(a,b))^r].
\end{aligned}$$

6 Applications for Hadamard fractional integrals

If we take $g(t) = \ln t$ and $0 \leq a < x \leq b$, then by Theorem 3 for Hadamard fractional integrals H_{a+}^α and H_{b-}^α we have for $f : [a, b] \rightarrow \mathbb{C}$, a function of bounded variation on $[a, b]$ that

$$\left| H_{a+}^\alpha f(x) + H_{b-}^\alpha f(x) - \frac{[\ln(\frac{x}{a})]^\alpha f(a) + [\ln(\frac{b}{x})]^\alpha f(b)}{\Gamma(\alpha+1)} \right|$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha)} \left[\int_a^x \frac{[\ln(\frac{x}{t})]^{\alpha-1} V_a^t(f) dt}{t} + \int_x^b \frac{[\ln(\frac{t}{x})]^{\alpha-1} V_t^b(f) dt}{t} \right] \\
&\leq \frac{1}{\Gamma(\alpha+1)} \left[\left[\ln\left(\frac{x}{a}\right) \right]^\alpha \bigvee_a^x(f) + \left[\ln\left(\frac{b}{x}\right) \right]^\alpha \bigvee_x^b(f) \right] \\
&\leq \frac{1}{\Gamma(\alpha+1)} \begin{cases} \left[\frac{1}{2} \ln\left(\frac{b}{a}\right) + \left| \ln\left(\frac{x}{G(a,b)}\right) \right| \right]^\alpha V_a^b(f); \\ \left((\ln(\frac{x}{a}))^{\alpha p} + (\ln(\frac{b}{x}))^{\alpha p} \right)^{1/p} \left((V_a^x(f))^q + (V_x^b(f))^q \right)^{1/q} \\ \text{with } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \left((\ln(\frac{x}{a}))^\alpha + (\ln(\frac{b}{x}))^\alpha \right) \left[\frac{1}{2} V_a^b(f) + \frac{1}{2} |V_a^x(f) - V_x^b(f)| \right] \end{cases} \quad (28)
\end{aligned}$$

and

$$\begin{aligned}
&\left| H_{x-}^\alpha f(a) + H_{x+}^\alpha f(b) - \frac{[\ln(\frac{x}{a})]^\alpha f(a) + [\ln(\frac{b}{x})]^\alpha f(b)}{\Gamma(\alpha+1)} \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \left[\int_a^x \frac{[\ln(\frac{t}{a})]^{\alpha-1} V_a^t(f) dt}{t} + \int_x^b \frac{[\ln(\frac{b}{t})]^{\alpha-1} V_t^b(f) dt}{t} \right] \\
&\leq \frac{1}{\Gamma(\alpha+1)} \left[\left(\ln\left(\frac{x}{a}\right) \right)^\alpha \bigvee_a^x(f) + \left(\ln\left(\frac{b}{x}\right) \right)^\alpha \bigvee_x^b(f) \right] \\
&\leq \frac{1}{\Gamma(\alpha+1)} \begin{cases} \left[\frac{1}{2} \ln\left(\frac{b}{a}\right) + \left| \ln\left(\frac{x}{G(a,b)}\right) \right| \right]^\alpha V_a^b(f); \\ \left((\ln(\frac{x}{a}))^{\alpha p} + (\ln(\frac{b}{x}))^{\alpha p} \right)^{1/p} \left((V_a^x(f))^q + (V_x^b(f))^q \right)^{1/q} \\ \text{with } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \left((\ln(\frac{x}{a}))^\alpha + (\ln(\frac{b}{x}))^\alpha \right) \left[\frac{1}{2} V_a^b(f) + \frac{1}{2} |V_a^x(f) - V_x^b(f)| \right] \end{cases} \quad (29)
\end{aligned}$$

for any $x \in (a, b)$

We also have

$$\begin{aligned}
&\left| \frac{H_{b-}^\alpha f(a) + H_{a+}^\alpha f(b)}{2} - \frac{1}{\Gamma(\alpha+1)} \left[\ln\left(\frac{b}{a}\right) \right]^\alpha \frac{f(b) + f(a)}{2} \right| \\
&\leq \frac{1}{2\Gamma(\alpha)} \left[\int_a^b \frac{[\ln(\frac{b}{t})]^{\alpha-1} V_t^b(f) dt}{t} + \int_a^b \frac{[\ln(\frac{t}{a})]^{\alpha-1} g'(t) V_a^t(f) dt}{t} \right] \\
&\leq \frac{1}{\Gamma(\alpha+1)} \left[\ln\left(\frac{b}{a}\right) \right]^\alpha \bigvee_a^b(f).
\end{aligned}$$

If we take in (28) and (29) $x = G(a, b)$, then we get

$$\begin{aligned}
 & \left| H_{a+}^{\alpha} f(G(a, b)) + H_{b-}^{\alpha} f(G(a, b)) - \frac{f(a) + f(b)}{2^{\alpha} \Gamma(\alpha + 1)} \left[\ln \left(\frac{b}{a} \right) \right]^{\alpha} \right| \\
 & \leq \frac{1}{\Gamma(\alpha)} \left[\int_a^{G(a, b)} \frac{\left[\ln \left(\frac{G(a, b)}{t} \right) \right]^{\alpha-1} V_a^t(f) dt}{t} + \int_{G(a, b)}^b \frac{\left[\ln \left(\frac{t}{G(a, b)} \right) \right]^{\alpha-1} V_t^b(f) dt}{t} \right] \\
 & \leq \frac{1}{2^{\alpha} \Gamma(\alpha + 1)} \left[\ln \left(\frac{b}{a} \right) \right]^{\alpha} V_a^b(f)
 \end{aligned}$$

and

$$\begin{aligned}
 & \left| H_{G(a, b)-}^{\alpha} f(a) + H_{G(a, b)+}^{\alpha} f(b) - \frac{f(a) + f(b)}{2^{\alpha} \Gamma(\alpha + 1)} \left[\ln \left(\frac{b}{a} \right) \right]^{\alpha} \right| \\
 & \leq \frac{1}{\Gamma(\alpha)} \left[\int_a^{G(a, b)} \frac{\left[\ln \left(\frac{t}{a} \right) \right]^{\alpha-1} V_a^t(f) dt}{t} + \int_{G(a, b)}^b \frac{\left[\ln \left(\frac{b}{t} \right) \right]^{\alpha-1} V_t^b(f) dt}{t} \right] \\
 & \leq \frac{1}{2^{\alpha} \Gamma(\alpha + 1)} \left[\ln \left(\frac{b}{a} \right) \right]^{\alpha} V_a^b(f).
 \end{aligned}$$

Assume that $f : [a, b] \rightarrow \mathbb{C}$ is r -H-Hölder continuous on $[a, b]$ with $r \in (0, 1]$ and $H > 0$. If we take $g(t) = \ln t$ and $0 \leq a < x \leq b$ in Theorem 4, then we get

$$\begin{aligned}
 & \left| H_{a+}^{\alpha} f(x) + H_{b-}^{\alpha} f(x) - \frac{\left[\ln \left(\frac{x}{a} \right) \right]^{\alpha} f(a) + \left[\ln \left(\frac{b}{x} \right) \right]^{\alpha} f(b)}{\Gamma(\alpha + 1)} \right| \\
 & \leq \frac{H}{\Gamma(\alpha)} \left[\int_a^x \frac{\left[\ln \left(\frac{x}{t} \right) \right]^{\alpha-1} (t-a)^r dt}{t} + \int_x^b \frac{\left[\ln \left(\frac{t}{x} \right) \right]^{\alpha-1} (b-t)^r dt}{t} \right] \\
 & \leq \frac{H}{\Gamma(\alpha + 1)} \left[\left[\ln \left(\frac{x}{a} \right) \right]^{\alpha} (x-a)^r + \left[\ln \left(\frac{b}{x} \right) \right]^{\alpha} (b-x)^r \right] \\
 & \leq \frac{H}{\Gamma(\alpha + 1)} \begin{cases} \left[\frac{1}{2} \ln \left(\frac{b}{a} \right) + \left| \ln \left(\frac{x}{G(a, b)} \right) \right| \right]^{\alpha} V_a^b(f); \\ \left(\left(\ln \left(\frac{x}{a} \right) \right)^{\alpha p} + \left(\ln \left(\frac{b}{x} \right) \right)^{\alpha p} \right)^{1/p} \left((V_a^x(f))^q + (V_x^b(f))^q \right)^{1/q} \\ \text{with } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \left(\left(\ln \left(\frac{x}{a} \right) \right)^{\alpha} + \left(\ln \left(\frac{b}{x} \right) \right)^{\alpha} \right) \left[\frac{1}{2} V_a^b(f) + \frac{1}{2} \left| V_a^x(f) - V_x^b(f) \right| \right] \end{cases}
 \end{aligned}$$

and

$$\begin{aligned}
& \left| H_{x-}^{\alpha} f(a) + H_{x+}^{\alpha} f(b) - \frac{[\ln(\frac{x}{a})]^{\alpha} f(a) + [\ln(\frac{b}{x})]^{\alpha} f(b)}{\Gamma(\alpha+1)} \right| \\
& \leq \frac{H}{\Gamma(\alpha)} \left[\int_a^x \frac{[\ln(\frac{t}{a})]^{\alpha-1} (t-a)^r dt}{t} + \int_x^b \frac{[\ln(\frac{b}{t})]^{\alpha-1} (b-t)^r dt}{t} \right] \\
& \leq \frac{H}{\Gamma(\alpha+1)} \left[\left[\ln\left(\frac{x}{a}\right) \right]^{\alpha} (x-a)^r + \left[\ln\left(\frac{b}{x}\right) \right]^{\alpha} (b-x)^r \right] \\
& \leq \frac{H}{\Gamma(\alpha+1)} \begin{cases} \left[\frac{1}{2} \ln\left(\frac{b}{a}\right) + \left| \ln\left(\frac{x}{G(a,b)}\right) \right| \right]^{\alpha} V_a^b(f); \\ \left((\ln(\frac{x}{a}))^{\alpha p} + (\ln(\frac{b}{x}))^{\alpha p} \right)^{1/p} \left((V_a^x(f))^q + (V_x^b(f))^q \right)^{1/q} \\ \text{with } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \left((\ln(\frac{x}{a}))^{\alpha} + (\ln(\frac{b}{x}))^{\alpha} \right) \left[\frac{1}{2} V_a^b(f) + \frac{1}{2} |V_a^x(f) - V_x^b(f)| \right] \end{cases} \quad (30)
\end{aligned}$$

for any $x \in (a, b)$.

We also have

$$\begin{aligned}
& \left| \frac{H_{b-}^{\alpha} f(a) + H_{a+}^{\alpha} f(b)}{2} - \frac{1}{\Gamma(\alpha+1)} \left[\ln\left(\frac{b}{a}\right) \right]^{\alpha} \frac{f(b) + f(a)}{2} \right| \\
& \leq \frac{H}{2\Gamma(\alpha)} \left[\int_a^b \frac{[\ln(\frac{b}{t})]^{\alpha-1} (b-t)^r dt}{t} + \int_a^b \frac{[\ln(\frac{t}{a})]^{\alpha-1} (t-a)^r dt}{t} \right] \quad (31) \\
& \leq \frac{H}{\Gamma(\alpha+1)} (b-a)^r \left[\ln\left(\frac{b}{a}\right) \right]^{\alpha}.
\end{aligned}$$

If we take in (30) and (31) $x = G(a, b)$, then we get

$$\begin{aligned}
& \left| H_{a+}^{\alpha} f(G(a, b)) + H_{b-}^{\alpha} f(G(a, b)) - \frac{f(a) + f(b)}{2^{\alpha} \Gamma(\alpha+1)} \left[\ln\left(\frac{b}{a}\right) \right]^{\alpha} \right| \\
& \leq \frac{H}{\Gamma(\alpha)} \left[\int_a^{G(a,b)} \frac{[\ln(\frac{G(a,b)}{t})]^{\alpha-1} (t-a)^r dt}{t} + \int_{G(a,b)}^b \frac{[\ln(\frac{b}{G(a,b)})]^{\alpha-1} (b-t)^r dt}{t} \right] \\
& \leq \frac{1}{2^{\alpha} \Gamma(\alpha+1)} \left[\ln\left(\frac{b}{a}\right) \right]^{\alpha} (b-a)^r
\end{aligned}$$

and

$$\left| H_{G(a,b)-}^{\alpha} f(a) + H_{G(a,b)+}^{\alpha} f(b) - \frac{f(a) + f(b)}{2^{\alpha} \Gamma(\alpha+1)} \left[\ln\left(\frac{b}{a}\right) \right]^{\alpha} \right|$$

$$\begin{aligned} &\leq \frac{1}{\Gamma(\alpha)} \left[\int_a^{G(a,b)} \frac{[\ln(\frac{t}{a})]^{\alpha-1} (t-a)^r dt}{t} + \int_{G(a,b)}^b \frac{[\ln(\frac{b}{t})]^{\alpha-1} (b-t)^r dt}{t} \right] \\ &\leq \frac{1}{2^\alpha \Gamma(\alpha+1)} \left[\ln\left(\frac{b}{a}\right) \right]^\alpha (b-a)^r. \end{aligned}$$

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