On certain subclasses of analytic functions associated with Poisson distribution series

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Abstract. In this paper, we find the necessary and sufficient conditions, inclusion relations for Poisson distribution series $K(m,z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n$ to be in the subclasses $S(k,\lambda)$ and $C(k,\lambda)$ of analytic functions with negative coefficients. Further, we obtain necessary and sufficient conditions for the integral operator $G(m,z) = \int_0^z \frac{F(m,t)}{t} dt$ to be in the above classes.

1 Introduction and definitions

Let $\mathcal{A}$ denote the class of the normalized functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. Further, let $\mathcal{T}$ be a subclass of $\mathcal{A}$ consisting of functions of the form,

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad z \in \mathcal{U}.$$
A function $f$ of the form (2) is in $S(k, \lambda)$ if it satisfies the condition

$$\left| \frac{zf'(z)}{(1-\lambda)f(z) + \lambda zf'(z)} - 1 \right| < k, \quad (0 < k \leq 1, \ 0 \leq \lambda < 1, \ z \in U)$$

and $f \in C(k, \lambda)$ if and only if $zf' \in S(k, \lambda)$. The class $S(k, \lambda)$ was introduced by Frasin et al. [3].

We note that $S(k, 0) = S(k)$ and $C(k, 0) = C(k)$, where the classes $S(k)$ and $C(k)$ were introduced and studied by Padmanabhan [9] (see also, [5], [8]).

A function $f \in A$ is said to be in the class $R^\tau(A, B), \tau \in \mathbb{C}\{0\}, -1 \leq B < A \leq 1$, if it satisfies the inequality

$$\left| \frac{f'(z) - 1}{(A - B)\tau - B[f'(z) - 1]} \right| < 1, \quad z \in U.$$

This class was introduced by Dixit and Pal [2].

A variable $x$ is said to be Poisson distributed if it takes the values $0, 1, 2, 3, \ldots$ with probabilities $e^{-m}, \ m e^{-m}, \ m^2 e^{-m}, \ m^3 e^{-m}, \ldots$ respectively, where $m$ is called the parameter. Thus

$$P(x = r) = \frac{m^r e^{-m}}{r!}, \ r = 0, 1, 2, 3, \ldots.
$$

Very recently, Porwal [10] (see also, [6, 7]) introduce a power series whose coefficients are probabilities of Poisson distribution

$$K(m, z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n, \quad z \in U,$$

where $m > 0$. By ratio test the radius of convergence of above series is infinity. In [10], Porwal also defined the series

$$F(m, z) = 2z - K(m, z) = z - \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n, \quad z \in U.$$

Using the Hadamard product, Porwal and Kumar [12] introduced a new linear operator $I(m, z) : A \rightarrow A$ defined by

$$I(m, z) f = K(m, z) * f(z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} a_n z^n, \quad z \in U,$$
where * denote the convolution or Hadamard product of two series.

Motivated by several earlier results on connections between various subclasses of analytic and univalent functions by using hypergeometric functions (see [1, 4, 13, 14]) and by the recent investigations of Porwal ([10, 12, 11]), in the present paper we determine the necessary and sufficient conditions for $F(m,z)$ to be in our new classes $S(k,\lambda)$ and $C(k,\lambda)$ and connections of these subclasses with $R^\tau(A,B)$. Finally, we give conditions for the integral operator $G(m,z) = \int_0^z \frac{F(m,t)}{t} \, dt$ to be in the classes $S(k,\lambda)$ and $C(k,\lambda)$.

To establish our main results, we will require the following Lemmas.

**Lemma 1** [3] A function $f$ of the form (2) is in $S(k,\lambda)$ if and only if it satisfies

$$\sum_{n=2}^\infty \left| n [(1 - \lambda) + k(1 + \lambda)] - (1 - \lambda)(1 - k) \right| a_n \leq 2k$$

(3)

where $0 < k \leq 1$ and $0 \leq \lambda < 1$. The result is sharp.

**Lemma 2** [3] A function $f$ of the form (2) is in $C(k,\lambda)$ if and only if it satisfies

$$\sum_{n=2}^\infty n [n [(1 - \lambda) + k(1 + \lambda)] - (1 - \lambda)(1 - k)] a_n \leq 2k$$

(4)

where $0 < k \leq 1$ and $0 \leq \lambda < 1$. The result is sharp.

**Lemma 3** [2] If $f \in R^\tau(A,B)$ is of the form, then

$$|a_n| \leq (A - B) \frac{|\tau|}{n}, \quad n \in \mathbb{N} - \{1\}.$$

The result is sharp.

2 The necessary and sufficient conditions

**Theorem 1** If $m > 0$, $0 < k \leq 1$ and $0 \leq \lambda < 1$, then $F(m,z)$ is in $S(k,\lambda)$ if and only if

$$((1 - \lambda) + k(1 + \lambda))me^m \leq 2k.$$  

(5)

**Proof.** Since

$$F(m,z) = z - \sum_{n=2}^\infty \frac{m^{n-1}}{(n-1)!} e^{-m} z^n$$

(6)
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according to (3) of Lemma 1, we must show that
\[
\sum_{n=2}^{\infty} \left[ n\left(1 - \lambda + k(1 + \lambda)\right) + (1 - \lambda)(k - 1) \right] \frac{m^{n-1}}{(n-1)!} \leq 2ke^m. \tag{7}
\]

Writing \(n = (n - 1) + 1\), we have
\[
\sum_{n=2}^{\infty} \left[ n\left(1 - \lambda + k(1 + \lambda)\right) + (1 - \lambda)(k - 1) \right] \frac{m^{n-1}}{(n-1)!} \leq 2ke^m \tag{8}
\]
\[
= \sum_{n=2}^{\infty} \left[ (n - 1)\left(1 - \lambda + k(1 + \lambda)\right) + 2k \right] \frac{m^{n-1}}{(n-1)!} + 2k \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!}
\]
\[
= \left( (1 - \lambda) + k(1 + \lambda) \right)m^2e^m + 2(1 + 2k + k\lambda - \lambda)me^m \leq 2k. \tag{9}
\]

But this last expression is bounded above by \(2ke^m\) if and only if (5) holds. □

**Theorem 2** If \(m > 0\), \(0 < k \leq 1\) and \(0 \leq \lambda < 1\), then \(F(m, z)\) is in \(C(k, \lambda)\) if and only if
\[
((1 - \lambda) + k(1 + \lambda))m^2e^m + 2(1 + 2k + k\lambda - \lambda)me^m \leq 2k. \tag{9}
\]

**Proof.** In view of Lemma 2, it suffices to show that
\[
\sum_{n=2}^{\infty} n\left[ n\left(1 - \lambda + k(1 + \lambda)\right) + (1 - \lambda)(k - 1) \right] \frac{m^{n-1}}{(n-1)!} \leq 2ke^m.
\]

Now
\[
\sum_{n=2}^{\infty} \left[ n\left(1 - \lambda + k(1 + \lambda)\right) + (1 - \lambda)(k - 1) \right] \frac{m^{n-1}}{(n-1)!} \leq 2ke^m \tag{10}
\]
\[
= \sum_{n=2}^{\infty} n^2((1 - \lambda) + k(1 + \lambda)) + n(1 - \lambda)(k - 1) \frac{m^{n-1}}{(n-1)!}.
\]

Writing \(n = (n - 1) + 1\) and \(n^2 = (n - 1)(n - 2) + 3(n - 1) + 1\, in \,(10)\) we see that
\[
\sum_{n=2}^{\infty} n^2((1 - \lambda) + k(1 + \lambda)) + n(1 - \lambda)(k - 1) \frac{m^{n-1}}{(n-1)!}.
\]

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\[ \sum_{n=2}^{\infty} (n-1)(n-2)((1-\lambda)+k(1+\lambda)) \frac{m^{n-1}}{(n-1)!} \]
\[ + \sum_{n=2}^{\infty} (n-1)[3((1-\lambda)+k(1+\lambda)+(1-\lambda)(k-1)] \frac{m^{n-1}}{(n-1)!} + \sum_{n=2}^{\infty} 2k \frac{m^{n-1}}{(n-1)!} \]
\[ = ((1-\lambda)+k(1+\lambda)) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-3)!} + 2(1+2k+k\lambda-\lambda) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-2)!} \]
\[ + 2k \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} \]
\[ = ((1-\lambda)+k(1+\lambda))m^2e^m + 2(1+2k+k\lambda-\lambda)m^2e^m + 2k(e^m-1). \]

But this last expression is bounded above by \(2ke^m\) if and only if (9) holds. □

By specializing the parameter \(\lambda = 0\) in Theorems 1 and 2, we have the following corollaries.

**Corollary 1** If \(m > 0\) and \(0 < k \leq 1\), then \(\mathcal{F}(m, z)\) is in \(S(k)\) if and only if
\[(1+k)me^m \leq 2k. \tag{11}\]

**Corollary 2** If \(m > 0\) and \(0 < k \leq 1\), then \(\mathcal{F}(m, z)\) is in \(C(k)\) if and only if
\[(1+k)m^2e^m + 2(1+2k)m^2e^m \leq 2k. \tag{12}\]

### 3 Inclusion properties

**Theorem 3** Let \(m > 0\), \(0 < k \leq 1\) and \(0 \leq \lambda < 1\). If \(f \in \mathcal{R}^\tau(A, B)\), then \(I(m, z)f\) is in \(S[k, \lambda]\) if and only if
\[
(A-B)|\tau| \left[ ((1-\lambda)+k(1+\lambda))(1-e^{-m}) \right.
\]
\[
+ \left. \frac{(1-\lambda)(k-1)}{m} (1-e^{-m}(1+m)) \right] \leq 2k. \tag{13}\]

**Proof.** In view of Lemma 1, it suffices to show that
\[
\sum_{n=2}^{\infty} n((1-\lambda)+k(1+\lambda)+(1-\lambda)(k-1)] \frac{m^{n-1}}{(n-1)!} |a_n| \leq 2ke^m. \]

Since \(f \in \mathcal{R}^\tau(A, B)\), then by Lemma 3, we get
\[
|a_n| \leq \frac{(A-B)|\tau|}{n}. \tag{14}\]
Thus, we have
\[ \sum_{n=2}^{\infty} \left[ n((1 - \lambda) + k(1 + \lambda)) + (1 - \lambda)(k - 1) \right] \frac{m^{n-1}}{(n-1)!} |a_n| \]
\[ \leq (A - B) |\tau| \sum_{n=2}^{\infty} \left[ n((1 - \lambda) + k(1 + \lambda)) + (1 - \lambda)(k - 1) \right] \frac{m^{n-1}}{n!} \]
\[ = (A - B) |\tau| \left[ ((1 - \lambda) + k(1 + \lambda)) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} + (1 - \lambda)(k - 1) \sum_{n=2}^{\infty} \frac{m^n}{n!} \right] \]
\[ = (A - B) |\tau| \left[ ((1 - \lambda) + k(1 + \lambda))(e^m - 1) + (1 - \lambda)(k - 1)\frac{e^m - 1 - m}{m} \right]. \]

But this last expression is bounded above by \(2ke^m\) if and only if (13) holds. □

**Theorem 4** Let \( m > 0, 0 < k \leq 1 \) and \( 0 \leq \lambda < 1 \). If \( f \in \mathcal{R}^*(A, B) \), then \( \mathcal{F}(m, z)f \) is in \( \mathcal{C}(k, \lambda) \) if and only if
\[ (A - B) |\tau| \left[ ((1 - \lambda) + k(1 + \lambda))m + 2k(1 - e^{-m}) \right] \leq 2k. \] (15)

**Proof.** In view of Lemma 2, it suffices to show that
\[ \sum_{n=2}^{\infty} n\left[ n((1 - \lambda) + k(1 + \lambda)) + (1 - \lambda)(k - 1) \right] \frac{m^{n-1}}{(n-1)!} |a_n| \leq 2ke^m. \]

Using (14), we have
\[ \sum_{n=2}^{\infty} n\left[ n((1 - \lambda) + k(1 + \lambda)) + (1 - \lambda)(k - 1) \right] \frac{m^{n-1}}{(n-1)!} |a_n| \]
\[ \leq \sum_{n=2}^{\infty} n\left[ n((1 - \lambda) + k(1 + \lambda)) + (1 - \lambda)(k - 1) \right] \frac{m^{n-1}}{(n-1)!} \frac{(A - B) |\tau|}{n} \]
\[ = (A - B) |\tau| \sum_{n=2}^{\infty} \left[ n((1 - \lambda) + k(1 + \lambda)) + (1 - \lambda)(k - 1) \right] \frac{m^{n-1}}{(n-1)!} \]
\[ = (A - B) |\tau| \sum_{n=2}^{\infty} \left[ (n - 1)((1 - \lambda) + k(1 + \lambda)) + 2k \right] \frac{m^{n-1}}{(n-1)!} \]
\[ = (A - B) |\tau| \left[ ((1 - \lambda) + k(1 + \lambda)) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-2)!} + 2k \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} \right]\]
\[ (A - B) |\tau| \left[ ((1 - \lambda) + k(1 + \lambda)) m e^m + 2k(e^m - 1) \right]. \]

But this last expression is bounded above by \(2ke^m\) if and only if (15) holds. \(\Box\)

By taking \(\lambda = 0\) in Theorems 3 and 4, we obtain the following corollaries.

**Corollary 3** Let \(m > 0\) and \(0 < k \leq 1\). If \(f \in \mathcal{R}^\tau(A, B)\), then \(I(m, z)f\) is in \(S(k)\) if and only if
\[ (A - B) |\tau| [(1 + k)m + 2k(1 - e^{-m})] \leq 2k. \quad (16) \]

**Corollary 4** Let \(m > 0\) and \(0 < k \leq 1\). If \(f \in \mathcal{R}^\tau(A, B)\), then \(I(m, z)f\) is in \(C(k)\) if and only if
\[ (A - B) |\tau| [(1 + k)m + 2k(1 - e^{-m})] \leq 2k. \quad (17) \]

### 4 An integral operator

In this section, we obtain the necessary and sufficient conditions for the integral operator \(G(m, z)\) defined by
\[ G(m, z) = \int_0^z \frac{F(m, t)}{t} \, dt \quad (18) \]

to be in the class \(C(k, \lambda)\).

**Theorem 5** If \(m > 0\), \(0 < k \leq 1\) and \(0 \leq \lambda < 1\), then the integral operator \(G(m, z)\) defined by (18) is in \(C(k, \lambda)\) if and only if (5) is satisfied.

**Proof.** Since
\[ G(m, z) = z - \sum_{n=2}^{\infty} \frac{e^{-m}m^{n-1}}{n!} z^n \]

then by Lemma 2, we need only to show that
\[ \sum_{n=2}^{\infty} n[(1 - \lambda) + k(1 + \lambda)](1 - \lambda)(k - 1) \frac{m^{n-1}}{n!} \leq 2ke^m. \]

or, equivalently
\[ \sum_{n=2}^{\infty} \frac{n[(1 - \lambda) + k(1 + \lambda)](1 - \lambda)(k - 1)m^{n-1}}{(n - 1)!} \leq 2ke^m. \]
From (8) it follows that
\[
\sum_{n=2}^{\infty} \left( n((1-\lambda) + k(1+\lambda)) + (1-\lambda)(k-1) \right) \frac{m^{n-1}}{(n-1)!} = ((1-\lambda) + k(1+\lambda))me^m + 2k(e^m - 1)
\]
and this last expression is bounded above by \(2ke^m\) if and only if (5) holds. □

The proof of Theorem 6 (below) is much similar to that of Theorem 5 and so the details are omitted.

**Theorem 6** If \(m > 0\), \(0 < k \leq 1\) and \(0 \leq \lambda < 1\), then the integral operator \(G(m, z)\) defined by (18) is in \(S(k, \lambda)\) if and only if
\[
((1-\lambda) + k(1+\lambda))(1-e^{-m}) + \frac{(1-\lambda)(k-1)}{m}(1-e^{-m} - me^{-m}) \leq 2k.
\]

By taking \(\lambda = 0\) in Theorems 5 and 6, we obtain the following corollaries.

**Corollary 5** If \(m > 0\) and \(0 < k \leq 1\), then the integral operator defined by (18) is in \(C(k)\) if and only if (11) is satisfied.

**Corollary 6** If \(m > 0\) and \(0 < k \leq 1\), then the integral operator defined by (18) is in \(S(k)\) if and only if
\[
(1+k)(1-e^{-m}) + \frac{(k-1)}{m}(1-e^{-m} - me^{-m}) \leq 2k.
\]

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**References**


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