On the sum of the Lah numbers and zeros of the Kummer confluent hypergeometric function

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Abstract. In the paper, the authors find the sum of the Lah numbers and make sure that the Kummer confluent hypergeometric function \( _1F_1(n + 1; 2; z) \) has only \( n - 1 \) real and negative zeros.

1 Notation and main results

In combinatorics, the Bell numbers, usually denoted by \( B_n \) for \( n \in \{0\} \cup \mathbb{N} \), count the number of ways a set with \( n \) elements can be partitioned into disjoint

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and nonempty subsets. These numbers have been studied by mathematicians since the 19th century, and their roots go back to medieval Japan, but they are named after Eric Temple Bell, who wrote about them in the 1930s. Every Bell number $B_n$ can be generated by

$$e^{e^x-1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k$$

or, equivalently, by

$$e^{e^{-x} - 1} = \sum_{k=0}^{\infty} (-1)^k \frac{B_k}{k!} x^k.$$  

In combinatorics, the Stirling numbers arise in a variety of combinatorics problems. They are introduced in the eighteenth century by James Stirling. There are two kinds of the Stirling numbers: the Stirling numbers of the first and second kinds. Every Stirling number of the second kind, usually denoted by $S(n,k)$, is the number of ways of partitioning a set of $n$ elements into $k$ nonempty subsets, can be computed by

$$S(n,k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n,$$

and can be generated by

$$\frac{(e^x - 1)^k}{k!} = \sum_{n=k}^{\infty} S(n,k) \frac{x^n}{n!}, \quad k \in \{0\} \cup \mathbb{N}.$$  

In combinatorics, the Lah numbers, discovered by Ivo Lah in 1955 and usually denoted by $L(n,k)$, count the number of ways a set of $n$ elements can be partitioned into $k$ nonempty linearly ordered subsets and have an explicit formula

$$L(n,k) = \binom{n-1}{k-1} \frac{n!}{k!}.$$  

The Lah numbers $L(n,k)$ can also be interpreted as coefficients expressing rising factorials $(x)_n$ in terms of falling factorials $\langle x \rangle_n$, where

$$(x)_n = \begin{cases} x(x+1)(x+2)\ldots(x+n-1), & n \geq 1, \\ 1, & n = 0 \end{cases}$$
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\[ \langle x \rangle_n = \begin{cases} 
  x(x-1)(x-2) \ldots (x-n+1), & n \geq 1, \\
  1, & n = 0. 
\end{cases} \]

In combinatorics and the theory of polynomials, the partial Bell polynomials \( B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) \) for \( n \geq k \geq 0 \) can be defined by

\[
B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum_{1 \leq i_1 \leq n, \ell_i \in \{0, n\}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} x_i^{\ell_i}
\]

and satisfy

\[ B_{n,k}(1!, 2!, \ldots, (n-k+1)!) = L(n, k). \quad (1) \]

The complete Bell polynomials \( Y_n(x_1, x_2, \ldots, x_n) \) are defined [3, p. 134] by

\[
Y_n(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{n} B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) \quad (2)
\]

and

\[ Y_0(x_1, x_2, \ldots, x_n) = 1. \quad (3) \]

In the theory of special functions, the generalized hypergeometric series

\[
pFq(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}
\]

is defined for complex numbers \( a_i \in \mathbb{C} \) and \( b_i \in \mathbb{C} \setminus \{0, -1, -2, \ldots\} \) and for positive integers \( p, q \in \mathbb{N} \). The generalized hypergeometric series \( pFq(a_1, \ldots, a_p; b_1, \ldots, b_q; z) \) converges absolutely for all \( z \in \mathbb{C} \) if \( p \leq q \), for \( |z| < 1 \) if \( p = q+1 \), and for \( |z| = 1 \) if \( p = q+1 \) and \( \Re[b_1 + \cdots + b_q - (a_1 + \cdots + a_p)] > 0 \). Specially, the series

\[
_1F_1(a; b; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!}
\]

is called the Kummer confluent hypergeometric function and it is analytic for all \( z \in \mathbb{C} \). See [4, pp. 3–5].

In [5] and [7], two explicit formulas for the Bell numbers \( B_n \) in terms of the Stirling numbers of the second kind \( S(n, k) \) together with the Kummer confluent hypergeometric function \( _1F_1(k+1; 1; 1) \) and the Lah numbers \( L(n, k) \) are given.
respectively were established as follows. For $n \in \mathbb{N}$, the Bell numbers $B_n$ can be expressed as

$$B_n = \frac{1}{e} \sum_{k=1}^{n} (-1)^{n-k} \frac{1}{k!} F_1(k+1;2;1) k! S(n,k)$$  \hspace{1cm} (4)$$

and

$$B_n = \sum_{k=1}^{n} (-1)^{n-k} \left[ \sum_{\ell=1}^{k} L(k,\ell) \right] S(n,k).$$  \hspace{1cm} (5)$$

Comparing the formulas (4) with (5) motivates us to conjecture that

$$\frac{k!}{e} F_1(k+1;2;1) = \sum_{\ell=1}^{k} L(k,\ell), \quad k \in \mathbb{N}.$$  \hspace{1cm} (6)$$

With the help of the famous software MATHEMATICA 9, we can verify that the equality (6) holds true for $1 \leq k \leq 9$ and they equal the following values respectively:

$$e, \quad \frac{3}{2} e, \quad \frac{13}{6} e, \quad \frac{73}{24} e, \quad \frac{167}{40} e, \quad \frac{4051}{720} e, \quad \frac{37633}{5040} e, \quad \frac{43817}{4480} e, \quad \frac{4596553}{362880} e.$$ 

This hints us that the above conjecture is true.

The aim of this paper is to prove a more general conclusion than the above conjecture. This general conclusion can be restated as the following theorems.

**Theorem 1** For $z \in \mathbb{C}$ and $n \in \mathbb{N}$, the formula

$$\sum_{k=1}^{n} L(n,k) z^{k-1} = \frac{n!}{e^z} F_1(n+1;2;z)$$  \hspace{1cm} (7)$$

is true. Specially, for $n \in \mathbb{N}$, the Lah number $L(n,k)$ and the complete Bell polynomials $Y_n(x_1,x_2,\ldots,x_n)$ satisfy

$$\sum_{k=1}^{n} L(n,k) = \frac{n!}{e} F_1(n+1;2;1)$$  \hspace{1cm} (8)$$

and

$$Y_n(1!,2!,\ldots,n!) = \frac{n!}{e} F_1(n+1;2;1)$$  \hspace{1cm} (9)$$

respectively.
Theorem 2  The Kummer confluent hypergeometric function \( _1F_1(n + 1; 2; z) \) has only \( n - 1 \) real and negative zeros.

Remark 1  The equations in (4) can be rewritten as

\[
\sum_{k=1}^{n} (-1)^{n-k} a_k S(n, k) = B_n,
\]

where \( a_k \) is sequence \( A000262 \) in the Online Encyclopedia of Integer Sequences. Such a sequence \( a_k \) has a nice combinatorial interpretation: it counts “the sets of lists, or the number of partitions of \( \{1, 2, \ldots, k\} \) into any number of lists, where a list means an ordered subset.” This reveals the combinatorial interpretation of the special sequence \( k!_1F_1(k + 1; 2; 1) \) and the total sum \( L_k = \sum_{\ell=1}^{k} L(k, \ell) \) of the Lah numbers \( L(k, \ell) \).

2  Proofs of theorems

We now start out to prove Theorems 1 and 2.

Proof.  [Proof of Theorem 1]  It is easy to see that the equality (9) follows from substituting (1) into (8) and making use of (2) and (3). Hence, in what follows, we pay our attention to the proof of the formula (7).

In [6, p. 79, Theorem 2.1], we obtained

\[
\sum_{k=1}^{n} L(n, k)x^k = \frac{e^{-x}}{x^n} \int_{0}^{\infty} I_1(2\sqrt{t})t^{n-1/2}e^{-t/x} dt
\]

(10)

for \( n \in \mathbb{N} \) and \( x > 0 \), where the modified Bessel function of the first kind \( I_\nu(z) \) can be defined by

\[
I_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(\nu + k + 1)} \left( \frac{z}{2} \right)^{2k+\nu}
\]

(11)

for \( \nu \in \mathbb{R} \) and \( z \in \mathbb{C} \). See [1, p. 375, 9.6.10]. Substituting (11) for \( \nu = 1 \)
into (10) and straightforward computing arrive at

\[
\sum_{k=1}^{n} L(n,k)x^k = \frac{e^{-x}}{x^n} \int_{0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{k!(k+1)!} t^{n+k}e^{-t/x} \, dt
\]

\[
= \frac{e^{-x}}{x^n} \sum_{k=0}^{\infty} \frac{1}{k!(k+1)!} \int_{0}^{\infty} t^{n+k}e^{-t/x} \, dt
\]

\[
= \frac{e^{-x}}{x^n} \sum_{k=0}^{\infty} \frac{1}{k!(k+1)!} x^{n+k+1} \Gamma(n+k+1)
\]

\[
= e^{-x} \sum_{k=0}^{\infty} \frac{(n+k)!}{k!(k+1)!} x^{k+1}
\]

\[
= n!xe^{-x} \sum_{k=0}^{\infty} \frac{(n+1)_k x^k}{(2)_k k!}
\]

\[
= n!xe^{-x} \text{F}_1(n+1;2;x).
\]

Therefore, it follows that

\[
\sum_{k=1}^{n} L(n,k)x^{k-1} = \frac{n!}{e^x} \text{F}_1(n+1;2;x)
\]  

(12)

for \(x > 0\) and \(n \in \mathbb{N}\).

Since the functions

\[
\sum_{k=1}^{n} L(n,k)z^{k-1}
\]

and

\[
\frac{n!}{e^z} \text{F}_1(n+1;2;z)
\]

are entire functions, that is, they are analytic on the whole complex plane \(\mathbb{C}\), by the uniqueness theorem of analytic functions in the theory of complex functions, see [17, p. 210, Corollary], and by the formula (12), we easily derive the formula (7) for \(z \in \mathbb{C}\) and \(n \in \mathbb{N}\). The proof of Theorem 1 is complete. \(\square\)

\textbf{Proof.} [Proof of Theorem 2] In [2, Lemma], the authors stated that if

\[
P_{m,k}(x) = \sum_{n=1}^{m} L_k(m,n)x^n,
\]

then the \(m\) roots of \(P_{m,k}(x)\) are real, distinct, and non-positive for all \(m \in \mathbb{N}\), where the associated Lah numbers \(L_k(m,n)\) for \(k > 0\) can be defined by

\[
L_k(m,n) = \frac{m!}{n!} \sum_{r=1}^{n} (-1)^{n-r} \binom{n}{r} \binom{m+rk-1}{m}
\]
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and \( L_k(m, n) = 0 \) for \( n > m \). Since \( L_1(m, n) = L(m, n) \), see [2, p. 158, Eq. (4)], when \( k = 1 \), the polynomial \( P_{m,1}(x) \) becomes

\[
P_{m,1}(x) = \sum_{n=1}^{m} L(m, n)x^n.
\]

The formula (7) implies that the integer polynomial \( \frac{P_{m,1}(x)}{x} \) have the same zeros as the Kummer confluent hypergeometric function \( \text{$_1F_1(n+1; 2; z)$} \). Since the Kummer confluent hypergeometric function \( \text{$_1F_1(n+1; 2; z)$} \) has no positive zero, the zeros of \( P_{m,1}(x) \) are non-positive, and then the Kummer confluent hypergeometric function \( \text{$_1F_1(n+1; 2; z)$} \) has only \( n-1 \) real and negative zeros. The proof of Theorem 2 is complete.

**Remark 2** The formula (5) has been generalized by R. B. Corcinoy, J. T. Malusay, J. A. Cillar, G. J. Rama, O. V. Silang, and I. M. Tacoloy in Philippines. There are more new results in [12] and [13, Section 5] for the Bell numbers \( B_n \).

**Remark 3** There are some new and closely related results published in [9, 10, 11, 14, 15, 16, 18] and references cited therein.

**Remark 4** This paper is a revised version of the preprint [8].

**References**


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