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Subclasses of bi-univalent functions related to shell-like curves connected with Fibonacci numbers

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Abstract. In this paper, we introduce and investigate new subclasses of bi-univalent functions related to shell-like curves connected with Fibonacci numbers. Furthermore, we find estimates of first two coefficients of functions in these classes. Also, we determine Fekete-Szegö inequalities for these function classes.

1 Introduction

Let $\mathbb{U} = \{z : |z| < 1\}$ denote the unit disc on the complex plane. The class of all analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

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in the open unit disc $\mathbb{U}$ with normalization $f(0) = f'(0) - 1 = 0$ is denoted by $\mathcal{A}$ and the class $\mathcal{S} \subset \mathcal{A}$ is the class which consists of univalent functions in $\mathbb{U}$.

The Koebe one quarter theorem [3] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{A}$ contains a disk of radius $\frac{1}{4}$. Thus every univalent function $f \in \mathcal{A}$ has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z, \ (z \in \mathbb{U}) \text{ and } \ f(f^{-1}(w)) = w\left(|w| < r_0(f), \ r_0(f) \geq \frac{1}{4}\right).$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions defined in the unit disk $\mathbb{U}$. Since $f \in \Sigma$ has the Maclaurin series given by (1), a computation shows that its inverse $g = f^{-1}$ has the expansion

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 + \cdots . \tag{2}$$

One can see a short history and examples of functions in the class Σ in [12]. Several authors have introduced and investigated subclasses of bi-univalent functions and obtained bounds for the initial coefficients (see [1, 2, 8, 12, 13, 14]).

An analytic function f is subordinate to an analytic function F in $\mathbb{U}$, written as $f \prec F$ $(z \in \mathbb{U})$, provided there is an analytic function ω defined on $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ satisfying $f(z) = F(\omega(z))$. It follows from Schwarz Lemma that

$$f(z) \prec F(z) \quad \Longleftrightarrow f(0) = F(0) \text{ and } \ f(\mathbb{U}) \subset F(\mathbb{U}) \ , z \in \mathbb{U}$$

(for details see [3], [7]). We recall important subclasses of $\mathcal{S}$ in geometric function theory such that if $f \in \mathcal{A}$ and

$$\frac{zf'(z)}{f(z)} \prec p(z) \quad \text{and} \quad 1 + \frac{zf''(z)}{f'(z)} \prec p(z)$$

where $p(z) = \frac{1+z}{1-z}$, then we say that f is starlike and convex, respectively. These functions form known classes denoted by $\mathcal{S}^*$ and $\mathcal{C}$, respectively. Recently,in [11], Sokół introduced the class $\mathcal{SL}$ of shell-like functions as the set of functions $f \in \mathcal{A}$ which is described in the following definition:

Definition 1 *The function $f \in \mathcal{A}$ belongs to the class $\mathcal{SL}$ if it satisfies the condition that*

$$\frac{zf'(z)}{f(z)} \prec \tilde{p}(z)$$

with

$$\tilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2},$$

where $\tau = (1-\sqrt{5})/2 \approx -0.618$.

It should be observed $\mathcal{SL}$ is a subclass of the starlike functions $\mathcal{S}^*$.

Later, Dziok et al. in [4] and [5] defined and introduced the class $\mathcal{KSL}$ and $\mathcal{SLM}_\alpha$ of convex and $\alpha-$convex functions related to a shell-like curve connected with Fibonacci numbers, respectively. These classes can be given in the following definitions.

Definition 2 *The function* $f \in \mathcal{A}$ *belongs to the class* $\mathcal{KSL}$ *of convex shell-like functions if it satisfies the condition that*

$$1+\frac{zf''(z)}{f'(z)} \prec \tilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2},$$

where $\tau = (1-\sqrt{5})/2 \approx -0.618$.

Definition 3 *The function* $f \in \mathcal{A}$ *belongs to the class* $\mathcal{SLM}_\alpha$, $(0 \leq \alpha \leq 1)$ *if it satisfies the condition that*

$$\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\alpha)\frac{zf'(z)}{f(z)} \prec \tilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2},$$

where $\tau = (1-\sqrt{5})/2 \approx -0.618$.

The class $\mathcal{SLM}_\alpha$ is related to the class $\mathcal{KSL}$ only through the function $\tilde{p}$ and $\mathcal{SLM}_\alpha \neq \mathcal{KSL}$ for all $\alpha \neq 1$. It is easy to see that $\mathcal{KSL} = \mathcal{SLM}_1$.

Besides, let's define the class $\mathcal{SLG}_\gamma$ of so-called gamma-starlike functions related to a shell-like curve connected with Fibonacci numbers as follows.

Definition 4 *The function* $f \in \mathcal{A}$ *belongs to the class* $\mathcal{SLG}_\gamma$, $(\gamma \geq 0)$, *if it satisfies the condition that*

$$\left(\frac{zf'(z)}{f(z)}\right)^{\gamma}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\gamma} \prec \tilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2},$$

where $\tau = (1-\sqrt{5})/2 \approx -0.618$.

The function $\tilde{p}$ is not univalent in $\mathbb{U}$, but it is univalent in the disc $|z| < (3-\sqrt{5})/2 \approx 0.38$. For example, $\tilde{p}(0) = \tilde{p}(-1/2\tau) = 1$ and $\tilde{p}(e^{\mp i \arccos(1/4)}) = \sqrt{5}/5$, and it may also be noticed that

$$\frac{1}{|\tau|} = \frac{|\tau|}{1-|\tau|},$$

which shows that the number $|\tau|$ divides $[0,1]$ such that it fulfils the golden section. The image of the unit circle $|z| = 1$ under $\tilde{p}$ is a curve described by the equation given by

$$\left(10x - \sqrt{5}\right)y^2 = \left(\sqrt{5} - 2x\right)\left(\sqrt{5}x - 1\right)^2,$$

which is translated and revolved trisectrix of Maclaurin. The curve $\tilde{p}(re^{it})$ is a closed curve without any loops for $0 < r \le r_0 = (3-\sqrt{5})/2 \approx 0.38$. For $r_0 < r < 1$, it has a loop, and for $r = 1$, it has a vertical asymptote. Since τ satisfies the equation $\tau^2 = 1 + \tau$, this expression can be used to obtain higher powers τ^n as a linear function of lower powers, which in turn can be decomposed all the way down to a linear combination of τ and 1. The resulting recurrence relationships yield Fibonacci numbers u_n:

$$\tau^n = u_n\tau + u_{n-1}.$$

In [10], taking $\tau z = t$, Raina and Sokół showed that

$$\begin{aligned}
\tilde{p}(z) &= \frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2} = \left(t + \frac{1}{t}\right)\frac{t}{1-t-t^2} \\
&= \frac{1}{\sqrt{5}}\left(t+\frac{1}{t}\right)\left(\frac{1}{1-(1-\tau)t} - \frac{1}{1-\tau t}\right) \\
&= \left(t+\frac{1}{t}\right)\sum_{n=1}^{\infty}\frac{(1-\tau)^n - \tau^n}{\sqrt{5}}t^n \\
&= \left(t+\frac{1}{t}\right)\sum_{n=1}^{\infty}u_n t^n = 1 + \sum_{n=1}^{\infty}(u_{n-1}+u_{n+1})\tau^n z^n,
\end{aligned} \tag{3}$$

where

$$u_n = \frac{(1-\tau)^n - \tau^n}{\sqrt{5}}, \tau = \frac{1-\sqrt{5}}{2} \quad (n = 1, 2, \ldots). \tag{4}$$

This shows that the relevant connection of $\tilde{p}$ with the sequence of Fibonacci numbers u_n, such that $u_0 = 0$, $u_1 = 1$, $u_{n+2} = u_n + u_{n+1}$ for $n = 0, 1, 2, \cdots$.

And they got

$$\begin{aligned}\tilde{p}(z) &= 1+\sum_{n=1}^{\infty}\tilde{p}_n z^n = 1+(u_0+u_2)\tau z+(u_1+u_3)\tau^2z^2\\ &\quad+\sum_{n=3}^{\infty}(u_{n-3}+u_{n-2}+u_{n-1}+u_n)\tau^n z^n\\ &= 1+\tau z+3\tau^2z^2+4\tau^3z^3+7\tau^4z^4+11\tau^5z^5+\cdots.\end{aligned} \tag{5}$$

Let $\mathcal{P}(\beta)$, $0 \le \beta < 1$, denote the class of analytic functions p in $\mathbb{U}$ with $p(0) = 1$ and $\mathrm{Re}\{p(z)\} > \beta$. Especially, we will use $\mathcal{P}$ instead of $\mathcal{P}(0)$.

Theorem 1 [5] *The function* $\tilde{p}(z) = \frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2}$ *belongs to the class* $\mathcal{P}(\beta)$ *with* $\beta = \sqrt{5}/10 \approx 0.2236$.

Now we give the following lemma which will use in proving.

Lemma 1 [9] *Let* $p \in \mathcal{P}$ *with* $p(z) = 1 + c_1z + c_2z^2 + \cdots$, *then*

$$|c_n| \le 2, \qquad \text{for} \qquad n \ge 1. \tag{6}$$

In this present work, we introduce two subclasses of Σ associated with shell-like functions connected with Fibonacci numbers and obtain the initial Taylor coefficients $|a_2|$ and $|a_3|$ for these function classes. Also, we give bounds for the Fekete-Szegö functional $|a_3 - \mu a_2^2|$ for each subclass.

2 Bi-univalent function class $\mathcal{SLM}_{\alpha,\Sigma}(\tilde{p}(z))$

In this section, we introduce a new subclass of Σ associated with shell-like functions connected with Fibonacci numbers and obtain the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function class by subordination.

Firstly, let $p(z) = 1 + p_1z + p_2z^2 + \cdots$, and $p \prec \tilde{p}$. Then there exists an analytic function u such that $|u(z)| < 1$ in $\mathbb{U}$ and $p(z) = \tilde{p}(u(z))$. Therefore, the function

$$h(z) = \frac{1+u(z)}{1-u(z)} = 1 + c_1z + c_2z^2 + \ldots \tag{7}$$

is in the class $\mathcal{P}(0)$. It follows that

$$u(z) = \frac{c_1z}{2} + \left(c_2 - \frac{c_1^2}{2}\right)\frac{z^2}{2} + \left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right)\frac{z^3}{2} + \cdots \tag{8}$$

and

$$\begin{aligned}
\tilde{p}(u(z)) &= 1 + \tilde{p}_1 \left\{ \frac{c_1 z}{2} + \left(c_2 - \frac{c_1^2}{2} \right) \frac{z^2}{2} + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) \frac{z^3}{2} + \cdots \right\} \\
&\quad + \tilde{p}_2 \left\{ \frac{c_1 z}{2} + \left(c_2 - \frac{c_1^2}{2} \right) \frac{z^2}{2} + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) \frac{z^3}{2} + \cdots \right\}^2 \\
&\quad + \tilde{p}_3 \left\{ \frac{c_1 z}{2} + \left(c_2 - \frac{c_1^2}{2} \right) \frac{z^2}{2} + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) \frac{z^3}{2} + \cdots \right\}^3 + \cdots \qquad (9) \\
&= 1 + \frac{\tilde{p}_1 c_1 z}{2} + \left\{ \frac{1}{2} \left(c_2 - \frac{c_1^2}{2} \right) \tilde{p}_1 + \frac{c_1^2}{4} \tilde{p}_2 \right\} z^2 \\
&\quad + \left\{ \frac{1}{2} \left(c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) \tilde{p}_1 + \frac{1}{2} c_1 \left(c_2 - \frac{c_1^2}{2} \right) \tilde{p}_2 + \frac{c_1^3}{8} \tilde{p}_3 \right\} z^3 + \cdots .
\end{aligned}$$

And similarly, there exists an analytic function v such that $|v(w)| < 1$ in $\mathbb{U}$ and $p(w) = \tilde{p}(v(w))$. Therefore, the function

$$k(w) = \frac{1 + v(w)}{1 - v(w)} = 1 + d_1 w + d_2 w^2 + \ldots \qquad (10)$$

is in the class $\mathcal{P}(0)$. It follows that

$$v(w) = \frac{d_1 w}{2} + \left(d_2 - \frac{d_1^2}{2} \right) \frac{w^2}{2} + \left(d_3 - d_1 d_2 + \frac{d_1^3}{4} \right) \frac{w^3}{2} + \cdots \qquad (11)$$

and

$$\begin{aligned}
\tilde{p}(v(w)) &= 1 + \frac{\tilde{p}_1 d_1 w}{2} + \left\{ \frac{1}{2} \left(d_2 - \frac{d_1^2}{2} \right) \tilde{p}_1 + \frac{d_1^2}{4} \tilde{p}_2 \right\} w^2 \\
&\quad + \left\{ \frac{1}{2} \left(d_3 - d_1 d_2 + \frac{d_1^3}{4} \right) \tilde{p}_1 + \frac{1}{2} d_1 \left(d_2 - \frac{d_1^2}{2} \right) \tilde{p}_2 + \frac{d_1^3}{8} \tilde{p}_3 \right\} w^3 + \cdots .
\end{aligned} \qquad (12)$$

Definition 5 *For $0 \leq \alpha \leq 1$, a function $f \in \Sigma$ of the form (1) is said to be in the class $\mathcal{SLM}_{\alpha,\Sigma}(\tilde{p}(z))$ if the following subordination hold:*

$$\alpha \left(1 + \frac{z f''(z)}{f'(z)} \right) + (1 - \alpha) \left(\frac{z f'(z)}{f(z)} \right) \prec \tilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2} \qquad (13)$$

and

$$\alpha \left(1 + \frac{w g''(w)}{g'(w)} \right) + (1 - \alpha) \left(\frac{w g'(w)}{g(w)} \right) \prec \tilde{p}(w) = \frac{1 + \tau^2 w^2}{1 - \tau w - \tau^2 w^2} \qquad (14)$$

where $\tau = (1 - \sqrt{5})/2 \approx -0.618$ where $z, w \in \mathbb{U}$ and g is given by (2).

Specializing the parameter $\alpha = 0$ and $\alpha = 1$ we have the following, respectively:

Definition 6 *A function* $f \in \Sigma$ *of the form* (1) *is said to be in the class* $\mathcal{SL}_{\Sigma}(\tilde{p}(z))$ *if the following subordination hold:*

$$\frac{zf'(z)}{f(z)} \prec \tilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2} \tag{15}$$

and

$$\frac{wg'(w)}{g(w)} \prec \tilde{p}(w) = \frac{1+\tau^2 w^2}{1-\tau w-\tau^2 w^2} \tag{16}$$

where $\tau = (1-\sqrt{5})/2 \approx -0.618$ *where* $z, w \in \mathbb{U}$ *and* g *is given by* (2).

Definition 7 *A function* $f \in \Sigma$ *of the form* (1) *is said to be in the class* $\mathcal{KL}_{\Sigma}(\tilde{p}(z))$ *if the following subordination hold:*

$$1+\frac{zf''(z)}{f'(z)} \prec \tilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2} \tag{17}$$

and

$$1+\frac{wg''(w)}{g'(w)} \prec \tilde{p}(w) = \frac{1+\tau^2 w^2}{1-\tau w-\tau^2 w^2} \tag{18}$$

where $\tau = (1-\sqrt{5})/2 \approx -0.618$ *where* $z, w \in \mathbb{U}$ *and* g *is given by* (2).

In the following theorem we determine the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function class $\mathcal{SLM}_{\alpha,\Sigma}(\tilde{p}(z))$. Later we will reduce these bounds to other classes for special cases.

Theorem 2 *Let* f *given by* (1) *be in the class* $\mathcal{SLM}_{\alpha,\Sigma}(\tilde{p}(z))$. *Then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{(1+\alpha)^2-(1+\alpha)(2+3\alpha)\tau}} \tag{19}$$

and

$$|a_3| \leq \frac{|\tau|\left[(1+\alpha)^2-(3\alpha^2+9\alpha+4)\tau\right]}{2(1+2\alpha)(1+\alpha)\left[(1+\alpha)-(2+3\alpha)\tau\right]}. \tag{20}$$

Proof. Let $f \in \mathcal{SLM}_{\alpha,\Sigma}(\tilde{p}(z))$ and $g = f^{-1}$. Considering (13) and (14), we have

$$\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right) = \tilde{p}(u(z)) \tag{21}$$

and

$$\alpha\left(1+\frac{wg''(w)}{g'(w)}\right)+(1-\alpha)\left(\frac{wg'(w)}{g(w)}\right)=\tilde{p}(v(w)) \tag{22}$$

where $\tau=(1-\sqrt{5})/2\approx-0.618$ where $z,w\in\mathbb{U}$ and g is given by (2). Since

$$\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right)=1+(1+\alpha)a_2z+(2(1+2\alpha)a_3 - (1+3\alpha)a_2^2)z^2+\cdots$$

and

$$\alpha\left(1+\frac{wg''(w)}{g'(w)}\right)+(1-\alpha)\left(\frac{wg'(w)}{g(w)}\right)=1-(1+\alpha)a_2w+((3+5\alpha)a_2^2 - 2(1+2\alpha)a_3)w^2+\cdots$$

Thus we have

$$\begin{aligned}&1+(1+\alpha)a_2z+(2(1+2\alpha)a_3-(1+3\alpha)a_2^2)z^2+\Delta\Delta\Delta\\&=1+\frac{\tilde{p}_1c_1z}{2}+\left[\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)\tilde{p}_1+\frac{c_1^2}{4}\tilde{p}_2\right]z^2\\&\quad+\left[\frac{1}{2}\left(c_3-c_1c_2+\frac{c_1^3}{4}\right)\tilde{p}_1+\frac{1}{2}c_1\left(c_2-\frac{c_1^2}{2}\right)\tilde{p}_2+\frac{c_1^3}{8}\tilde{p}_3\right]z^3+\cdots.\end{aligned} \tag{23}$$

and

$$\begin{aligned}&1-(1+\alpha)a_2w+((3+5\alpha)a_2^2-2(1+2\alpha)a_3)w^2+\Delta\Delta\Delta,\\&=1+\frac{\tilde{p}_1d_1w}{2}+\left[\frac{1}{2}\left(d_2-\frac{d_1^2}{2}\right)\tilde{p}_1+\frac{d_1^2}{4}\tilde{p}_2\right]w^2\\&\quad+\left[\frac{1}{2}\left(d_3-d_1d_2+\frac{d_1^3}{4}\right)\tilde{p}_1+\frac{1}{2}d_1\left(d_2-\frac{d_1^2}{2}\right)\tilde{p}_2+\frac{d_1^3}{8}\tilde{p}_3\right]w^3+\cdots.\end{aligned} \tag{24}$$

It follows from (23) and (24) that

$$(1+\alpha)a_2=\frac{c_1\tau}{2}, \tag{25}$$

$$2(1+2\alpha)a_3-(1+3\alpha)a_2^2=\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)\tau+\frac{c_1^2}{4}3\tau^2, \tag{26}$$

and

$$-(1+\alpha)a_2=\frac{d_1\tau}{2}, \tag{27}$$

$$(3+5\alpha)a_2^2-2(1+2\alpha)a_3=\frac{1}{2}\left(d_2-\frac{d_1^2}{2}\right)\tau+\frac{d_1^2}{4}3\tau^2. \tag{28}$$

From (25) and (27), we have

$$c_1 = -d_1, \tag{29}$$

and

$$2a_2^2 = \frac{(c_1^2 + d_1^2)}{4(1+\alpha)^2}\tau^2. \tag{30}$$

Now, by summing (26) and (28), we obtain

$$2(1+\alpha)a_2^2 = \frac{1}{2}(c_2+d_2)\tau - \frac{1}{4}(c_1^2+d_1^2)\tau + \frac{3}{4}(c_1^2+d_1^2)\tau^2. \tag{31}$$

By putting (30) in (31), we have

$$2(1+\alpha)\left[(-2-3\alpha)\tau + (1+\alpha)\right] a_2^2 = \frac{1}{2}(c_2+d_2)\tau^2. \tag{32}$$

Therefore, using Lemma (1) we obtain

$$|a_2| \leq \frac{|\tau|}{\sqrt{(1+\alpha)^2 - (1+\alpha)(2+3\alpha)\tau}}. \tag{33}$$

Now, so as to find the bound on $|a_3|$, let's subtract from (26) and (28). So, we find

$$4(1+2\alpha)a_3 - 4(1+2\alpha)a_2^2 = \frac{1}{2}\left(c_2 - d_2\right)\tau. \tag{34}$$

Hence, we get

$$4(1+2\alpha)|a_3| \leq 2|\tau| + 4(1+2\alpha)|a_2|^2. \tag{35}$$

Then, in view of (33), we obtain

$$|a_3| \leq \frac{|\tau|\left[(1+\alpha)^2 - (3\alpha^2+9\alpha+4)\tau\right]}{2(1+2\alpha)(1+\alpha)\left[(1+\alpha)-(2+3\alpha)\tau\right]}. \tag{36}$$

□

If we can take the parameter $\alpha = 0$ and $\alpha = 1$ in the above theorem, we have the following the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function classes $\mathcal{SL}_\Sigma(\tilde{p}(z))$ and $\mathcal{KSL}_\Sigma(\tilde{p}(z))$, respectively.

Corollary 1 *Let* f *given by* (1) *be in the class* $\mathcal{SL}_\Sigma(\tilde{p}(z))$. *Then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{1-2\tau}} \tag{37}$$

and

$$|a_3| \leq \frac{|\tau|(1-4\tau)}{2(1-2\tau)}. \tag{38}$$

Corollary 2 *Let* f *given by* (1) *be in the class* $\mathcal{KSL}_\Sigma(\tilde{p}(z))$. *Then*

$$|a_2| \le \frac{|\tau|}{\sqrt{4-10\tau}} \tag{39}$$

and

$$|a_3| \le \frac{|\tau|(1-4\tau)}{3(2-5\tau)}. \tag{40}$$

3 Bi-univalent function class $\mathcal{SLG}_{\gamma,\Sigma}(\tilde{p}(z))$

In this section, we define a new class $\mathcal{SLG}_{\gamma,\Sigma}(\tilde{p}(z))$ of $\gamma-$ bi-starlike functions associated with Shell-like domain.

Definition 8 For $\gamma \ge 0$, we let a function $f \in \Sigma$ given by (1) is said to be in the class $\mathcal{SLG}_{\gamma,\Sigma}(\tilde{p}(z))$, if the following conditions are satisfied:

$$\left(\frac{zf'(z)}{f(z)}\right)^{\gamma}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\gamma} \prec \tilde{p}(z) = \frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2} \tag{41}$$

and

$$\left(\frac{wg'(w)}{g(w)}\right)^{\gamma}\left(1+\frac{wg''(w)}{g'(w)}\right)^{1-\gamma} \prec \tilde{p}(w) = \frac{1+\tau^2w^2}{1-\tau z-\tau^2w^2}, \tag{42}$$

where $\tau = (1-\sqrt{5})/2 \approx -0.618$ where $z, w \in \mathbb{U}$ and g is given by (2).

Remark 1 Taking $\gamma = 1$, we get $\mathcal{SLG}_{1,\Sigma}(\tilde{p}(z)) \equiv \mathcal{SL}_\Sigma(\tilde{p}(z))$ the class as given in Definition 5 satisfying the conditions given in (15) and (16).

Remark 2 Taking $\gamma = 0$, we get $\mathcal{SLG}_{0,\Sigma}(\tilde{p}(z)) \equiv \mathcal{KL}_\Sigma(\tilde{p}(z))$the class as given in Definition 6 satisfying the conditions given in (17) and (18).

Theorem 3 Let f given by (1) be in the class $\mathcal{SLG}_{\gamma,\Sigma}(\tilde{p}(z))$. Then

$$|a_2| \le \frac{\sqrt{2}|\tau|}{\sqrt{2(2-\gamma)^2-(5\gamma^2-21\gamma+20)\tau}}$$

and

$$|a_3| \le \frac{|\tau|\left[2(2-\gamma)^2-(5\gamma^2-29\gamma+32)\tau\right]}{2(3-2\gamma)\left[2(2-\gamma)^2-(5\gamma^2-21\gamma+20)\tau\right]}.$$

Proof. Let $f \in \mathcal{SLG}_{\gamma,\Sigma}(\tilde{p}(z))$ and $g = f^{-1}$ given by (2) Considering (41) and (42), we have

$$\left(\frac{zf'(z)}{f(z)}\right)^{\gamma}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\gamma} \prec \tilde{p}(u(z)) \tag{43}$$

and

$$\left(\frac{wg'(w)}{g(w)}\right)^{\gamma}\left(1+\frac{wg''(w)}{g'(w)}\right)^{1-\gamma} \prec \tilde{p}(v(w)) \tag{44}$$

where $\tau = (1-\sqrt{5})/2 \approx -0.618$ where $z, w \in \mathbb{U}$ and g is given by (2). Since,

$$\begin{aligned}\left(\frac{zf'(z)}{f(z)}\right)^{\gamma}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\gamma} &= 1+(2-\gamma)a_2 z \\ + \left(2(3-2\gamma)a_3 + \frac{1}{2}[(\gamma-2)^2 - 3(4-3\gamma)]a_2^2\right) z^2 + \cdots &\prec \tilde{p}(u(z))\end{aligned} \tag{45}$$

and

$$\begin{aligned}\left(\frac{wg'(w)}{g(w)}\right)^{\gamma}\left(1+\frac{wg''(w)}{g'(w)}\right)^{1-\gamma} &= 1-(2-\gamma)a_2 w \\ + \left([8(1-\gamma) + \frac{1}{2}\gamma(\gamma+5)]a_2^2 - 2(3-2\gamma)a_3\right) w^2 + \cdots &\prec \tilde{p}(v(w)).\end{aligned} \tag{46}$$

Equating the coefficients in(45) and (46), with (9) and (12) respectively we get,

$$(2-\gamma)a_2 = \frac{c_1\tau}{2}, \tag{47}$$

$$2(3-2\gamma)a_3 + \frac{1}{2}[(\gamma-2)^2 - 3(4-3\gamma)]a_2^2 = \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)\tau + \frac{c_1^2}{4}3\tau^2, \tag{48}$$

and

$$-(2-\gamma)a_2 = \frac{d_1\tau}{2}, \tag{49}$$

$$-2(3-2\gamma)a_3 + [8(1-\gamma) + \frac{1}{2}\gamma(\gamma+5)]a_2^2 = \frac{1}{2}\left(d_2 - \frac{d_1^2}{2}\right)\tau + \frac{d_1^2}{4}3\tau^2. \tag{50}$$

From (47) and (49), we have

$$a_2 = \frac{c_1\tau}{2(2-\gamma)} = -\frac{d_1\tau}{2(2-\gamma)}, \tag{51}$$

which implies

$$c_1 = -d_1 \tag{52}$$

and

$$a_2^2 = \frac{(c_1^2 + d_1^2)\tau^2}{8(2-\gamma)^2}. \tag{53}$$

Now, by summing (48) and (50), we obtain

$$(\gamma^2 - 3\gamma + 4)a_2^2 = \frac{1}{2}(c_2 + d_2)\tau - \frac{1}{4}(c_1^2 + d_1^2)\tau + \frac{3}{4}(c_1^2 + d_1^2)\tau^2. \tag{54}$$

Proceeding similarly as in the earlier proof of Theorem 2, using Lemma (1) we obtain

$$|a_2| \le \frac{\sqrt{2}|\tau|}{\sqrt{2(2-\gamma)^2 - (5\gamma^2 - 21\gamma + 20)\tau}}. \tag{55}$$

Now, so as to find the bound on $|a_3|$, let's subtract from (48) and (50). So, we find

$$4(3-2\gamma)a_3 - 4(3-2\gamma)a_2^2 = \frac{1}{2}\,(c_2 - d_2)\,\tau. \tag{56}$$

Hence, we get

$$4(1+2\gamma)|a_3| \le 2|\tau| + 4(1+2\gamma)|a_2|^2. \tag{57}$$

Then, in view of (55), we obtain

$$|a_3| \le \frac{|\tau|\left[2(2-\gamma)^2 - (5\gamma^2 - 29\gamma + 32)\tau\right]}{2(3-2\gamma)\left[2(2-\gamma)^2 - (5\gamma^2 - 21\gamma + 20)\tau\right]}. \tag{58}$$

□

Remark 3 *By taking $\gamma = 1$ and $\gamma = 0$ in the above theorem, we have the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function classes $\mathcal{SL}_\Sigma(\tilde{p}(z))$ and $\mathcal{KSL}_\Sigma(\tilde{p}(z))$, as stated in Corollary 1 and Corollary 2 respectively.*

4 Fekete-Szegö inequalities for the function classes $\mathcal{SLM}_{\alpha,\Sigma}(\tilde{p}(z))$ and $\mathcal{SLG}_{\gamma,\Sigma}(\tilde{p}(z))$

Fekete and Szegö [6] introduced the generalized functional $|a_3 - \mu a_2^2|$, where μ is some real number. Due to Zaprawa [15], in the following theorem we determine the Fekete-Szegö functional for $f \in \mathcal{SLM}_{\alpha,\Sigma}(\tilde{p}(z))$.

Theorem 4 *Let* f *given by (1) be in the class* $\mathcal{SLM}_{\alpha,\Sigma}(\tilde{p}(z))$ *and* $\mu \in \mathbb{R}$. *Then we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\tau|}{2(1+2\alpha)}, & |\mu - 1| \leq \frac{(1+\alpha)[(1+\alpha)-(2+3\alpha)\tau]}{2(1+2\alpha)|\tau|} \\ \frac{|1-\mu|\tau^2}{(1+\alpha)[(1+\alpha)-(2+3\alpha)\tau]}, & |\mu - 1| \geq \frac{(1+\alpha)[(1+\alpha)-(2+3\alpha)\tau]}{2(1+2\alpha)|\tau|} \end{cases}.$$

Proof. From (32) and (34)we obtain

$$\begin{aligned} a_3 - \mu a_2^2 &= (1-\mu)\frac{\tau^2(c_2+d_2)}{4(1+\alpha)\left[(1+\alpha)-(2+3\alpha)\tau\right]} + \frac{\tau(c_2-d_2)}{8(1+2\alpha)} \\ &= \left(\frac{(1-\mu)\tau^2}{4(1+\alpha)\left[(1+\alpha)-(2+3\alpha)\tau\right]} + \frac{\tau}{8(1+2\alpha)}\right)c_2 \qquad (59) \\ &\quad + \left(\frac{(1-\mu)\tau^2}{4(1+\alpha)\left[(1+\alpha)-(2+3\alpha)\tau\right]} - \frac{\tau}{8(1+2\alpha)}\right)d_2. \end{aligned}$$

So we have

$$a_3 - \mu a_2^2 = \left(h(\mu) + \frac{|\tau|}{8(1+2\alpha)}\right)c_2 + \left(h(\mu) - \frac{|\tau|}{8(1+2\alpha)}\right)d_2 \qquad (60)$$

where

$$h(\mu) = \frac{(1-\mu)\tau^2}{4(1+\alpha)\left[(1+\alpha)-(2+3\alpha)\tau\right]}. \qquad (61)$$

Then, by taking modulus of (60), we conclude that

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\tau|}{2(1+2\alpha)}, & 0 \leq |h(\mu)| \leq \frac{|\tau|}{8(1+2\alpha)} \\ 4|h(\mu)|, & |h(\mu)| \geq \frac{|\tau|}{8(1+2\alpha)} \end{cases}.$$

Taking $\mu = 1$, we have the following corollary.

Corollary 3 *If* $f \in \mathcal{SLM}_{\alpha,\Sigma}(\tilde{p}(z))$, *then*

$$|a_3 - a_2^2| \leq \frac{|\tau|}{2(1+2\alpha)}. \qquad (62)$$

If we can take the parameter $\alpha = 0$ and $\alpha = 1$ in the above theorem, we have the following the Fekete-Szegö inequalities for the function classes $\mathcal{SL}_\Sigma(\tilde{p}(z))$ and $\mathcal{KSL}_\Sigma(\tilde{p}(z))$, respectively.

Corollary 4 *Let* f *given by (1) be in the class* $\mathcal{SL}_\Sigma(\tilde{p}(z))$ *and* $\mu \in \mathbb{R}$. *Then we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\tau|}{2}, & |\mu - 1| \leq \frac{1-2\tau}{2|\tau|} \\ \frac{|1-\mu|\tau^2}{1-2\tau}, & |\mu - 1| \geq \frac{1-2\tau}{2|\tau|} \end{cases}.$$

Corollary 5 *Let* f *given by (1) be in the class* $\mathcal{KSL}_\Sigma(\tilde{p}(z))$ *and* $\mu \in \mathbb{R}$. *Then we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\tau|}{6}, & |\mu - 1| \leq \frac{2-5\tau}{3|\tau|} \\ \frac{|1-\mu|\tau^2}{2(2-5\tau)}, & |\mu - 1| \geq \frac{2-5\tau}{3|\tau|} \end{cases}.$$

In the following theorem, we find the Fekete-Szegö functional for $f \in \mathcal{SLG}_{\gamma,\Sigma}(\tilde{p}(z))$.

Theorem 5 *Let* f *given by (1) be in the class* $\mathcal{SLG}_{\gamma,\Sigma}(\tilde{p}(z))$ *and* $\mu \in \mathbb{R}$. *Then we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\tau|}{2(3-2\gamma)}, & |\mu - 1| \leq \frac{2(2-\gamma)^2+(-5\gamma^2+21\gamma-20)\tau}{4(3-2\gamma)|\tau|} \\ \frac{2|1-\mu|\tau^2}{2(2-\gamma)^2+[(-5\gamma^2+21\gamma-20)\tau]}, & |\mu - 1| \geq \frac{2(2-\gamma)^2+(-5\gamma^2+21\gamma-20)\tau}{4(3-2\gamma)|\tau|} \end{cases}.$$

Taking $\mu = 1$, we have the following corollary.

Corollary 6 *If* $f \in \mathcal{SLG}_{\gamma,\Sigma}(\tilde{p}(z))$, *then*

$$|a_3 - a_2^2| \leq \frac{|\tau|}{2(3-2\gamma)}. \tag{63}$$

By taking $\gamma = 1$ and $\gamma = 0$ in the above theorem, we have the Fekete-Szegö inequality for the function classes $\mathcal{SL}_\Sigma(\tilde{p}(z))$ and $\mathcal{KSL}_\Sigma(\tilde{p}(z))$, as stated in Corollary 4 and Corollary 5, respectively.

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