Finite groups with a certain number of cyclic subgroups II

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Abstract. In this note we describe the finite groups $G$ having $|G| - 2$ cyclic subgroups. This partially solves the open problem in the end of [3].

Let $G$ be a finite group and $C(G)$ be the poset of cyclic subgroups of $G$. The connections between $|C(G)|$ and $|G|$ lead to characterizations of certain finite groups $G$. For example, a basic result of group theory states that $|C(G)| = |G|$ if and only if $G$ is an elementary abelian 2-group. Recall also the main theorem of [3], which states that $|C(G)| = |G| - 1$ if and only if $G$ is one of the following groups: $\mathbb{Z}_3$, $\mathbb{Z}_4$, $S_3$ or $D_8$.

In what follows we shall continue this study by describing the finite groups $G$ for which

$$|C(G)| = |G| - 2.$$  \hfill (*)

First, we observe that certain finite groups of small orders, such as $\mathbb{Z}_6$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $D_{12}$ and $\mathbb{Z}_2 \times D_8$, have this property. Our main theorem proves that in fact these groups exhaust all finite groups $G$ satisfying (*).

**Theorem 1** Let $G$ be a finite group. Then $|C(G)| = |G| - 2$ if and only if $G$ is one of the following groups: $\mathbb{Z}_6$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $D_{12}$ or $\mathbb{Z}_2 \times D_8$.

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Proof. We will use the same technique as in the proof of Theorem 2 in [3]. Assume that $G$ satisfies (*), let $n = |G|$ and denote by $d_1, d_2, \ldots, d_k$ the positive divisors of $n$. If $n_i = |\{H \in C(G) \mid |H| = d_i\}|$, $i = 1, 2, \ldots, k$, then

$$\sum_{i=1}^{k} n_i \phi(d_i) = n.$$ 

Since $|C(G)| = \sum_{i=1}^{k} n_i = n - 2$, one obtains

$$\sum_{i=1}^{k} n_i (\phi(d_i) - 1) = 2,$$

which implies that we have the following possibilities:

Case 1. There exists $i_0 \in \{1, 2, \ldots, k\}$ such that $n_{i_0} (\phi(d_{i_0}) - 1) = 2$ and $n_i (\phi(d_i) - 1) = 0, \forall i \neq i_0$.

Since the image of the Euler’s totient function does not contain odd integers $>1$, we infer that $n_{i_0} = 2$ and $\phi(d_{i_0}) = 2$, i.e. $d_{i_0} \in \{3, 4, 6\}$. We remark that $d_{i_0}$ cannot be equal to 6 because in this case $G$ would also have a cyclic subgroup of order 3, a contradiction. Also, we cannot have $d_{i_0} = 3$ because in this case $G$ would contain two cyclic subgroups of order 3, contradicting the fact that the number of subgroups of a prime order $p$ in $G$ is $\equiv 1 \pmod{p}$ (see e.g. the note after Problem 1C.8 in [1]). Therefore $d_{i_0} = 4$, i.e. $G$ is a 2-group containing exactly two cyclic subgroups of order 4. Let $n = 2^m$ with $m \geq 3$. If $m = 3$ we can easily check that the unique group $G$ satisfying (*) is $\mathbb{Z}_2 \times \mathbb{Z}_4$. If $m \geq 4$ by Proposition 1.4 and Theorems 5.1 and 5.2 of [2] we infer that $G$ is isomorphic to one of the following groups:

- $M_{2^m}$;
- $\mathbb{Z}_2 \times \mathbb{Z}_{2^{m-1}}$;
- $\langle a, b \mid a^{2^{m-2}} = b^8 = 1, a^b = a^{-1}, a^{2^{m-3}} = b^4 \rangle$, where $m \geq 5$;
- $\mathbb{Z}_2 \times D_{2^{m-1}}$;
- $\langle a, b \mid a^{2^{m-2}} = b^2 = 1, a^b = a^{-1}+2^{m-4}c, c^2 = [c, b] = 1, a^c = a^{1+2^{m-3}} \rangle$, where $m \geq 5$. 

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All these groups have cyclic subgroups of order 8 for \( m \geq 5 \) and thus they do not satisfy (\( \ast \)). Consequently, \( m = 4 \) and the unique group with the desired property is \( \mathbb{Z}_2 \times D_8 \).

**Case 2.** There exist \( i_1, i_2 \in \{1, 2, \ldots, k\} \), \( i_1 \neq i_2 \), such that \( n_{i_1}(\phi(d_{i_1})-1) = n_{i_2}(\phi(d_{i_2})-1) = 1 \) and \( n_i(\phi(d_i)-1) = 0, \forall i \neq i_1, i_2 \).

Then \( n_i = 1 \) and \( \phi(d_i) = 2 \), i.e. \( d_i \in \{3, 4, 6\} \). Assume that \( d_{i_1} < d_{i_2} \). If \( d_{i_2} = 4 \), then \( d_{i_1} = 3 \), that is \( G \) contains normal cyclic subgroups of order 3 and 4. We infer that \( G \) also contains a cyclic subgroup of order 12, a contradiction. If \( d_{i_2} = 6 \), then we necessarily must have \( d_{i_1} = 3 \). Since \( G \) has a unique subgroup of order 3, it follows that a Sylow 3-subgroup of \( G \) must be cyclic and therefore of order 3. Let \( n = 3 \cdot 2^m \), where \( m \geq 1 \). Denote by \( n_2 \) the number of Sylow 2-subgroups of \( G \) and let \( H \) be such a subgroup. Then \( H \) is elementary abelian because \( G \) does not have cyclic subgroups of order \( 2^i \) with \( i \geq 2 \). By Sylow’s Theorems,

\[
n_2 \equiv 1 \pmod{3}
\]

implying that either \( n_2 = 1 \) or \( n_2 = 3 \). If \( n_2 = 1 \), then \( G \cong \mathbb{Z}_2^m \times \mathbb{Z}_3 \), a group that satisfies (\( \ast \)) if and only if \( m = 1 \), i.e. \( G \cong \mathbb{Z}_6 \). If \( n_2 = 3 \), then \( |\text{Core}_G(H)| = 2^{m-1} \) because \( G/\text{Core}_G(H) \) can be embedded in \( S_3 \). It follows that \( G \) contains a subgroup isomorphic with \( \mathbb{Z}_2^{m-1} \times \mathbb{Z}_3 \). If \( m \geq 3 \) this has more than one cyclic subgroup of order 6, contradicting our assumption. Hence either \( m = 1 \) or \( m = 2 \). For \( m = 1 \) one obtains \( G \cong S_3 \), a group that does not have cyclic subgroups of order 6, a contradiction, while for \( m = 2 \) one obtains \( G \cong D_{12} \), a group that satisfies (\( \ast \)). This completes the proof. \( \square \)

**References**

