On vertex independence number of uniform hypergraphs

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Abstract. Let H be an r-uniform hypergraph with r \geq 2 and let \( \alpha(H) \) be its vertex independence number. In the paper bounds of \( \alpha(H) \) are given for different uniform hypergraphs: if H has no isolated vertex, then in terms of the degrees, and for triangle-free linear H in terms of the order and average degree.

1 Introduction to independence in graphs

Let \( n \) be a positive integer. A graph \( G \) on vertex set \( V = \{v_1, v_2, \ldots, v_n\} \) is a pair \( (V, E) \), where the edge set \( E \) is a subset of \( V \times V \). \( n \) is the order of \( G \) and \( |E| \) is the size of \( G \).
Let $v \in V$ and $N(v)$ be the neighborhood of $v$, namely, the set of vertices $x$ so that there is an edge which contains both $v$ and $x$. Let $U$ be a subset of $V$, then the subgraph of $G$ induced by $U$ is defined as a graph on vertex set $U$ and edge set $E_U = \{(u, v) | u \in U$ and $v \in U\}$.

The degree $d(v)$ of a vertex $v \in V$ is the number of edges that contains $v$. Let $d(G)$ be the average degree of $G$, then $nd(G) = \sum_{v \in V} d(v) = 2|E|$ for any graph $G$. Let $\delta(G)$ be the minimal degree, $\Delta(G)$ the maximal degree of $G$ A graph $G$ is regular, if $\Delta(G) = \delta(G)$, and it is semi-regular, if $\Delta(G) - \delta(G) = 1$.

Three vertices $v_1, v_2, v_3$ form a triangle in $G$ if there are distinct vertices $e_1, v_2, v_3 \in F$ such that $\{v_i, v_{i+1}\} \subseteq E$, where the indices are taken mod 3. If $G$ does not contain a triangle, then it is trianglefree.

A subset $U \subseteq V$ of vertices in a graph $G$ is called a vertex independent set if no two vertices in $U$ are adjacent. The maximum-size vertex independent set is called maximum vertex independent set. The size of the maximum vertex independent set is called vertex independence number and is denoted by $\alpha(G)$. The problem of finding a vertex maximum independent set and vertex independence number are NP-hard optimization problems [73, 167].

A maximal vertex independent set is a vertex independent set such that adding any other vertex to the set forces the set to contain an edge. The problem of finding a maximal vertex independent set can be solved in polynomial time (see e.g. the algorithms due to Tarjan and Trojanowski [155], Karp and Widgerson [101], further the improved algorithms due to Luby [128] and Alon [9].

There are exponential time exact (as Alon [9]) and polynomial time approximate algorithms (as Boppana and Haldórsson [30], Agnarsson, Haldórsson, and Losievskaja [4, 5], Losievskaja [126]) determining $\alpha(G)$. Also there are known algorithms producing the list of all maximum independent sets of graphs (see e.g. Johnson and Yannakakis [93], Lawler, Lenstra, Rinnooy Kan [121]).

An independent edge set of a graph $G$ is a subset of the edges such that no two edges in the subset share a vertex of $G$ [166]. An independent edge set of maximum size is called a maximum independent edge set, and an independent edge set that cannot be expanded to another independent edge set by addition of any other edge in the graph is called a maximal independent edge set. The size of the largest independent edge set (i.e., of any maximum independent edge set) in a graph is known as its edge independence number (or matching number), and is denoted by $\nu(G)$. The determination of $\nu(G)$ is an easy task for bipartite graphs [49, 50], but it is a polynomially solvable problem for general graphs too [10, 101, 161, 162].

Let $G = (V, E)$ be an $n$-order graph. The classical Turán theorem [159] gives
a simple lower bound for $\alpha(G)$.

**Theorem 1** (Turán [159]) If $n \geq 1$ and $G$ is an $n$-order graph, then

$$\alpha(G) \geq \frac{n}{d(G) + 1}. \quad (1)$$

This result was strengthened independently in 1979 by Caro and in 1981 by Wei.

**Theorem 2** (Caro [36], Wei, [165]) If $G(V, E)$ is a graph, then

$$\alpha(G) \geq \sum_{v \in V} \frac{1}{d(v) + 1}. \quad (2)$$

**Proof.** See [36, 165].

A nice probabilistic proof of the result can be found in the paper of Alon and Spencer [11]. Since the function $\frac{1}{x+1}$ is convex, $\sum_{v \in V} \frac{1}{d(v) + 1} \geq \frac{n}{d + 1}$ [170].

Since this bound is the best-possible only for graphs which are unions of cliques, additional structural assumptions excluding these graphs allow improvement of 2 [80, 81]. A natural candidate for such assumptions is connectivity. In 2013 Angel, Campigotto, and Laforest [14] improved (2) for some connected graphs. For locally sparse graphs Ajtai, Erdős, Komlós and Szemerédi improved Turán’s bound greatly.

**Theorem 3** (Ajtai, Erdős, Komlós and Szemerédi [6, 7, 8]) If $G$ is an $n$-order triangle-free graph with average degree $d$, then

$$\alpha(G) \geq cn \ln \frac{d}{d + 1}. \quad (3)$$

**Proof.** See [6, 7, 8].

They conjectured that $c = 1 - o(1)$ when $d$ tends to $\infty$. Griggs [72] improved that $c$ can be $\frac{5}{12}$. Shearer [152] finally proved $c = 1 - o(1)$, thus confirming the conjecture. In 1994 Selkow improved the bound due to Caro and Wei supposing that the degrees of the neighbors of the vertices are also known.

**Theorem 4** (Selkow [150]) If $G(V, E)$ is a graph, then

$$\alpha(G) \geq \sum_{v \in V} \frac{1}{d(v) + 1} \left(1 + \max \left(0, 1 \cdot \frac{d(v)}{d(v) + 1} - \sum_{u \in N(v)} \frac{1}{d(u) + 1}\right)\right). \quad (4)$$
Proof. See [150].

The bound of Selkow is equal to Caro–Wei bound for regular graph and always less then twice the Caro–Wei bound. A recent review on lower bounds for 3-order graphs was published by Henning and Yeo [89].

Let $j$ and $k$ be positive integers. A subset $I \subseteq V(G)$ is a vertex-$k$-independent set of $G$, if every vertex in $I$ has at most $k - 1$ neighbors in $I$. The vertex-$k$-independence number $\alpha_k(G)$ of $G$ is the cardinality of the largest vertex-$k$-independent set of $G$.

A subset $D \subseteq V(G)$ is a vertex-$j$-dominating set of $G$, if every vertex of $D$ has at least $j - 1$ neighbors in $D$. The vertex-$j$-dominance number $\gamma_j(G)$ of $G$ is the cardinality of the largest vertex-$j$-dominating set of $G$.


Last year Hansberg and Pepper [79] investigated the connection between $\alpha_k(G)$ and $\gamma_j(G)$. They proved the following theorems.

**Theorem 5** (Hansberg, Pepper [79]) If Let $G$ be an $n$-order graph, $j$, $k$ and $m$ be positive integers such that $m = j + k - 1$ and let $H_m$ and $G_m$ denote, respectively, the subgraphs induced by the vertices of degree at least $m$ and the vertices of degree at least $m$. Then

$$\alpha_k(H_m) + \gamma_j(G_m) \leq n$$

and

$$\alpha_k(G) + \gamma_j(G) \leq n(G).$$

**Proof.** See [79].

**Theorem 6** (Hansberg, Pepper [79]) Let $G$ be a connected $n$-order graph with maximum degree $\Delta$ and minimum degree $\delta \geq 1$. Then

$$\alpha_k(G) + \gamma_j(G) = n(G) \quad \text{and} \quad \alpha'_k(G) + \gamma'_j(G) = n(G)$$
for every pair of integers \( j, k \) and \( j', k' \) such that \( j + k - 1 = \delta \) and \( j' + k' - 1 = \Delta \) if and only if \( G \) is regular.

**Proof.** See [79].

**Theorem 7** (Hansberg, Pepper [79]) For any graph \( G \) the following two statements are equivalent:

\[
\gamma(G) + \alpha_\delta(G) = \nu(G) \tag{8}
\]

and

\[
G \text{ is regular or } \gamma(G) + \gamma_2(G) = \nu(G). \tag{9}
\]

**Proof.** See [79].

Spencer [153] also published some extension of Turán theorem.

In 2014 Henning, Löwenstein, Southey and Yeo [87] proved the following theorem, which is an improvement of the result due to Fajtlowicz [53].

**Theorem 8** (Henning et al. [87]) If \( G \) is a graph of order \( n \) and \( p \) is an integer, such that for every clique \( X \) in \( G \) there exists a vertex \( x \in X \) such, that \( d(x) < p - |X| \), then \( \alpha(G) \geq 2n/p \).

There are results on the independence number of random graphs (e.g. Balogh, Morris, Samotij [18] and Frieze [60], Henning, Löwenstein, Southey and Yeo [87], on the weighted independence number (see e.g. Halldórsson [75], Kako, Ono, Hirata, and Halldórsson [98], further Sakai, Mitsunori, and Yamazaki [149]), and on the enumeration of maximum independent sets (see e.g. Gaspers, Kratsch, and Liedloff [69]).

Let \( G(n, p) = (V, E) \) the random graph with vertex set \( V = \{v_1, \ldots, v_n\} \), \( p \), \( \alpha(G_{n,p}) \) denote the independence number of \( G_{n,p} \). In 1990 Frieze [60] proved, that if \( d = np \) and \( \epsilon > 0 \) is fixed, then with probability going to 1 as \( n \to \infty \)

\[
\left| \alpha(G_n, p) - \frac{2n(\ln d - \ln \ln d - \ln 2 + 1)}{d} \right| \leq \frac{\epsilon n}{d}, \tag{10}
\]

provided \( d_\epsilon \leq d = o(n) \), where \( d_\epsilon \) is some fixed constant and \( p \) is the join probability for each edge to be included in \( E \).

In 1983 Shearer proved the following lower bound.

**Theorem 9** (Shearer [152]) If \( G \) is triangle-free, then

\[
\alpha(G) \geq nf(d), \tag{11}
\]
where
\[ f(x) = \frac{x \ln x - x + 1}{(x - 1)^2}, \]
\[ f(0) = 1 \text{ and } f(1) = \frac{1}{2}. \]

According to the proof of Shearer for \( 0 < x < \infty \) hold \( 0 < f(d) < 1 \), \( f'(d) < 0 \)
and \( f''(d) < 0 \). Further \( f(x) \) satisfies the differential equation
\[ (x + 1)f(x) = (x + 1)d^2f'(x). \]

It is easy to see that
\[ \lim_{x \to \infty} \frac{f(x)}{x} = \frac{\ln x}{x}. \]

In 1995 Füredi [62] determined the number of different vertex maximal independent set in path graphs.

It is known [22] a minimum covering set of \( G \) is also a maximum vertex independent set of \( G \). Therefore we are interested in the results on dominating sets (see e.g. [41, 54, 79, 82, 143].

The structure of the paper is as follows. After this introduction in Section 2 we present a review of results connected with the vertex and edge independence number of hypergraphs, then in Section 3 a lower bound of \( \alpha(H) \) is presented for \( n \)-order \( r \)-uniform hypergraphs with average degree \( d(H) \), and finally in Section 4 a similar bound is proved for hypergraphs not containing isolated vertex.

2 Introduction to independence in hypergraphs

Let \( n \geq 1 \) and \( W = \{w_1, w_2, \ldots, w_n\} \) be a finite set called vertex set. A hypergraph \( H \) on vertex set \( W \) is a pair \((W, F)\), where the edge set \( F \) is a family of the elements of \( W \). We always assume that distinct edges are distinct as subsets. If each edge in \( F \) contains exactly \( r \geq 2 \) vertices, then \( H \) is \( r \)-uniform. So any graph \( G \) is a 2-uniform hypergraph.

Let \( w \in W \) and \( N(w) \) be the neighborhood of \( w \), namely, the set of vertices \( x \) so that there is an edge which contains both \( w \) and \( x \). Let \( U \) be a subset of \( W \). The sub-hypergraph of \( H \) induced by \( U \) is defined as a hypergraph on vertex set \( U \) with edge set \( F_U = \{f \in F : f \subseteq U\} \).

The degree \( d(w) \) of a vertex \( w \in W \) is the number of edges that contain \( w \). Let \( d(H) = d \) be the average degree of an \( r \)-uniform \( H \), then \( nd = \sum_{w \in W} d(w) = r|F| \).
For the simplicity we usually omit $G$ and $H$ as arguments of $d(H)$ and similar notations.

A hypergraph $H$ is linear, if any two edges of $H$ have at most one vertex in common. Note that a graph $G$ is always linear. Three vertices $w_1, w_2, w_3$ form a triangle in $H$, if there are distinct edges $f_1, f_2, f_3 \in F$ such that $\{f_i, f_{i+1}\} \subseteq F$, where the indices are taken mod 3.

A subset $U \subseteq W$ of vertices in a hypergraph $H$ is called a vertex independent set if no two vertices in $U$ are adjacent. The maximum-size vertex independent set of $H$ is called maximum vertex independent set. The size of the maximum vertex independent set is called vertex independence number and is denoted by $\alpha(H)$. The problem of finding a maximum vertex-independent set and vertex independence number are NP-hard optimization problems [73, 167].

There are exponential time exact (as Alon [9], Tarjan and Trojanowski [155]) and polynomial time approximate algorithms (as Boppana and Haldórsson [30], Agnarsson, Haldórsson, and Losievskaja [4, 5], Losievskaja [126]). Also there are known algorithms producing the list of all maximum independent sets of graphs (see e.g. Johnson and Yannakakis [93], Lawler, Lenstra, Rinnooy Kan [121]) and hypergraphs (see e.g. Kelsen [107]).

A maximal vertex independent set is a vertex independent set such that adding any other vertex to the set forces the set to contain an edge. The problem of finding a maximal vertex independent set can be solved in polynomial time (see e.g. the algorithms due to Tarjan and Trojanowski [155], Karp and Widgerson [101], further the improved algorithms due to Luby [128] and Noga [9]).

In 2012 Dutta, Mubayi, and Subramanian [48] gave new lower bond for the vertex independence number of sparse hypergraphs.

In 2013 Eustis devoted a PhD dissertation to the problems of hypergraph independence numbers [51, 52].

An independent edge set of a hypergraph $H$ is a subset of the edges such that no two edges in the subset share a vertex of $H$ [136]. An independent edge set of maximum size is called a maximum independent edge set, and an independent edge set that cannot be expanded to another independent edge set by addition of any other edge in the hypergraph is called a maximal independent edge set. The size of the largest independent edge set (i.e., of any maximum independent edge set) in a hypergraph is known as its edge independence number (or matching number), and is denoted by $\nu(H)$. The determination of $\nu(H)$ is an easy task for bipartite graphs [49, 50], but it is a polynomially solvable problem for general graphs too [10].

There are many results on the characterization of hypergraph score se-
quences and on their reconstruction (see e.g. [20, 110, 140, 171, 139, 164, 172]), on the enumeration of different hypergraphs (see e.g. [21, 47, 138, 144, 145]) and directed hypergraphs (see e.g. [15]).

An \( r \)-uniform hypergraph with \( n \) vertices is called \textit{complete}, if its set of edges has the cardinality \( \binom{n}{r} \). The \textit{complement} of an \( r \)-uniform hypergraph \( H \) is \( \overline{H} = (W, \overline{F}) \), if \( |F \cup \overline{F}| = \binom{n}{r} \) and \( |F \cap \overline{F}| = 0 \).

A set \( P \subseteq W \) is called an \textit{edge cover} of \( H \), if for any non-isolated vertex \( x \in W \) there exists an edge \( f_i \in P \) that \( x \in f_i \). The cardinality of a minimum set which is an edge covering of \( H \) is called the \textit{edge covering number} of \( H \), and is denoted by \( \nu(H) \).

The following lemma, proved in [97], gives a relation between the edge covering number and the edge independence number in an \( r \)-uniform hypergraph \( H \) without isolated vertices.

**Lemma 10** (Jucovič, Olejník [97]) For an \( r \)-uniform \( n \)-order hypergraph \( H \) with \( n \) without isolated vertices the following inequalities hold:

\[
\alpha(H) \leq n - (kr - 1)\nu(H), \quad (15)
\]

\[
\alpha(H) + (r - 1)\nu(H) \leq n. \quad (16)
\]

\[
\nu(H) + (r - 1)r - 1\nu(H) \geq n, \quad (17)
\]

**Proof.** See [97]. \( \square \)


In 1989 Olejník proved the following three theorems characterizing \( \alpha(H) \) and \( \nu(H) \).

**Theorem 11** (Olejník [136]) For an \( r \)-uniform \( n \)-order hypergraph \( H = (W, F) \) with \( n \) and its complement \( \overline{H} = (W, \overline{F}) \)

\[
\left\lfloor \frac{n}{r} \right\rfloor \leq \nu(H) + \nu(\overline{H}) \leq 2 \left\lfloor \frac{n}{r} \right\rfloor \quad (18)
\]

and

\[
0 \leq \nu(H)\nu(\overline{H}) \leq \left\lfloor \frac{n}{r} \right\rfloor^2. \quad (19)
\]

**Proof.** See [136]. \( \square \)

This bounds are direct generalizations of the bounds published by Chartrand and Schuster in 1974 [40].
Theorem 12 (Olejník [136]) For an \( r \)-uniform \( n \)-order hypergraph \( H = (W, F) \) and its complement \( \overline{H} = (W, \overline{F}) \), where neither \( H \) nor \( \overline{F} \) have isolated vertices,

\[
\left\lfloor \frac{n}{r} \right\rfloor \leq \nu(H) + \nu(\overline{H}) \leq 2 \left\lfloor \frac{n}{r} \right\rfloor
\]

and

\[
0 \leq \nu(H)\nu(\overline{H}) \leq \left\lfloor \frac{n}{r} \right\rfloor^2.
\]

Proof. See [136].

This result is an extension of the work of R. Laskar and B. Auerbach published in 1978 [120].

Theorem 13 (Olejník [136]) For an \( r \)-uniform \( n \)-order hypergraph \( H = (W, F) \) and its complement \( \overline{H}, \overline{F} \), where neither \( H \) nor \( \overline{H} \) have isolated vertices and \( n \neq 2r \)

\[
2 \left\lfloor \frac{n}{r} \right\rfloor \leq \alpha_H + \alpha_{\overline{H}} \leq 2n - (r - 1) \left\lfloor \frac{n}{r} \right\rfloor - r + 1
\]

and

\[
\left\lfloor \frac{n}{r} \right\rfloor^2 \leq \alpha(H)\alpha(\overline{H}) \leq \frac{1}{4} \left( 2n - (r - 1) \left\lfloor \frac{n}{r} \right\rfloor - k + 1 \right)^2.
\]

Proof. See [136].


Let

\[
B(p, q) = \int_0^1 (1 - t)^{p-1}t^{q-1}\,dt
\]

denote the beta-function with \( p, q > 0 \). Set constants \( 0 < a \leq 1, 0 < b \leq 1 \), and let

\[
f_r(x) = \frac{1}{B} \int_0^1 \frac{1 - t}{{(t^b[1 + (x-1)t])}}\,dr.
\]

In 2004 Zhou and Li [170] proved the following theorem on sparse hypergraphs.

Theorem 14 (Zhou, Li [170]) Let \( H \) be a triangle-free, \( r \)-uniform \( (r \geq 2) \) \( n \)-order linear hypergraph with average degree \( d \). Then its strong vertex independence number \( \alpha_s(G) \) is at least \( nf_r(d) \).
Proof. See [170].


Shearer’s result ([152], further (11) and (12)) was generalized in [170] with the function \( g_r(x) \) satisfying

\[
(r - 1)^2 x(x - 1) g'_r(x) + [(r - 1)x + 1] g_r(x) = 1 
\]

for \( r \)-uniform, triangle-free linear hypergraphs, with sparse neighborhood and in [125] with the function \( g_{r,m}(x) \) satisfying

\[
(r - 1)^2 x(x - m) g'_{r,m}(x) + [(r - 1)x + 1] g_{r,m}(x) = 1 
\]

for \( r \)-uniform, triangle-free, and double linear hypergraphs, in which each subhypergraph induced by a neighborhood, has maximum degree less than \( m \). A linear hypergraph is called double linear if for any non-adjacent distinct vertices \( w \) and \( z \), each edge containing \( w \) has at most one neighbor of \( z \).

From the uniqueness of solutions of the differential equations, we see that \( g_2(x) = g(x) \) and \( g_{r,1}(x) = g_r(x) \). It is shown [125] that \( g_{2,m}(x) \sim \frac{\log x}{x} \), and for \( g_{r,m}(x) \sim \frac{c}{d^{r/(r-1)}} \) for \( r \geq 3 \), where \( c = c(r, m) > 0 \) is a constant without knowing exact values.

Independent sets and numbers are studied in many papers (see e.g. the papers of Abraham [1], Alon, Uri and Azar [12], Berger and Ziv [23], Bollobás, Daykin and Erdős [27], Bonato, Brown, Mitsche and Pralat [28, 29], Bordewich, Dyer and Karpiński [31], Boros, Gurvich, Elbassioni, Gurvich and Khachiyan [32, 33], Borowiecki and Michalak [34], Cutler and Radcliffe [45], Greenhill [70], Halldórson and Losievskaja [76], Hofmeister and Lehman [90], Johnson and Yannakakis [93], Khachiyan, Boros, Gurvich, and Elbassioni [108], Lepin [122], Li and Zhang [125], Losievskaja [126], Shachnai and Srinivasan [151], Tarjan and Trojanowski [155], Yuster [168]).

Since independence number and matching number are closely connected, we are interested in the results on maximum matching algorithms too (see e.g. [25, 26, 46, 47, 49, 50, 56, 57, 61, 65, 66, 77, 78, 86, 88, 89, 91, 92, 100, 104, 105, 109, 112, 113, 118, 119, 127, 131, 132, 133, 135, 137, 142, 146, 147, 148, 154, 157, 158, 169]).
Minimum dominating set of $H$ and maximum vertex independent set of $H$ are connected concepts, therefore we are interested in the results on dominating sets of hypergraphs (see e.g. [2, 96]).

Further connected problems are also often analyzed (see e.g. e.g. in the papers of Agnarsson, Egilsson, and Halldórson [3], Alon, Frankl, Huan, Rödl, Ruciński [10], Alon and Yuster [13], Baranyai [19], Balogh, Butterfield, Hu and Lenz [17], Bertram-Kretchberg and Letzma [24], Bujtás and Tuza [35], Cockayne, Hedetniemi, and Laskar [43], Frank, Király and Király [55], Frankl and Rödl [58, 59], Füredi, Ruszinkó, and Selver [63, 64], Hán, Person and Schacht [78], Henning and Yeo [89], Huang, Loh and Sudakov [92], Johnson and Yannakakis [93], Johnston and Lu [94, 95], Jucovič and Olejník [97], Karonki and Luczak [99], Katona [102, 103], Keevash and Sudakov [106], Kelsen [107], Kohayakawa, Rödl, Skokan [111], Krivelevich [115], Kühn and Loose [117], Kostochka, Mubayi, Verstraëte [114], Krivelevich, Nathaniel, and Sudakov [116], Li, Rousseau and Zang [123, 124], Luczak and Szymańska [129, 134], Szymańska [154], Treglown and Zhao [157, 158], Tuza [160], Yuster [169]).

Although hypergraphs are less often used in the practice than the graphs, they also have different applications in the practice.

For example Bailey, Manoukian, Ramamohanaro [16], further Gunopolus, Khardon, Mannila and Toivonen [74] reported on the applications in data mining, Gallo, Longo, Nguyen, and Pallottino [68], further and Maier [130] in relational databases.

In 2000 Carr, Lancia, Istrail, and Genomics [39] reported on Branch-and-Cut algorithms for vertex independent set problem and on their application to solve problems connected with protein structure alignment.

In this paper, we obtain $\alpha(H) \geq \sum_{v \in V} \frac{1-1/r}{d(v)^{1/(r-1)}}$ for any $r$-uniform hypergraph $H$ without the condition of being triangle-free. The algorithm is naive: it deletes a vertex of maximum degree repeatedly. In order to get a large independent set, a commonly used algorithm is to find a suitable vertex $v$, then delete $v$ and its neighbors, and then do the iterations. Deleting all neighbors seems to be of no use for hypergraphs as in [125, 170]. After deleting a vertex $v$, we delete only one vertex other than $v$ from each edge containing $v$. Our new function $f_r(x)$ satisfies

$$[(r - 1)x^2 - x]f_r'(x) + (x + 1)f_r(x) = 1. \quad (28)$$

Then $f_r(x) \sim \frac{c}{x^{r/(r-1)}}$ as $x \to \infty$. We do not know the exact value of $c = c(r)$. However, when we run the algorithm, we note that for a vertex $v$, we delete $1 + d(v)$ vertices instead of deleting $1 + (r - 1)d(v)$ vertices as in [125, 170]. So
if \( c \) is the constant such that \( g_r(x) \sim \frac{c}{x^{1/(r-1)}} \) as \( x \to \infty \), then the new constant seems to be \((r-1)c\), namely, \( f_r(x) \sim \frac{(r-1)c}{x^{1/(r-1)}} \).

3 Bound for uniform hypergraphs without isolated vertex

The following Theorem 15 is a corollary of Theorem 18, but it has an easy probabilistic proof.

**Theorem 15** Let \( H = (V, E) \) be an \( r \)-uniform hypergraph of order \( n \) and average degree \( d \geq 1 \), then

\[
\alpha(H) \geq \left(1 - \frac{1}{r}\right) \frac{n}{d^{1/(r-1)}}. \tag{29}
\]

**Proof.** Define a random subset \( U \subseteq V \) by \( \Pr(v \in U) = p \) for some \( 0 \leq p \leq 1 \) with all these events being mutually independent over \( v \in V \).

Let \( X(U) \) be the number of vertices in \( U \) and let \( Y(U) \) be the number of edges in the subgraph induced by \( U \). Note that for one of the edges of \( H \), the probability that all of its vertices belong to \( U \) is \( p^r \). By linearity of expectation, we have

\[
E(X - Y) = E(X) - E(Y) = np - \frac{nd}{r} p^r. \tag{30}
\]

Thus there exists a set \( U \) satisfying

\[
X(U) - Y(U) \geq E(X) - E(Y). \tag{31}
\]

Note that \( U \) is not that we require, since the sub-hypergraph of \( H \) induced by \( U \) may have edges. However, if we delete one vertex from each edge contained in \( U \), then at most \( Y(U) \) vertices are deleted, we thus obtain a new set with at least \( E(X) - E(Y) \) vertices and whose induced sub-hypergraph has no edges. The desired lower bound follows by taking \( p = \frac{1}{d^{1/(r-1)}} \). \( \square \)

For hypergraphs that are not regular, Theorem 18 is stronger than Theorem 15. We need two lemmas for the proof of Theorem 18.

**Lemma 16** Let \( r \geq 2 \) be an integer and define

\[
h_r(x) = \begin{cases} 
1 - \frac{x}{r} & \text{if } 0 \leq x < 1 \\
\frac{1}{x^{1/(r-1)}} & \text{if } x \geq 1,
\end{cases} \tag{32}
\]

then \( h_r(x) \) is positive, decreasing and convex. Furthermore, for \( x \geq 1 \), the function \( h_r(x) \) satisfies that \((r-1)x h'(x) + h_r(x) = 0\).
Proof. It is easy to see that $h_r(x)$ is positive and
\[
h'_r(x) = \begin{cases} 
-\frac{1}{r} & \text{if } 0 \leq x < 1 \\
\frac{1}{x^{r/(r-1)}} & \text{if } x \geq 1.
\end{cases}
\] (33)

So $h'_r(x)$ is continuous, negative and increasing, thus $h_r(x)$ is decreasing and convex. The fact that $h_r(x)$ satisfies the mentioned differential equation is straightforward. \qed

Let $\Delta = \Delta(H)$ denote the maximal degree in $H$ and define
\[
S(G) = \sum_{x \in V} h(d(x)), \quad S(H) = \sum_{x \in W} h(d(x)).
\] (34)

**Lemma 17** If $\Delta(H) \geq 1$, $w \in W$, $d(w) = \Delta(H)$, and $H_1 = H - \{w\}$, then $S(H_1) \geq S(G)$.

**Proof.** For each $x \in V \setminus \{v\}$, denote by $n_x$ the number of edges of $H$ that contain both $x$ and $v$. Then $n_x = 0$ if $x$ and $v$ are not adjacent, and $n_x \geq 1$ otherwise. It is easy to see
\[
\sum_{x \in V \setminus \{v\}} n_x = (r-1)\Delta
\] (35)
since $H$ is $r$-uniform. On the other hand, we have
\[
S(H_1) = S(H) - h(\Delta) + \sum_{x \in V \setminus \{v\}} [h(d(x) - n_x) - h(d(x))].
\] (36)

From the fact that $h'(x)$ is negative and increasing, we have
\[
h(d(x) - n_x) - h(d(x)) = -h'(\theta_x)n_x \geq -h'(\Delta)n_x,
\] (37)
where $\theta_x \in [d(x) - n_x, d(x)]$, thus
\[
S(H_1) \geq S(H) - h(\Delta) - h'(\Delta) \sum_{x \in V \setminus \{v\}} n_x
= S(H) - h(\Delta) - (r-1)\Delta h'(\Delta)
= S(H),
\]
proving the claim. \qed
Theorem 18 Let $H = (V, E)$ be an $r$-uniform hypergraph without isolated vertex, then

$$\alpha(H) \geq \left(1 - \frac{1}{r}\right) \sum_{v \in V} \frac{1}{d(v)^{1/(r-1)}}.$$  

(38)

Proof. We write $h_r(x)$ as $h(x)$ for simplicity and define

$$S(H) = \sum_{x \in V} h(d(x)).$$  

(39)

Repeat the algorithm by deleting the vertex of maximum degree if the degree is at least one, terminate the algorithm if there are no edges. Denote by $H_0 = H, H_1, \ldots, H_\ell$ for the sequence of hypergraphs, where $H_\ell$ has no edge. We get $S(H_\ell) = n - \ell$ since $h(0) = 1$, where $n - \ell$ is the order of $H_\ell$, and $\alpha(H) \geq n - \ell$. So

$$\alpha(H) \geq S(H_\ell) \geq S(H_{\ell-1}) \geq \cdots \geq S(H_0) = S(H),$$  

(40)

the assertion follows immediately. \hfill \Box

Since the function $\frac{1}{x^{1/(r-1)}}$ is convex, Theorem 15 is truly a corollary of Theorem 18.

Remark. Theorem 18 gives $\alpha(G) \geq \sum_v \frac{1}{2d[v]}$ for a graph $G$ with $\delta(G) \geq 1$, which is weaker than $\alpha(G) \geq \sum_v \frac{1}{d[v] + 1}$. However, the later can be proved similarly by replacing the function $h(x)$ with $1/(x + 1)$. For details of this algorithm, see Griggs [72].

4 Bound for uniform linear triangle-free hypergraphs

In this section triangle-free hypergraphs are considered. To generalize Shearer’s method [152] and to delete less vertices for a hypergraph, we have a definition as follows.

Let $H = (V, E)$ be an $r$-uniform hypergraph and let $v$ be a vertex of $H$, denote by $E_v = \{e \in E : v \in e\} = \{e_1, e_2, \ldots, e_{d(v)}\}$ for the set of edges containing $v$. A claw of $v$ is a set of neighbors of $v$ of the form $\{u_1, u_2, \ldots, u_{d(v)}\}$ such that each $u_i \in e_i - v$. For a claw $T$ of $v$, we write as $Q_T$, the number of edges that intersect $T$.

When we run the algorithm in each step, we will delete $v$ and a claw $T$, so $Q_T$ edges will be deleted. The new function is as follows.
Let \( r \geq 2 \) be and integer and let \( b = \frac{r^2 - 2}{r - 1} \). Define
\[
f_r(x) = \frac{1}{r - 1} \int_0^1 \frac{1 - t}{t^b[1 + ((r - 1)x - 1)t]} \, dt.
\] (41)

**Lemma 19** The function \( f_r(x) \) satisfies the differential equation
\[
[(r - 1)x^2 - x]f'_r(x) + (x + 1)f_r(x) = 1,
\] (42)
and it is positive, decreasing and convex.

**Proof.** By differentiating under the integral and then integrating by parts, we have
\[
[(r - 1)x^2 - x]f'_r(x) = -[(r - 1)x^2 - x] \int_0^1 \frac{1 - t}{t^{1-b}[1 + ((r - 1)x - 1)t]} \, dt
\]
\[
= x \int_0^1 (1 - t)t^{1-b} \frac{d}{dt} \left( \frac{1}{1 + [(r - 1)x - 1]t} \right)
\]
\[
= -x \int_0^1 \frac{1}{1 + [(r - 1)x - 1]t} \left[ (1 - t)(1 - b)t^{-b} - t^{1-b} \right] \, dt
\]
\[
= -(r - 1)(1 - b)xf_r(x) + x \int_0^1 \frac{t^{1-b}}{1 + [(r - 1)x - 1]t} \, dt
\]
\[
= -xf_r(x) + \frac{1}{r - 1} \int_0^1 \left( \frac{1}{1 - t} - \frac{1}{1 + [(r - 1)x - 1]t} \right) (1 - t)t^{-b} \, dt
\]
\[
= -xf_r(x) + 1 - f_r(x)
\]
\[
= 1 - (x + 1)f_r(x)
\]
which follows by the differential equation. The monotonicity and convexity of \( f_r(x) \) can be seen by repeated differentiation under the integral. \( \square \)

**Theorem 20** Let \( H \) be an \( r \)-uniform \( n \)-order hypergraph with average degree \( d \). If it is triangle-free and linear, then \( \alpha(H) \geq nf_r(d) \).

**Proof.** We apply induction on \(|V|\), the number of vertices of \( H \). The result is trivial for \(|V| = 1\), since \( f(0) = 1 \). Since the case \( r = 2 \) is exactly what Shearer has given, we suppose that \( r \geq 3 \).
For each \( v \in H \), let \( T = \{ u_1, u_2, \ldots, u_{d(v)} \} \) be a claw of \( v \). Since \( H \) is \( r \)-uniform, linear and triangle-free, we have

\[
Q_T = d(v) + \sum_{i=1}^{d(v)} (d(u_i) - 1) = \sum_{i=1}^{d(v)} d(u_i). \tag{43}
\]

Let \( T_v \) be the set of all claws of \( v \), then \( |T_v| = (r - 1)^{d(v)} \). Therefore

\[
\sum_{T \in T_v} Q_T = \sum_{T \in T_v} \sum_{i=1}^{d(v)} d(u_i) = \sum_{u \in n(v)} (r - 1)^{d(v) - 1} d(u), \tag{44}
\]

and

\[
\frac{1}{|T_v|} \sum_{T \in T_v} Q_T = \sum_{u \in n(v)} \frac{d(u)}{r - 1}. \tag{45}
\]

We write \( f(x) \) for \( f_r(x) \) and set

\[
R_T(v) = 1 - (d(v) + 1)f(d) + (dd(v) + d - rQ_T)f'(d). \tag{46}
\]

Then the average of \( R_T(v) \) among \( T \in T_v \) is

\[
\frac{1}{|T_v|} \sum_{T \in T_v} R_T(v) = 1 - (d(v) + 1)f(d) + (dd(v) + d)f'(d) - r \sum_{u \in n(v)} \frac{d(u)}{r - 1} f'(d). \tag{47}
\]

Note that

\[
\frac{1}{n} \sum_{v \in V} \sum_{u \in n(v)} \frac{d(u)}{r - 1} = \frac{1}{n} \sum_{v \in V} d^2(v) \geq d^2 \tag{48}
\]

as \( x^2 \) is a convex function. Since \( f'(x) < 0 \), we have

\[
\frac{1}{n} \sum_{v \in V} \frac{1}{|T_v|} \sum_{T \in T_v} R_T(v) \geq 1 - (d + 1)f(d) + (d^2 + d - rd^2)f'(d) = 0. \tag{49}
\]

Hence there exists a vertex, say \( v \), and a claw of \( v \), say \( T = \{ u_1, u_2, \ldots, u_{d(v)} \} \), such that \( R(v) \geq 0 \). Now by deleting \( v \) and \( u_1, u_2, \ldots, u_{d(v)} \), we obtain a new hypergraph \( H' \) with \( n - d(v) - 1 \) vertices and \( \frac{nd}{r} - Q_T \) edges. For an edge \( e \) containing \( v \), it contains \( r \geq 3 \) vertices, and we delete exactly two vertices from \( e \), so \( H' \) has some vertices. Note that the average degree \( \bar{d} \) of \( H' \) is \( \frac{nd - rQ_T}{n - d(v) - 1} \). By induction hypothesis, we have

\[
\alpha(H) \geq (n - d(v) - 1)f(\bar{d}) = (n - d(v) - 1)f\left( \frac{nd - rQ_T}{n - d(v) - 1} \right). \tag{50}
\]
Combining the facts that $\alpha(H) \geq 1 + \alpha(H')$ and $f(x) \geq f(d) + f'(d)(x - d)$ for all $x \geq 0$ as $f(x)$ is convex, we obtain

$$\alpha(H) = 1 + (n - d(v) - 1)f\left(\frac{nd - rQ_T}{n - d(v) - 1}\right) \geq 1 + (n - d(v) - 1)f(d) + (dd(v) + d - rQ_T)f'(d) = nf(d) + R(v) \geq nf(d)$$

completing the proof. \(\square\)

We now get an asymptotic form of $f_r(x)$ as $c \frac{x}{x^{1/(r-1)}}$ without knowing exact expression of $c = c(r)$ in hope of improving the old constant based on analysis of the algorithm as mentioned.

**Lemma 21** Let $r \geq 3$ be an integer. Then

$$\lim_{x \to \infty} f_r(x) = \frac{c}{x^{1/(r-1)}}, \quad (51)$$

where $c = c(r)$ is a positive constant.

**Proof.** Recall that a first order linear differential equation $\frac{dy}{dx} = p(x)y + q(x)$ has the unique solution of the form

$$y = e^{\phi(x)}\left(y_0 + \int_{x_0}^{x} q(t)e^{-\phi(t)} dt\right) \quad (52)$$

satisfying $y_0 = y(x_0)$, where $\phi(x) = \int_{x_0}^{x} p(t) dt$. From the differential equation that $f_r(x)$ satisfies, we set

$$p(x) = -\frac{x + 1}{(r - 1)x^2 - x}, \quad \text{and} \quad q(x) = \frac{1}{(r - 1)x^2 - x} \quad (53)$$

For $x_0 = 2$,

$$\phi(x) = -\int_{2}^{x} \frac{t + 1}{(r - 1)t^2 - t} dt = \ln \frac{c_1 x}{[(r - 1)x - 1]^\frac{r}{r-1}} \quad (54)$$

Hence

$$e^{\phi(x)} = \frac{c_1 x}{[(r - 1)x - 1]^\frac{r}{r-1}} \sim \frac{c_2}{x^{1/(r-1)}} \quad (55)$$

Then we have

$$q(t)e^{-\phi(t)} \sim \frac{1}{c_2 (r - 1)^x^{1/(r-1) - 2}}, \quad (56)$$
implying that \( c_3 = \int_2^\infty q(t)e^{-\phi(t)}\,dt < \infty \), and \( \int_x^\infty q(t)e^{-\phi(t)}\,dt = c_3 + o(1) \) as \( x \to \infty \). Therefore,

\[
fr(x) = e^{\phi(x)}(y_0 + c_3 + o(1)) \sim \frac{c}{x^{\frac{1}{(r-1)}}},
\]

(57)

where \( c = c_2(y_0 + c_3) \) and \( y_0 = fr(2) \) are positive constants. \( \square \)

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**References**


[77] J. Han, Near perfect matchings in k-uniform hypergraphs, arXiv:1404.1136, 2014, 7 pages. ⇒ 141
On vertex independence number of uniform hypergraphs


[113] D. König, Graphs and matrices (Hungarian), Matematikai és Fizikai Lapok 38 (1931) 116–119. ⇒ 141


⇒ 141, 142


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