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I-Rad-$\oplus$-supplemented modules

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Abstract. Let M be an R-module and I be an ideal of R. We say that M is I-Rad-$\oplus$-*supplemented*, provided for every submodule N of M, there exists a direct summand K of M such that $M = N + K$, $N \cap K \subseteq IK$ and $N \cap K \subseteq \mathrm{Rad}(K)$. The aim of this paper is to show new properties of I-Rad-$\oplus$-supplemented modules. Especially, we show that any finite direct sum of I-Rad-$\oplus$-supplemented modules is I-Rad-$\oplus$-supplemented. We also prove that an R-module M is I-Rad-$\oplus$-supplemented if and only if K and $\frac{M}{K}$ are I-Rad-$\oplus$-supplemented for a fully invariant direct summand K of M. Finally, we determine the structure of I-Rad-$\oplus$-supplemented modules over a discrete valuation ring.

1 Introduction

Throughout the whole text, all rings are to be associative, unit and all modules are left unitary. Let R be such a ring and M be an R-module. The notation $K \subseteq M$ ($K \subset M$) means that K is a (proper) submodule of M. A module M is called *extending* if every submodule is essential in a direct summand of M [4]. Here a submodule $K \leq M$ is said to be *essential* in M, denoted as $K \trianglelefteq M$, if $K \cap N \neq 0$ for every non-zero submodule $N \leq M$. Dually, a submodule S of M is called *small (in M)*, denoted as $S \ll M$, if $M \neq S + L$ for every proper submodule L of M [17]. If all non-zero submodules of M are essential in M,

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then M is called *uniform* [4, 1.5]. The Jacobson radical of M will be denoted by Rad(M). It is known that Rad(M) is the sum of all small submodules of M.

A non-zero module M is said to be *hollow* if every proper submodule of M is small in M, and it is said to be *local* if it is hollow and is finitely generated. A module M is local if and only if it is finitely generated and Rad(M) is the maximal submodule of M (see [4, 2.12 §2.15]). A ring R is said to be *local* if J is the maximal ideal of R, where J is the Jacobson radical of R.

An R-module M is called *supplemented* if every submodule of M has a supplement in M. Here a submodule $K \subseteq M$ is said to be a *supplement* of N in M if K is minimal with respect to $N+K = M$, or equivalently, if $N+K = M$ and $N \cap K \ll K$ [17]. A supplement submodule X of M is then defined when X is a supplement of some submodule of M. Every direct summand of a module M is a supplement submodule of M, and supplemented modules are a generalization of semisimple modules. In addition, every factor module of a supplemented module is again supplemented.

A module M is called *lifting* (or D_1*-module*) if, for every submodule N of M, there exists a direct summand K of M such that $K \leq N$ and $\frac{N}{K} \ll \frac{M}{K}$. Mohamed and Müller have generalized the concept of lifting modules to ⊕-supplemented modules. M is called *⊕-supplemented* if every submodule N of M has a supplement that is a direct summand of M [12]. Clearly every ⊕-supplemented module is supplemented, but a supplemented module need not be ⊕-supplemented in general (see [12, Lemma A.4 (2)]). It is shown in [12, Proposition A.7 and Proposition A.8] that if R is a Dedekind domain, every supplemented R-module is ⊕-supplemented. Hollow modules are ⊕-supplemented.

Weakening the notion of "supplement", one calls a submodule K of M a Rad-*supplement* of N in M if $M = N + K$ and $N \cap K \subseteq \text{Rad}(K)$ ([4, pp.100]).

Recall from [6] that a module M is called *Rad-⊕-supplemented*(or *generalized ⊕-supplemented* in [5]) if for every $N \subseteq M$, there exists a direct summand K of M such that $M = N + K$ and $N \cap K \subseteq \text{Rad}(K)$. In [15], various properties of Rad-⊕-supplemented modules are given. In addition, a ring R is semiperfect if and only if every finitely generated free R-module is generalized ⊕-supplemented (see [5]).

In this paper, we define I-Rad-⊕-supplemented modules which is specialized of Rad-⊕-supplemented modules. We obtain various properties of this modules adapting by [14]. We show that every finite direct sum of I-Rad-⊕-supplemented modules is a I-Rad-⊕-supplemented module. We prove that the class of I-Rad-⊕-supplemented modules is closed under extension in some constriction. Finally, we characterize I-Rad-⊕-supplemented modules over a discrete valuation ring.

2 Some results of I-Rad-$\oplus$-supplemented modules

A module M is called *semilocal* if $\frac{M}{\mathrm{Rad}(M)}$ is semisimple, and a ring R is called *semilocal* if ${}_RR$ (or R_R) is semilocal. Lomp proved in [11, Theorem 3.5] that a ring R is semilocal if and only if every left R-module is semilocal. Using this fact we obtain the following:

Lemma 1 *Let M be a module over a semilocal ring R. Then M is* Rad-$\oplus$-*supplemented if and only if for every submodule $N \subseteq M$, there exists a direct summand K of M such that $M = N + K$, $N \cap K \subseteq JK$.*

Proof. Clear by [1, Corollary 15.18]. □

By using the above lemma, we have a specialized notion which is strong of Rad-$\oplus$-supplemented modules. Now we define this notion.

Definition 1 *Let M be an R-module and I be an ideal of R. We say that M is a* I-Rad-$\oplus$-supplemented module*, provided for every submodule N of M, there exists a direct summand K of M such that $M = N + K$, $N \cap K \subseteq IK$ and $N \cap K \subseteq \mathrm{Rad}(K)$.*

Lemma 2 *Let M be an R-module and I be an ideal of R such that $IM = 0$. Then, M is* I-Rad-$\oplus$-*supplemented if and only if M is semisimple.*

Proof. ($\Rightarrow$) Let N be a submodule of M. By the hypothesis, there exists a direct summand K of M such that $M = N+K$, $N\cap K \subseteq IK$ and $N\cap K \subseteq \mathrm{Rad}(K)$. Since $IK \subseteq IM = 0$, we obtain that $M = N \oplus K$. Hence M is semisimple.

($\Leftarrow$) Let N be a submodule of M. Then there exists a submodule N' of M such that $M = N \oplus N'$. So $M = N + N'$, $N \cap N' = 0 \subseteq IN'$ and $N \cap N' = 0 \subseteq \mathrm{Rad}(N')$. Therefore M is a I-Rad-$\oplus$-supplemented module. □

Lemma 3 [14, Lemma 3.4] *Let M be an R-module and I be an ideal of R. If K is a direct summand of M, then we have $IK = K \cap IM$.*

Proposition 1 *Let M be an arbitrary R-module and I be an ideal of R such that $\mathrm{Rad}(M) \subseteq IM$. Then M is* I-Rad-$\oplus$-*supplemented if and only if M is* Rad-$\oplus$-*supplemented.*

Proof. ($\Rightarrow$) It is clear.

($\Leftarrow$) Suppose that M is I-Rad-$\oplus$-supplemented. Let N be a submodule of M. Then there exists a direct summand K of M such that $M = N + K$ and

$N \cap K \subseteq \mathrm{Rad}(K)$. Note that $IK = K \cap IM$ by Lemma 3. Since $\mathrm{Rad}(M) \subseteq IM$, we have $N \cap K \subseteq \mathrm{Rad}(K) \subseteq K \cap \mathrm{Rad}(M) \subseteq K \cap IM = IK$. Therefore M is I-Rad-⊕-supplemented. This completes the proof. □

Recall from [17] that a ring R is called *a left good ring* if $\mathrm{Rad}(M) = JM$ for every R-module M. A semilocal ring is an example of a left good ring.

Corollary 1 *Let M be an R-module. Suppose further that either*

(1) *R is a left good ring, or*

(2) *M is a projective module.*

If an ideal I of R contains the Jacobson radical J of R, then M is Rad-⊕-supplemented if and only if M is I-Rad-⊕-supplemented.

Proof. Note that $\mathrm{Rad}(M) = JM$ by [1, Proposition 17.10]. The result follows from Proposition 1. □

It is clear that every I-Rad-⊕-supplemented module is Rad-⊕-supplemented module, but the following example shows that the converse is not be always true. Firstly, we need the following crucial proposition.

Proposition 2 *Let M be an indecomposable R-module with $\mathrm{Rad}(M) \ll M$ and I be an ideal of R. Then the following statements are equivalent.*

(1) *M is I-Rad-⊕-supplemented;*

(2) *M is local with $IM = M$ or $IM = \mathrm{Rad}(M)$.*

Proof. (1)$\Longrightarrow$(2) Let N be a proper submodule of M. By hypothesis, there exists a direct summand K of M such that $M = N + K$, $N \cap K \subseteq IK$ and $N \cap K \subseteq \mathrm{Rad}(K)$. Since M is indecomposable, we have $K = M$. Hence, $N \subseteq IM$ and $N \subseteq \mathrm{Rad}(M)$. Since $\mathrm{Rad}(M) \ll M$, we have $N \ll M$. Thus, M is a local module. Moreover, note that if $IM \neq M$, then IM contains all other proper submodules of M. Hence M is a local module and $IM = \mathrm{Rad}(M)$.

(2)$\Longrightarrow$(1) Let N be a proper submodule of M. Then $M = N + M$ and $N \cap M = N \subseteq \mathrm{Rad}(M) \subseteq IM$. So M is I-Rad-⊕-supplemented. □

Example 1 (See [14, Example 3.8]) *Let p and q be two different prime integers. Consider the local $\mathbb{Z}$-module $M = \frac{\mathbb{Z}}{\mathbb{Z}p^3}$. We have $\mathrm{Rad}(M) = \frac{\mathbb{Z}p}{\mathbb{Z}p^3} \ll M$. Let $I_1 = \mathbb{Z}p$, $I_2 = \mathbb{Z}q$ and $I_3 = \mathbb{Z}p^2$. Then $I_1M = \mathrm{Rad}(M)$, $I_2M = M$ and $I_3M = \frac{\mathbb{Z}p^2}{\mathbb{Z}p^3}$. By Proposition 2, M is I_i-Rad-⊕-supplemented for each $i = 1, 2$ but not I_3-Rad-⊕-supplemented. On the other hand, it is clear that M is Rad-⊕-supplemented.*

Proposition 3 *Let* I *be an ideal of* R *and* M *be an* R*-module. If* M *is an* I-Rad-$\oplus$*-supplemented* R*-module, then* $\frac{M}{IM}$ *is semisimple.*

Proof. Let N be a submodule of M such that $IM \subseteq N$. By assumption, there exists a direct summand K of M such that $M = N + K$, $N \cap K \subseteq IK$ and $N \cap K \subseteq \mathrm{Rad}(K)$. Then $\frac{N}{IM} + \frac{K+IM}{IM} = \frac{M}{IM}$. Clearly, we have $N \cap (K + IM) = IM + N \cap K = IM$ and so $\frac{N}{IM} \cap \frac{K+IM}{IM} = \frac{IM}{IM}$. Therefore $\frac{M}{IM} = \frac{N}{IM} \oplus \frac{K+IM}{IM}$. It means that $\frac{M}{IM}$ is semisimple. □

Corollary 2 *Let* M *be a* Rad-$\oplus$*-supplemented* R*-module such that* $IM = M$*, where* I *is an ideal of* R*. Then* M *is* I-Rad-$\oplus$*-supplemented.*

Corollary 3 *Let* m *be a maximal ideal of a commutative ring* R *and* M *be an* R*-module. Assume that* I *is an ideal of* R *such that* $IM = mM$*. If* M *is a* Rad-$\oplus$*-supplemented* R*-module, then* M *is* I-Rad-$\oplus$*-supplemented.*

Proof. Note that $\mathrm{Rad}(M) \subseteq mM$ by [7, Lemma 3]. The result follows from Proposition 1. □

Recall from [17] that an R-module M is called *divisible* in case $rM = M$ for each non-zero element $r \in R$, where R is a commutative domain.

Proposition 4 *Let* M *be a divisible module over a commutative domain* R*. If* M *is* Rad-$\oplus$*-supplemented, then* M *is* I-Rad-$\oplus$*-supplemented for every non-zero ideal* I *of* R*.*

Proof. This follows from Corollary 2. □

Corollary 4 *Let* R *be a Dedekind domain and* M *be an injective* R*-module. Then,* M *is* I-Rad-$\oplus$*-supplemented for every non-zero ideal* I *of* R*.*

Proof. Since every injective module over a Dedekind domain is divisible, the proof follows from Proposition 4. □

Theorem 1 *Let* I *be an ideal of* R*. Then any finite direct sum of* I-Rad-$\oplus$*-supplemented* R*-modules is* I-Rad-$\oplus$*-supplemented.*

Proof. Let n be any positive integer and M_i $(1 \leq i \leq n)$ be any finite collection of I-Rad-$\oplus$-supplemented R-modules. Let $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$. Suppose that $n = 2$, that is, $M = M_1 \oplus M_2$. Let K be any submodule of M. Then $M = M_1 + M_2 + K$ and so $M_1 + M_2 + K$ has a Rad-supplement 0 in M. Since M_1

is I-Rad-$\oplus$-supplemented, $M_1 \cap (M_2 + K)$ has a Rad-supplement X in M_1 such that X is a direct summand of M_1 and $X \cap (M_2 + K) = M_1 \cap (M_2 + K) \cap X \subseteq IX$. By [5, Lemma 3.2], X is a Rad-supplement of $M_2 + K$ in M. Since M_2 is I-Rad-$\oplus$-supplemented, $M_2 \cap (K + X)$ has a Rad-supplement Y in M_2 such that Y is a direct summand of M_2 and $Y \cap (K + X) = M_2 \cap (K + X) \cap Y \subseteq IY$. Again applying [5, Lemma 3.2], we obtain that $X + Y$ is a Rad-supplement of K in M. Since X is a direct summand of M_1 and Y is a direct summand of M_2, it follows that $X \oplus Y$ is a direct summand of M. Note that

$$\begin{aligned} K \cap (X + Y) &\subseteq X \cap (Y + K) + Y \cap (K + X) \\ &\subseteq X \cap (M_2 + K) + Y \cap (K + X) \\ &\subseteq IX \oplus IY = I(X \oplus Y) \end{aligned}$$

So $M_1 \oplus M_2$ is I-Rad-$\oplus$-supplemented. The proof is completed by induction on n. □

Recall from [17] that a submodule U of an R-module M is called *fully invariant* if $f(U)$ is contained in U for every R-endomorphism f of M. Let M be an R-module and τ be a preradical for the category of R-modules. Then $\tau(M)$ is fully invariant submodule of M. A module M is called *duo* if every submodule of M is fully invariant [13].

Proposition 5 *Let* I *be an ideal of* R *and* $M = \oplus_{\lambda \in \Lambda} M_\lambda$ *be a duo module where* M *is a direct sum of submodules* M_λ $(\lambda \in \Lambda)$. *Assume that* M_λ *is* I-Rad-$\oplus$-*supplemented for every* $\lambda \in \Lambda$. *Then* M *is* I-Rad-$\oplus$-*supplemented.*

Proof. By hypothesis, for every $\lambda \in \Lambda$, there exists a direct summand K_λ of M_λ such that $M_\lambda = (N \cap M_\lambda) + K_\lambda$, $N \cap K_\lambda \subseteq IK_\lambda$ and $N \cap K_\lambda \subseteq \mathrm{Rad}(K_\lambda)$. Put $K = \oplus_{\lambda \in \Lambda} K_\lambda$. Clearly K is a direct summand of M and $M = N + K$. Also, we have $N \cap K = \oplus_{\lambda \in \Lambda}(N \cap K_\lambda) \subseteq IK$ and $N \cap K \subseteq \mathrm{Rad}(K)$. This completes the proof. □

Now, we give an example showing that the I-Rad-$\oplus$-supplemented property doesn't always transfer from a module to each of its factor modules.

Example 2 (see [2, Example 4.1]) *Let* F *be a field. Consider the local ring* $R = \frac{F[x^2,x^3]}{(x^4)}$ *and let* m *be the maximal ideal of* R. *Let* n *be an integer with* $n \geq 2$ *and* $M = R^{(n)}$. *By Proposition 2 and Theorem 1,* M *is* m-Rad-$\oplus$-*supplemented. Note that* R *is an artinian local ring which is not a principal ideal ring. So, there exists a submodule* K *of* M *such that the factor module* $\frac{M}{K}$ *isn't* Rad-$\oplus$-*supplemented. Therefore* $\frac{M}{K}$ *isn't* m-Rad-$\oplus$-*supplemented.*

Recall from [17, 6.4] that a module M is called *distributive* if $(A+B)\cap C=(A\cap C)+(B\cap C)$ for all submodules A, B, C of M (or equivalently, $(A\cap B)+C=(A+C)\cap(B+C)$ for all submodules A, B, C of M).

Now, we show that a factor module of an I-Rad-$\oplus$-supplemented module is I-Rad-$\oplus$-supplemented under some conditions.

Proposition 6 *Let* I *be an ideal of* R *and* M *be an* I-Rad-$\oplus$-*supplemented module.*

(1) *Let* $X\subseteq M$ *be a submodule such that for every direct summand* K *of* M, $\frac{X+K}{X}$ *is a direct summand of* $\frac{M}{X}$. *Then* $\frac{M}{X}$ *is* I-Rad-$\oplus$-*supplemented;*

(2) *Let* $X\subseteq M$ *be a submodule such that for every decomposition* $M=M_1\oplus M_2$, *we have* $X=(X\cap M_1)\oplus(X\cap M_2)$. *Then* $\frac{M}{X}$ *is* I-Rad-$\oplus$-*supplemented;*

(3) *If* X *is a fully invariant submodule of* M, *then* $\frac{M}{X}$ *is* I-Rad-$\oplus$-*supplemented;*

(4) *If* M *is a distributive module, then* $\frac{M}{X}$ *is* I-Rad-$\oplus$-*supplemented for every submodule* X *of* M.

Proof. (1) Let N be a submodule of M such that $X\subseteq N$. Since M is I-Rad-$\oplus$-supplemented, there exists a direct summand K of M such that $M=N+K$, $N\cap K\subseteq IK$ and $N\cap K\subseteq \mathrm{Rad}(K)$. Therefore $\frac{M}{X}=\frac{N}{X}+\frac{X+K}{X}$ and $\frac{N}{X}\cap\frac{K+X}{X}=\frac{X+(N\cap K)}{X}\subseteq\frac{X+IK}{X}\subseteq I(\frac{X+K}{X})$. Consider the natural epimorphism $\pi:K\longrightarrow\frac{X+K}{X}$. Since $N\cap K\subseteq\mathrm{Rad}(K)$, we have $\pi(N\cap K)=\frac{X+(N\cap K)}{X}\subseteq\mathrm{Rad}(\frac{X+K}{X})$. Note that by assumption, $\frac{X+K}{X}$ is a direct summand of $\frac{M}{X}$. It follows that $\frac{M}{X}$ is I-Rad-$\oplus$-supplemented.

(2), (3) and (4) are consequences of (1). $\square$

Proposition 7 *Let* M *be an* R-*module,* I *be an ideal of* R *and* K *be a fully invariant direct summand of* M. *Then the following statements are equivalent:*

(1) M *is* I-Rad-$\oplus$-*supplemented;*

(2) K *and* $\frac{M}{K}$ *are* I-Rad-$\oplus$-*supplemented.*

Proof. (1)$\Rightarrow$(2) Let L be a submodule of K. By hypothesis, there exist submodules A and B of M such that $M=A\oplus B$, $M=A+L$, $A\cap L\subseteq IA$ and $A\cap L\subseteq\mathrm{Rad}(A)$. Clearly, we have $K=(A\cap K)+L$. Since K is fully invariant in M, we have $K=(A\cap K)\oplus(B\cap K)$. Hence $A\cap K$ is a direct summand of K. By Lemma 3, $I(A \cap K) = (A \cap K) \cap IM$. It follows that $(A \cap K) \cap L = A \cap L \subseteq (A \cap K) \cap IM = I(A \cap K)$. Since $A \cap K$ is a direct summand of K and K is a direct summand of M, $A \cap K$ is a direct summand of M such that $A \cap L \subseteq A \cap K$. Since $A \cap L \subseteq \mathrm{Rad}(M)$, we have $A \cap L \subseteq \mathrm{Rad}(A \cap K)$. Therefore, K is I-Rad-⊕-supplemented. Moreover, $\frac{M}{K}$ is I-Rad-⊕-supplemented by Proposition 6 (3).

(2) ⇒ (1) It follows from Theorem 1. □

Let I be an ideal of R. We call an R-module M is called *completely* I-Rad-⊕-*supplemented* if every direct summand of M is I-Rad-⊕-supplemented. Clearly, semisimple modules are completely I-Rad-⊕-supplemented. Also, every I-Rad-⊕-supplemented hollow module is completely I-Rad-⊕-supplemented.

Proposition 8 *Let $M = M_1 \oplus M_2$ be a direct sum of local submodules M_1 and M_2. Then the following statements are equivalent:*

(1) *M_1 and M_2 are* I-Rad-⊕-*supplemented modules;*

(2) *M is a completely* I-Rad-⊕-*supplemented module.*

Proof. (1) ⇒ (2) Let L be a non-zero direct summand of M. If $L = M$, then L is I-Rad-⊕-supplemented by Theorem 1. Assume that $L \neq M$. Let K be a submodule of M such that $M = L \oplus K$. Then L is a local module by [4, 5.4 (1)]. Let us prove that L is I-Rad-⊕-supplemented. To see this, it suffices to show that $IL = L$ or $IL = \mathrm{Rad}(L)$ by Proposition 2. Since M is I-Rad-⊕-supplemented, $\frac{M}{IM} \cong \frac{L}{IL} \oplus \frac{K}{IK}$ is semisimple by Proposition 3. Then $\frac{L}{IL}$ is semisimple and so $\mathrm{Rad}(L) \subseteq IL$. Since L is local, we get that $L = IL$ or $\mathrm{Rad}(L) = IL$.

(2) ⇒ (1) Obvious. □

Now, we determine the structure of all I-Rad-⊕-supplemented modules over a discrete valuation ring.

Theorem 2 *Assume that R is a discrete valuation ring with maximal ideal $\mathfrak{m}$. Let I be an ideal of R and M be an R-module.*

(1) *If $I = \mathfrak{m}$ or $I = R$, then the following statements are equivalent.*

   (i) *M is* I-Rad-⊕-*supplemented;*

   (ii) *M is* Rad-⊕-*supplemented;*

   (iii) *$M \cong R^a \oplus D \oplus B$, where $a \in \mathbb{N}$, B is a bounded R-module and D is an injective R-module.*

(2) *If* $I \notin \{\mathfrak{m}, R\}$, *then the following are equivalent:*

(i) M *is* I-Rad-$\oplus$-*supplemented;*

(ii) $M \cong D \oplus B$ *for some injective* R-*module* D *and some semisimple* R-*module* B.

Proof. It is well known that, for any module M over a discrete valuation ring, we have $Rad(M) = JM = \mathfrak{m}M$.
(1) (i) $\Leftrightarrow$ (ii) Since local rings are a good ring, by Corollary 1 and assumption, the proof follows.
(ii) $\Leftrightarrow$ (iii) Clear by [15, Corollary 3.3].
(2) (i) $\Rightarrow$ (ii) Suppose that M is I-Rad-$\oplus$-supplemented. Applying [15, Corollary 3.3], $M \cong R^a \oplus D \oplus B$ for some bounded R-module B, some natural numbers a and an injective R-module D. Since D is a fully invariant submodule of M, it follows from Proposition 7 that $N = R^a \oplus B$ is I-Rad-$\oplus$-supplemented. Using Lemma 3 and Proposition 3, we obtain that $\frac{N}{IN}$ is semisimple. Since $I \notin \{\mathfrak{m}, R\}$, we get that $a = 0$. Now we will prove that B is semisimple. Since $\frac{B}{IB}$ is semisimple and $I < \mathfrak{m}$, we can write $Rad(B) = JB = IB$. Note that B is bounded. Then, there exists an ideal H of R such that $HB = 0$. Therefore, $Rad(B) = JB = HB = 0$ and so B is semisimple by Lemma 2. This completes the proof.
(ii) $\Rightarrow$ (i) By Corollary 4, D is I-Rad-$\oplus$-supplemented. Since B is semisimple, B is I-Rad-$\oplus$-supplemented. Applying Theorem 1, we obtain that M is I-Rad-$\oplus$-supplemented. □

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