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Initial coefficient bounds for certain class of meromorphic bi-univalent functions

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Abstract. In this paper, we introduce and investigate an interesting subclass of meromorphic bi-univalent functions defined on $\Delta = \{z \in \mathbb{C} : 1 < |z| < \infty\}$. For functions belonging to this class, estimates on the initial coefficients are obtained. The results presented in this paper generalize and improve some recent works.

1 Introduction

Let Σ be the family of meromorphic functions f of the form

$$f(z) = z + b_0 + \sum_{n=1}^{\infty} b_n \frac{1}{z^n}, \tag{1}$$

that are univalent in $\Delta = \{z \in \mathbb{C} : 1 < |z| < \infty\}$. Since $f \in \Sigma$ is univalent, it has an inverse f^{-1} that satisfy

$$f^{-1}(f(z)) = z \qquad (z \in \Delta)$$

and

$$f(f^{-1}(w)) = w \qquad (M < |w| < \infty,\ M > 0).$$

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Furthermore, the inverse function f^{-1} has a series expansion of the form

$$f^{-1}(w) = w + \sum_{n=0}^{\infty} B_n \frac{1}{w^n}, \tag{2}$$

where $M < |w| < \infty$. A simple calculation shows that the function f^{-1}, is given by

$$f^{-1}(w) = w - b_0 - \frac{b_1}{w} - \frac{b_2 + b_0 b_1}{w^2} - \frac{b_3 + 2b_0 b_1 + b_0^2 b_1 + b_1^2}{w^3} + \dots . \tag{3}$$

A function $f \in \Sigma$ is said to be meromorphic bi-univalent if $f^{-1} \in \Sigma$. The family of all meromorphic bi-univalent functions is denoted by $\Sigma_{\mathfrak{B}}$.

Estimates on the coefficient of meromorphic univalent functions were widely investigated in the literature; for example, Schiffer [8] obtained the estimate $|b_2| \leq 2/3$ for meromorphic univalent functions $f \in \Sigma$ with $b_0 = 0$ and Duren [2] proved that $|b_n| \leq 2/(n+1)$ for $f \in \Sigma$ with $b_k = 0$, $1 \leq k \leq n/2$.

For the coefficients of inverses of meromorphic univalent functions, Springer [10] proved that

$$|B_3| \leq 1 \quad \text{and} \quad |B_3 + \frac{1}{2}B_1^2| \leq \frac{1}{2}$$

and conjectured that

$$|B_{2n-1}| \leq \frac{(2n-2)!}{n!(n-1)!} \qquad (n = 1, 2, \cdots).$$

In 1977, Kubota [6] proved that the Springer conjecture is true for $n = 3, 4, 5$ and subsequently Schober [9] obtained a sharp bounds for the coefficients B_{2n-1}, $1 \leq n \leq 7$.

A function f in the class $\Sigma_{\mathfrak{B}}$ is said to be memorphic bi-univalent starlike of order β where $0 \leq \beta < 1$, if it satisfies the flowing inequalities

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta \quad \text{and} \quad \mathrm{Re}\left(\frac{wg'(w)}{g(w)}\right) > \beta \quad (z, w \in \Delta),$$

where g is the inverse of f given by (3). We denote by $\Sigma^*_{\mathfrak{B}}(\beta)$ the class of all meromorphic bi-univalent starlike functions of order β. Similarly, a function f in the class $\Sigma_{\mathfrak{B}}$ is said to be meromorphic bi-univalent strongly starlike of order α where $0 < \alpha \leq 1$, if it satisfies the following conditions

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \quad \text{and} \quad \left|\arg\left(\frac{wg'(w)}{g(w)}\right) \arg\right| < \frac{\alpha\pi}{2} \quad (z, w \in \Delta),$$

where g is the inverse of f given by (3). We denote by $\widetilde{\Sigma}^*_{\mathfrak{B}}(\alpha)$ the class of all meromorphic bi-univalent strongly starlike functions of order α. The classes $\Sigma^*_{\mathfrak{B}}(\beta)$ and $\widetilde{\Sigma}^*_{\mathfrak{B}}(\alpha)$ were introduced and studied by Halim et al. [3].

Several researchers introduced and investigated some subclasses of meromorphically bi-univalent functions. (see, for detailes [3], [4], [5], [6], [9] and [13]).

Recently, Srivastava at al. [11] introduced the following subclasses of the meromorphic bi-univalent function and obtained non sharp estimates on the initial coefficients $|b_0|$ and $|b_1|$ as follow.

Definition 1 [11, Definition 2] *A function* $f(z) \in \Sigma_{\mathfrak{B}}$ *given by (1) is said to be in the class* $\Sigma_{B,\lambda^*}(\alpha)$, *if the following conditions are satisfied:*

$$\left|\arg\left(\frac{z[f'(z)]^\lambda}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \qquad (0 < \alpha \leq 1,\ \lambda \geq 1,\ z \in \Delta)$$

and

$$\left|\arg\left(\frac{w[g'(w)]^\lambda}{g(w)}\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1,\ \lambda \geq 1,\ w \in \Delta),$$

where the function g *is the inverse of* f *given by* (3).

Theorem 1 [11, Theorem 2.1] *Let* $f(z) \in \Sigma_{\mathfrak{B}}$ *given by* (1) *be in the class* $\Sigma_{B,\lambda^*}(\alpha)$. *Then*

$$|b_0| \leq 2\alpha, \qquad |b_1| \leq \frac{2\sqrt{5}\alpha^2}{1+\lambda}.$$

Definition 2 [11, Definition 3] *A function* $f(z) \in \Sigma_{\mathfrak{B}}$ *given by* (1) *is said to be in the class* $\Sigma_{B^*}(\lambda, \beta)$, *if the following conditions are satisfied:*

$$\mathrm{Re}\left(\frac{z[f'(z)]^\lambda}{f(z)}\right) > \beta \quad (0 \leq \beta < 1,\ \lambda \geq 1,\ z \in \Delta)$$

and

$$\mathrm{Re}\left(\frac{w[g'(w)]^\lambda}{g(w)}\right) > \beta \quad (0 \leq \beta < 1,\ \lambda \geq 1,\ w \in \Delta),$$

where the function g *is the inverse of* f *given by* (3).

Theorem 2 [11, Theorem 3.1] *Let* $f(z)$ *given by* (1) *be in the class* $\Sigma_{B^*}(\lambda,\beta)$*. Then*

$$|b_0| \leq 2(1-\beta), \qquad |b_1| \leq \frac{2(1-\beta)\sqrt{4\beta^2-8\beta+5}}{1+\lambda}.$$

The following subclass of the meromorphic bi-univalent functions was investigated by Hai-Gen Xiao and Qing-Hua Xu [12].

Definition 3 [12, Definition 3] *A function* $f(z) \in \Sigma_{\mathfrak{B}}$ *given by* (1) *is said to be in the class* $\Sigma^*_{\vartheta}(\mu,\alpha)$*, if the following conditions are satisfied:*

$$\left|\arg\left\{(1-\mu)\frac{zf'(z)}{f(z)} + \mu\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}\right| < \frac{\alpha\pi}{2} \quad (0<\alpha\leq 1,\ \mu\in\mathbb{R},\ z\in\Delta)$$

and

$$\left|\arg\left\{(1-\mu)\frac{wg'(w)}{g(w)} + \mu\left(1+\frac{wg''(w)}{g'(w)}\right)\right\}\right| < \frac{\alpha\pi}{2} \quad (0<\alpha\leq 1,\ \mu\in\mathbb{R},\ w\in\Delta),$$

where the function g *is the inverse of* f *given by* (3)*.*

Theorem 3 [12, Theorem 1] *Let* $f(z)$ *given by* (1) *be in the class* $\Sigma^*_{\vartheta}(\mu,\alpha)$*,* $\mu\in\mathbb{R}-\left\{\frac{1}{2},1\right\}$*. Then*

$$|b_0| \leq \frac{2\alpha}{|1-\mu|}, \qquad |b_1| \leq \frac{\sqrt{\mu^2-2\mu+5}}{|1-\mu||2\mu-1|}\alpha^2.$$

The object of the present paper is to introduce a new subclass of the function class $\Sigma_{\mathfrak{B}}$ and obtain estimates on the initial coefficients for functions in this new subclass which improve Theorem 1, Theorem 2 and Theorem 3. Our results generalize and improve those in related works of several earlier authors.

2 Coefficient bounds for the function class $M^{h,p}_{\Sigma_{\mathfrak{B}}}(\lambda,\mu)$

In this section, we introduce and investigate the general subclass $M^{h,p}_{\Sigma_{\mathfrak{B}}}(\lambda,\mu)$.

Definition 4 *Let the functions* $h, p : \Delta \to \mathbb{C}$ *be analytic functions and*

$$h(z) = 1 + \frac{h_1}{z} + \frac{h_2}{z^2} + \frac{h_3}{z^3} + \cdots, \quad p(z) = 1 + \frac{p_1}{z} + \frac{p_2}{z^2} + \frac{p_3}{z^3} + \cdots,$$

such that

$$\min\{\mathrm{Re}(h(z)),\ \mathrm{Re}\,(p(z))\} > 0, \quad z \in \Delta.$$

A function $f \in \Sigma_{\mathfrak{B}}$ *given by* (1) *is said to be in the class* $M^{h,p}_{\Sigma_{\mathfrak{B}}}(\lambda, \mu)$ $(\lambda \geq 1,$ $\mu \in \mathbb{R})$*, if the following conditions are satisfied:*

$$(1-\mu)\frac{z(f'(z))^{\lambda}}{f(z)} + \mu\left(1 + \frac{zf''(z)}{f'(z)}\right)^{\lambda} \in h(\Delta) \quad (\lambda \geq 1,\ \mu \in \mathbb{R},\ z \in \Delta) \tag{4}$$

and

$$(1-\mu)\frac{w(g'(w))^{\lambda}}{g(w)} + \mu\left(1 + \frac{wg''(w)}{g'(w)}\right)^{\lambda} \in p(\Delta) \quad (\lambda \geq 1,\ \mu \in \mathbb{R},\ w \in \Delta), \tag{5}$$

where the function g *is the inverse of* f *given by* (3)*.*

Remark 1 *There are many selections of the functions* $h(z)$ *and* $p(z)$ *which would provide interesting subclasses of the meromorphic function class* Σ*. For example, if we let*

$$h(z) = p(z) = \left(\frac{1+\frac{1}{z}}{1-\frac{1}{z}}\right)^{\alpha} = 1 + \frac{2\alpha}{z} + \frac{2\alpha^2}{z^2} + \cdots \quad (0 < \alpha \leq 1,\ z \in \Delta),$$

it is easy to verify that the functions $h(z)$ *and* $p(z)$ *satisfy the hypotheses of Definition 4.*

If $f \in M^{h,p}_{\Sigma_{\mathfrak{B}}}(\lambda, \mu)$*, then*

$$\left|\arg\left\{(1-\mu)\frac{z(f'(z))^{\lambda}}{f(z)} + \mu\left(1 + \frac{zf''(z)}{f'(z)}\right)^{\lambda}\right\}\right| < \frac{\alpha\pi}{2}$$
$$(0 < \alpha \leq 1,\ \lambda \geq 1,\ \mu \in \mathbb{R},\ z \in \Delta)$$

and

$$\left|\arg\left\{(1-\mu)\frac{w(g'(w))^{\lambda}}{g(w)} + \mu\left(1 + \frac{wg''(w)}{g'(w)}\right)^{\lambda}\right\}\right| < \frac{\alpha\pi}{2}$$
$$(0 < \alpha \leq 1,\ \lambda \geq 1,\ \mu \in \mathbb{R},\ w \in \Delta).$$

In this case, the function f *is said to be in the class* $M_{\Sigma_{\mathfrak{B}}}(\lambda, \mu, \alpha)$ *and in special case* $\lambda = 1$*, it reduces to Definition 3. We note that, by putting* $\mu = 0$*,* *the class* $M_{\Sigma_{\mathfrak{B}}}(\lambda, \mu, \alpha)$ *reduces to Definition 1, the class* $\Sigma_{B,\lambda^*}(\alpha)$ *introduced and studied by Srivastava et al.* [11].

If we let

$$h(z) = p(z) = \frac{1 + \frac{1-2\beta}{z}}{1 - \frac{1}{z}}$$
$$= 1 + \frac{2(1-\beta)}{z} + \frac{2(1-\beta)}{z^2} + \frac{2(1-\beta)}{z^3} + \dots \quad (0 \le \beta < 1,\ z \in \Delta),$$

it is easy to verify that the functions $h(z)$ *and* $p(z)$ *satisfy the hypotheses of Definition 4.*

If $f \in M_{\Sigma_{\mathfrak{B}}}^{h,p}(\lambda, \mu)$, *then*

$$\mathrm{Re}\left\{(1-\mu)\frac{z(f'(z))^{\lambda}}{f(z)} + \mu\left(1 + \frac{zf''(z)}{f'(z)}\right)^{\lambda}\right\} > \beta$$
$$(0 \le \beta < 1,\ \lambda \ge 1,\ \mu \in \mathbb{R},\ z \in \Delta)$$

and

$$\mathrm{Re}\left\{(1-\mu)\frac{w(g'(w))^{\lambda}}{g(w)} + \mu\left(1 + \frac{wg''(w)}{g'(w)}\right)^{\lambda}\right\} > \beta$$
$$(0 \le \beta < 1,\ \lambda \ge 1,\ \mu \in \mathbb{R},\ w \in \Delta).$$

Therefore for $h(z) = p(z) = \frac{1+\frac{1-2\beta}{z}}{1-\frac{1}{z}}$ *and* $\mu = 0$, *the class* $M_{\Sigma_{\mathfrak{B}}}^{h,p}(\lambda, \mu)$ *reduces to Definition 2.*

Now, we derive the estimates of the coefficients $|b_0|$ and $|b_1|$ for class $M_{\Sigma_{\mathfrak{B}}}^{h,p}(\lambda, \mu)$.

Theorem 4 *Let* $f(z) \in \Sigma_{\mathfrak{B}}$ *given by* (1) *be in the class* $M_{\Sigma_{\mathfrak{B}}}^{h,p}(\lambda, \mu)$ $(\lambda \ge 1,\ \mu \in \mathbb{R} - \{1\},\ (3\lambda\mu + \mu - \lambda) \ne 1)$. *Then*

$$|b_0| \le \min\left\{\sqrt{\frac{|h_1|^2 + |p_1|^2}{2(1-\mu)^2}}, \sqrt{\frac{|h_2| + |p_2|}{2|1-\mu|}}\right\} \tag{6}$$

and

$$|b_1| \le \min\left\{\frac{|h_2| + |p_2|}{2|3\lambda\mu + \mu - \lambda - 1|}, \frac{1}{|3\lambda\mu + \mu - \lambda - 1|}\sqrt{\frac{|h_2|^2 + |p_2|^2}{2} + \frac{(|h_1|^2 + |p_1|^2)^2}{4(1-\mu)^2}}\right\}. \tag{7}$$

Proof. First of all, we write the argument inequalities in (4) and (5) in their equivalent forms as follows:

$$(1-\mu)\frac{z(f'(z))^{\lambda}}{f(z)}+\mu\left(1+\frac{zf''(z)}{f'(z)}\right)^{\lambda}=h(z)\quad(z\in\Delta)\tag{8}$$

and

$$(1-\mu)\frac{w(g'(w))^{\lambda}}{g(w)}+\mu\left(1+\frac{wg''(w)}{g'(w)}\right)^{\lambda}=p(w)\quad(w\in\Delta),\tag{9}$$

respectively, where functions $h(z)$ and $p(w)$ satisfy the conditions of Definition 4.
Furtheremore, the functions $h(z)$ and $p(w)$ have the forms:

$$h(z)=1+\frac{h_1}{z}+\frac{h_2}{z^2}+\frac{h_3}{z^3}+\cdots$$

and

$$p(w)=1+\frac{p_1}{w}+\frac{p_2}{w^2}+\frac{p_3}{w^2}+\cdots,$$

respectively. Now, upon equating the coefficients of

$$\begin{aligned}&(1-\mu)\frac{z(f'(z))^{\lambda}}{f(z)}+\mu\left(1+\frac{zf''(z)}{f'(z)}\right)^{\lambda}\\&=1-\frac{(1-\mu)b_0}{z}+\frac{(1-\mu)b_0^2+(3\lambda\mu+\mu-\lambda-1)b_1}{z^2}+\dots\end{aligned}\tag{10}$$

with those of $h(z)$ and coefficients of

$$\begin{aligned}&(1-\mu)\frac{w(g'(w))^{\lambda}}{g(w)}+\mu\left(1+\frac{wg''(w)}{g'(w)}\right)^{\lambda}\\&=1+\frac{(1-\mu)b_0}{w}+\frac{(1-\mu)b_0^2-(3\lambda\mu+\mu-\lambda-1)b_1}{w^2}+\dots\end{aligned}\tag{11}$$

with those of $p(w)$, we get

$$-(1-\mu)b_0=h_1,\tag{12}$$

$$(1-\mu)b_0^2+(3\lambda\mu+\mu-\lambda-1)b_1=h_2,\tag{13}$$

$$(1-\mu)b_0=p_1\tag{14}$$

and

$$(1-\mu)b_0^2 - (3\lambda\mu + \mu - \lambda - 1)b_1 = p_2 \tag{15}$$

From (12) and (14), we get

$$h_1 = -p_1 \quad (b_0 = -\frac{h_1}{1-\mu})$$

and

$$2(1-\mu)^2 b_0^2 = h_1^2 + p_1^2. \tag{16}$$

Adding (13) and (15), we get

$$2(1-\mu)b_0^2 = h_2 + p_2. \tag{17}$$

Therefore, we find from the equations (16) and (17) that

$$|b_0|^2 \leq \frac{|h_1|^2 + |p_1|^2}{2(1-\mu)^2},$$

and

$$|b_0|^2 \leq \frac{|h_2| + |p_2|}{2|1-\mu|}$$

respectively. So we get the desired estimate on the coefficient $|b_0|$ as asserted in (6).

Next, in order to find the bound on the coefficient $|b_1|$, we subtract (15) from (13). We thus get

$$2(3\lambda\mu + \mu - \lambda - 1)b_1 = h_2 - p_2. \tag{18}$$

By squaring and adding (13) and (15), using (16) in the computation leads to

$$b_1^2 = \frac{1}{2(3\lambda\mu + \mu - \lambda - 1)^2}\left(h_2^2 + p_2^2 - \frac{(h_1^2 + p_1^2)^2}{2(1-\mu)^2}\right). \tag{19}$$

Therefore, we find from the equations (18) and (19) that

$$|b_1| \leq \frac{|h_2| + |p_2|}{2|3\lambda\mu + \mu - \lambda - 1|}$$

and

$$|b_1| \leq \frac{1}{|3\lambda\mu + \mu - \lambda - 1|}\sqrt{\frac{|h_2|^2 + |p_2|^2}{2} + \frac{(|h_1|^2 + |p_1|^2)^2}{4(1-\mu)^2}}.$$

This evidently completes the proof of Theorem 4. □

3 Corollaries and consequences

By setting

$$h(z) = p(z) = \frac{1 + \frac{1-2\beta}{z}}{1 - \frac{1}{z}} = 1 + \frac{2(1-\beta)}{z} + \frac{2(1-\beta)}{z^2} + \dots \quad (0 \le \beta < 1,\ z \in \Delta)$$

and $\mu = 0$ in Theorem 4, we conclude the following result.

Corollary 1 *Let the function* $f(z)$ *given by* (1) *be in the class* $\Sigma_{B^*}(\lambda, \beta)$, $(0 \le \beta < 1, \lambda \ge 1)$. *Then*

$$|b_0| \le \begin{cases} \sqrt{2(1-\beta)}; & \beta \le \frac{1}{2} \\ 2(1-\beta); & \beta > \frac{1}{2} \end{cases}$$

and

$$|b_1| \le \min\left\{ \frac{2(1-\beta)}{1+\lambda}, \frac{2(1-\beta)\sqrt{4\beta^2 - 8\beta + 5}}{1+\lambda} \right\} = \frac{2(1-\beta)}{1+\lambda}.$$

Remark 2 *The bounds on* $|b_0|$ *and* $|b_1|$ *given in Corollary 1 are better than those given in Theorem 2.*

By setting $\lambda = 1$ in Corollary 1, we conclude the following result.

Corollary 2 *Let the function* $f(z)$ *given by* (1) *be in the class* $\Sigma^*_{\mathfrak{B}}(\beta)$ $(0 \le \beta < 1)$. *Then*

$$|b_0| \le \begin{cases} \sqrt{2(1-\beta)}; & \beta \le \frac{1}{2} \\ 2(1-\beta); & \beta > \frac{1}{2} \end{cases}$$

and

$$|b_1| \le \min\{1-\beta, (1-\beta)\sqrt{1 + 4(1-\beta)^2}\} = 1 - \beta.$$

Remark 3 *The bounds on* $|b_0|$ *and* $|b_1|$ *given in Corollary 2 are better than those given by Halim et al.* [3, Theorem 1].

By setting

$$h(z) = p(z) = \left(\frac{1 + \frac{1}{z}}{1 - \frac{1}{z}} \right)^{\alpha} \quad (0 < \alpha \le 1, \quad z \in \Delta),$$

in Theorem 4, we conclude the following result.

Corollary 3 *Let the function* $f(z)$ *given by* (1) *be in the class* $M_{\Sigma_{\mathfrak{B}}}(\lambda,\mu,\alpha)$ $(0<\alpha\leq 1,\ \lambda\geq 1,\ \mu\in\mathbb{R}-\{1\},\ (3\lambda\mu+\mu-\lambda)\neq 1)$. *Then*

$$|b_0|\leq\begin{cases}\alpha\sqrt{\frac{2}{|1-\mu|}}; & |1-\mu|\leq 2\\ \frac{2\alpha}{|1-\mu|}; & |1-\mu|>2\end{cases}$$

and

$$|b_1|\leq\min\left\{\frac{2\alpha^2}{|3\lambda\mu+\mu-\lambda-1|},\frac{2\alpha^2}{|3\lambda\mu+\mu-\lambda-1|}\sqrt{1+\frac{4}{(1-\mu)^2}}\right\}$$
$$=\frac{2\alpha^2}{|3\lambda\mu+\mu-\lambda-1|}.$$

By setting $\mu=0$ in Corollary 3, we conclude the following result.

Corollary 4 *Let the function* $f(z)$ *given by* (1) *be in the class* $\Sigma_{B,\lambda^*}(\alpha)$ $(0<\alpha\leq 1,\ \lambda\geq 1)$. *Then*

$$|b_0|\leq\sqrt{2}\alpha$$

and

$$|b_1|\leq\frac{2\alpha^2}{\lambda+1}.$$

Remark 4 *The bounds on* $|b_0|$ *and* $|b_1|$ *given in Corollary 4 are better than those given in Theorem 2.*

By setting $\lambda=1$ in Corollary 3, we conclude the following result.

Corollary 5 *Let the function* $f(z)$ *given by* (1) *be in the class* $\Sigma^*_{\vartheta}(\mu,\alpha)$ $(0<\alpha\leq 1,\ \mu\in\mathbb{R}-\{\frac{1}{2},1\})$. *Then*

$$|b_0|\leq\begin{cases}\alpha\sqrt{\frac{2}{|1-\mu|}}; & |1-\mu|\leq 2\\ \frac{2\alpha}{|1-\mu|}; & |1-\mu|>2\end{cases}$$

and

$$|b_1|\leq\min\left\{\frac{\alpha^2}{|2\mu-1|},\frac{\sqrt{\mu^2-2\mu+5}}{|1-\mu||2\mu-1|}\alpha^2\right\}=\frac{\alpha^2}{|2\mu-1|}.$$

Remark 5 *The bounds on* $|b_0|$ *and* $|b_1|$ *given in Corollary 5 are better than those given in Theorem 3.*

By setting $\mu = 0$ in Corollary 5, we conclude the following result.

Corollary 6 *Let the function* $f(z)$ *given by* (1) *be in the class* $\widetilde{\Sigma}^*_{\mathfrak{B}}(\alpha)$ $(0 < \alpha \leq 1)$. *Then*

$$|b_0| \leq \sqrt{2}\alpha \qquad \text{and} \qquad |b_1| \leq \min\left\{\alpha^2, \sqrt{5}\alpha^2\right\} = \alpha^2.$$

Remark 6 *The bounds on* $|b_0|$ *and* $|b_1|$ *given in Corollary 6 are better than those given by Halim et al.* [3, Theorem 2].

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