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Parallel enumeration of degree sequences of simple graphs II

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Abstract. In the paper we report on the parallel enumeration of the degree sequences (their number is denoted by $G(n)$) and zerofree degree sequences (their number is denoted by $(G_z(n))$ of simple graphs on $n = 30$ and $n = 31$ vertices. Among others we obtained that the number of zerofree degree sequences of graphs on $n = 30$ vertices is $G_z(30) = 5\,876\,236\,938\,019\,300$ and on $n = 31$ vertices is $G_z(31) = 22\,974\,847\,474\,172\,374$. Due to Corollary 21 in [52] these results give the number of degree sequences of simple graphs on 30 and 31 vertices.

1 Introduction

In the practice an often occuring problem is the ranking of different objects (examples can be found e.g. in [52]), assigning points to the objects and then ranking of the objects on the base of the sum of the assigned to them points.

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Especially extensive bibliography has the case when the results are represented by a simple graph and the problem is the test, reconstruction and enumeration of the degree sequences. Havel in 1955 [42], Erdős and Gallai in 1960 [16, 32, 77], Hakimi in 1962 [39], Knuth in 2008 [61], Tripathi et al. in 2010 [89] proposed a method to decide, whether a sequence of nonnegative integers can be the degree sequence of a simple graph. Sierksma and Hoogeven in 1991 [83] compared seven known methods. The running time of their algorithms in worst case is $\Omega(n^2)$. In 2007 Takahashi [86], in 2009 Hell and Kirkpatrick [43], in 2011 Iványi et al. [52] and in April of 2012 Király [58] proposed an algorithm, whose worst running time is $\Theta(n)$.

There are several new proofs for the classical Havel-Hakimi and Erdős-Gallai theorems [26, 32, 63, 70, 75, 87, 88, 89].

Extensions of the algorithms for $(0, b)$-graphs [8, 9, 24, 23, 25, 27, 69, 75, 90, 92] and (a, b)-graphs [44, 45, 46, 53] are also known.

As an application of our linear time algorithm we describe Erdős-Gallai-Enumerative algorithm (EGE) and its parallel version used to enumerate the different degree sequences of simple graphs for 30 vertices. We also present the linear test version of Havel-Hakimi algorithm (HHL).

Let $n \geq 1$. We call a sequence $\mathbf{s} = (s_1, \ldots, s_n)$ (l, u, n)-*bounded*, if $0 \leq s_i \leq n$ for $i = 1, \ldots, n$, n-*bounded*, if it is $(0, n-1, n)$-bounded, *n-regular*, if the conditions $n - 1 \geq s_1 \geq \cdots \geq s_n \geq 0$ hold, and n-*even*, if the sum of the elements of $\mathbf{s}$ is even. If there exists a graph with n vertices which has the degree sequence $\mathbf{s}$, then we say that $\mathbf{s}$ is n-*graphical*. If such graph does not exist, then we say that $\mathbf{s}$ is *nongraphical*. A sequence is *zerofree*, if it does not contain zero. If n is not necessary, then we omit it in the terms n-bounded, n-regular, n-even and n-graphical. The first i elements of an n-regular $\mathbf{s}$ are called *the head*, and the last $n-i$ elements are called *the tail*, belonging to the element i of $\mathbf{s}$.

2 Earlier results

A classical problem of the graph theory is the enumeration of the sorted degree sequences of different graphs—among others simple graphs. For example *The On-Line Encyclopedia of Integer Sequences* contains for $n = 1, \ldots, 29$ vertices the number of degree sequences of simple graphs (the values for $n = 20, \ldots, 23$ were set in July of 2011 by Nathann Cohen [28], and for $24, \ldots, 29$ in 15 November, 2011 by us [48, 52]) and the number of zerofree degree sequences of simple graphs (the values for $n = 1, \ldots, 9$ were set in 12 June, 2004 by

N. J. Sloane, for $n = 10, \ldots, 20$ in 12 August, 2006 by Gordon Royle, for $n = 21, 22$, and 23 in August 31, 2011, and in December 10, 2012 by Frank Ruskey [80], and the values for $n = 24, \ldots, 29$ by us [50, 51].

In this section we review the theoretical and practical results connected with the enumeration of simple graphs.

2.1 Exact enumeration results

It is known [52, equation (23)] that if $n \geq 1$, then the number $R(n)$ of the regular sequences is

$$R(n) = \binom{2n-1}{n} \tag{1}$$

and the number $R_z(n)$ of the zerofree regular sequences is [52, equation (24)]

$$R_z(n) = \binom{2n-2}{n} \tag{2}$$

implying [52]

$$\lim_{n\to\infty} \frac{R(n+1)}{R(n)} = \lim_{n\to\infty} \frac{R_z(n+1)}{R_z(n)} = 4 \tag{3}$$

and

$$\lim_{n\to\infty} \frac{R_z(n)}{R(n)} = \frac{1}{2}, \tag{4}$$

and

$$R(n) = \frac{4^n}{2\sqrt{\pi n}} + O\left(\frac{4^n}{n^{3/2}}\right) \tag{5}$$

and

$$R_z(n) = \frac{4^n}{4\sqrt{\pi n}} + O\left(\frac{4^n}{n^{3/2}}\right). \tag{6}$$

Table 1 in [52] shows the values values of $R(n)$ for $n = 1, \ldots, 38$, Table 4 in [51] for $n = 39, \ldots, 60$, and in [47, 51, 68] the values are presented for $n = 1, \ldots, 1200$. Table 1 in Subsection 3.3 presents the values $R(n)/R(n+1)$ for $n = 1, \ldots, 32$ and [68] for $n = 1, \ldots, 1200$.

Figure 1 in Subsection 3.3 shows the values of $R_z(n)/R_z(n+1))$ for $n = 1, \ldots, 32$.

In 1987 Ascher derived the following explicit formula for the number $E(n)$ of even sequences.

Lemma 1 (Ascher [2], Sloane and Pfoffe [85]) *If* $n \geq 1$, *then the number of even sequences* $E(n)$ *is*

$$E(n) = \frac{1}{2}\left(\binom{2n-1}{n} + \binom{n-1}{\lfloor n/2 \rfloor}\right). \tag{7}$$

Proof. See [2]. □

Table 1 in [52] contains the values of $E(n)$ and $E(n+1)/E(n)$ for $n = 1, \ldots, 31$.

(7) implies (see [52])

$$\lim_{n\to\infty} \frac{E(n+1)}{E(n)} = 4 \tag{8}$$

and

$$E(n) = \frac{4^n}{8\sqrt{\pi n}} + O\left(\frac{4^n}{n^{3/2}}\right). \tag{9}$$

further (1) and (7) imply

$$\lim_{n\to\infty} \frac{E(n)}{R(n)} = \frac{1}{2}, \tag{10}$$

(2) and (7) imply

$$\frac{R_z(n)}{E(n)} = \frac{2n-2}{2n-1} = 1 - \frac{1}{2n-1} \quad \text{and} \quad \lim_{n\to\infty} \frac{R_z(n)}{E(n)} = 1. \tag{11}$$

Table 1 in [52] shows the values of $E(n)$ for $n = 1, \ldots, 38$, Table 4 in [51] for $n = 39, \ldots, 60$, the *list* of [64] for $n = 1, \ldots, 1000$, and [68] for $n = 31, \ldots, 1200$.

Figure 3 in [52] shows the values of $E_z(n)$ for $n = 1, \ldots, 20$, and [68] $n = 1, \ldots, 1200$. Table 5 in [51] shows the values of $E_z(n/R(n)$ for $n = 1, \ldots, 20$.

Using (1) and (7) we computed $E(n)$ and $E(n+1)/E(n)$ for $i = 1, \ldots, 750$ (see [52, 68]). Recently Librandi [64] published the values of $E(n)$ up to $n = 1000$ and we continued the computations up to 1200 [51, 68].

The following theorem gives a very useful connection between the values of $G(n)$ and $G_z(n)$: it helped to decrease the computing time of $G(29)$ with about 50 %.

Lemma 2 (Iványi, Lucz, Móri, Sótér [52]) *If* $n \geq 2$, *then the number of* n*-graphical sequences* $G(n)$ *can be computed from the number of* $(n-1)$*-graphical sequences* $G(n-1)$ *and the number of* n*-graphical zero-free sequences* G_z*:*

$$G(n) = G(n-1) + G_z(n),$$

and if $n \geq 1$ *then*

$$G(n) = 1 + \sum_{i=2}^{n} G_z(i).$$

Proof. If an even sequence $s = (s_1, \ldots, s_n)$ contains at least one zero, then $s_n = 0$ and $s' = (s_1, \ldots, s_{n-1})$ is graphical or not. If $\mathbf{a} = (a_1, \ldots, a_{n-1})$ is $(n-1)$-graphical, then $\mathbf{a}' = (a_1, \ldots, a_{n-1}, 0)$ is n-graphical.

The set of the n-graphical sequences $\mathcal{S}(n)$ consists of two subsets: set of zerofree sequences $\mathcal{S}_z(n)$ and the set of the remaining sequences $\mathcal{S}_0(n)$. There is a bijection between the set of the $(n-1)$-graphical sequences and such n-graphical sequences, which contain at least one zero. Therefore $|\mathcal{S}| = |\mathcal{S}_z| + |\mathcal{S}_0| = G_z(n) + G(n-1)$. □

Using the parallel version EGP (see the next section) of EGE we computed $G(n)$ up to $n = 29$. These numbers can be found in Table 2 of [52].

Theorem 3 (Burns [22]) *There exist positive constants* c *and* C *such that the following bounds of the function* $G(n)$ *are true for* $n \geq 1$*:*

$$\frac{4^n}{cn} < G(n) < \frac{4^n}{(\log n)^C \sqrt{n}}. \tag{12}$$

Proof. See [22]. □

This result implies that the asymptotic density of the graphical sequences is zero among the even sequences.

Corollary 4 *If* $n \geq 1$*, then there exists a positive constant* C *such that*

$$\frac{G(n)}{E(n)} < \frac{1}{(\log_2 n)^C} \tag{13}$$

and

$$\lim_{n\to\infty} \frac{G(n)}{E(n)} = 0. \tag{14}$$

Proof. (13) is a direct consequence of (7) and (12). □

Table 1 in [52] contains the values of $G(n)$ and $G(n+1)/G(n)$ for $n = 1, \ldots, 29$. Table 5 in [51] contains values of $G_z(n)$, $G_z(n)/R(n)$, and $G(n)/R(n)$ for $n = 1, \ldots, 29$.

We remark that a zerofree degree sequence belongs to a graph not containing isolated vertex, therefore the number of zerofree graphical degree sequences

$G_z(n)$ is at the same time also the number of degree sequences of simple graphs, not containing isolated vertex.

There are several classic asymptotic results, e.g. due to Bender and Canfield [7], Bollobás [17, 18, 19], Harary and Palmer [41], Kleitman and Winston [56, 60], Reid [78], Winston and Kleitman [91]. A modern direction is to get approximate results by sampling of random graphs (see e.g. the papers of Erdős, Király and Miklós [34], further of Miklós, Erdős and Soukup [**?**].

An interesting connected problem is the characterization of pairs of different directed graphs having a pair of prescribed indegree and outdegree sequences [8, 9, 10, 11, 12, 14, 15, 20, 40, 72, 76, 81].

Another interesting related questions are the unicity of the realizations of the degree sequences [29, 55, 62, 82] and the parallel realization of degree sequences [1].

Several recent papers consider the problem of approximate enumeration of the number of all realizations of simple graphs (see e.g. [13, 34, 35, 36, 37, 38, 59, 71]). In 1978 Bender and Canfield [7] characterized the asymptotic number of realizations of given graphical degree sequences, while in 2012 Zoltán Király [58] proposed an algorithm which with polynomial delay lists all realizations of a given graphical sequence.

2.2 Earlier algorithmic results

In this subsection the linear Havel-Hakimi algorithm (HHL) based on Havel-Hakimi theorem [39, 42] and the enumerating Erdős-Gallai algorithm (EGE) based on Erdős-Gallai theorem [32] are shortly described.

2.2.1 Linear Havel-Hakimi algorithm (HHL)

In a previous paper [52] we described the classical Havel-Hakimi [39, 42] and Erdős-Gallai [32] algorithms and their some improvements as linear Erdős-Gallai (EGL) and jumping Erdős-Gallai (EGLJ) algorithms.

it is worth to remark that our linear Erdős-Gallai algorithm is applied in the solution of different problems connected with degree sequences [5, 6, 21, 31].

Here we present the linear version of Havel-Hakimi algorithm (HHL) [46] and compare it with the previous linear algorithms EGL and EGLJ [52]. It is important to remark that this linear version of HH only tests the investigated sequences without their reconstruction.

In the worst case the original Havel-Hakimi algorithm requires quadratic time to test the $(0, 1, n)$-regular sequences. Using the new concepts weight

point and reserve we reduced the worst running time to $O(n)$.

Let $s = (s_1, \ldots, s_n)$ be a potential graphical sequence. The definition of the *weight point* w_i belonging to s_i was introduced in [52] in connection with ERDŐS-GALLAI-LINEAR: if $s_1 \geq i$, then w_i is the largest k ($1 \leq k \leq n$) having the property $s_k \geq i$. But if $s_1 < i$, then $w_i = 0$. EGL exploits the property w_i ensuring that if $i \leq w_i$, then the key expression $\min j, s_k$ in the Erdős-Gallai theorem equals i, otherwise equals s_k.

In HHL the weight point w_i determines the increment of the tail capacity when we switch to the investigation of the next element of s.

The reserve r_i belonging to s_i is defined as the unused part of the actual tail capacity and can be computed by the formulas

$$r_1 = w_1 - 1 - s_1 \tag{15}$$

and

$$r_i = w_i + r_{i-1} - s_i \quad \text{for} \quad 2 \leq i \leq n-1. \tag{16}$$

Theorem 5 *The running time of* HAVEL-HAKIMI-LINEAR *is in best case* $\Theta(1)$*, and in worst case it is* $\Theta(n)$.

Proof. If the condition in line 1 or 3 holds, then the running time is $\Theta(1)$. If not, then we decrease the actual w at most n times and the remaining operations require $O(1)$ operations for all reductions. □

2.2.2 Enumerating Erdős-Gallai algorithm (EGE)

A classical problem of the graph theory is the enumeration of the degree sequences of different graphs—among others simple graphs. For example *The On-Line Encyclopedia of Integer Sequences* [84] contains for $n = 1, \ldots, 30$ vertices the number of degree sequences of simple graphs (the values for $n = 20, \ldots, 23$ were set in July of 2011 by Nathann Cohen, in November 15, 2011 for $24, \ldots, 29$ and in 29 July of 2013 for $n = 30$ by us [48]).

We applied the new quick EGL to get these numbers for larger values of n.

Our starting point was to test all regular sequences and so to enumerate the graphical ones. Equation 1 gives the number of regular sequences.

According to Erdős-Gallai theorem [32] the sum of the elements of a graphical sequence is always even. Therefore it is sufficient to test only the even sequences. In 1987 Ascher [2] derived Lemma 1, containing an explicit formula for the number of even sequences $E(n)$.

According to Lemma 2 it is enough to test only the zerofree even sequences.

This lemma was the base of Erdős-Gallai Enumerative algorithm (EGE) used to enumerate the graphical sequences for $n = 23, \ldots, 29$ [51].

We enumerated the graphical sequences of simple graphs on $n = 30$ and 31 vertices using algorithm EGE2. The running time of EGE was substantially (with about 30 %) decreased due to Lemma 9.

We prepare the enumeration of degree sequences of simple graphs on 32 vertices. The running time of EGE2 would be about 320 years for a computer with one processor having 2,2 GHz speed. We wish to decrease the running time of EGE2 using Lemmas 10 and 11.

2.3 Earlier simulation results

The papers [44, 45, 46, 51, 52, 66] and OEIS [64, 73, 74] contain many simulation results. We describe them together with the new results in Subsection 3.3.

It is worth to mention other methods of enumeration of graph sequences as generation of random graphs (e.g. [65] and generation of graphical partitions (see e.g. [3, 4, 30, 33].

3 New results

In this section we describe the new mathematical and simulation results.

3.1 New enumerative results

At first we give a new formula for the number of zerofree even sequences. This formula is more sophisticated than Ascher's formula, and its application requires more time, but it has the adventage that we can extend it to a formula for $E_z(n)$. Let $\mathbf{s}$ be an n-even sequence and let $\mathbf{s}' = (s'_1, \ldots, s'_n)$ be defined by $s'_i = s_i + n - i$ for $i = 1, \ldots, n$. Then the number of different possible sequences $\mathbf{s}$ is $E(n)$ and the number of different sequences $\mathbf{s}'$ is $R(n)$.

If $j = 0,\ 1,\ 2$, or 3 and $n = 4k + j$, then let $E(n)$ be denoted by $E(k, j)$.

Lemma 6 *If* $n \geq 1$ *and* $n = 4k + j$, *then*

$$E(k, 0) = \sum_{i=0}^{2k-1} \binom{4k-1}{2i} \binom{4k}{4k-2i}, \tag{17}$$

$$E(k, 1) = \sum_{i=0}^{2k} \binom{4k}{2i} \binom{4k+1}{4k-2i+1}, \tag{18}$$

$$E(k,2) = \sum_{i=0}^{2k} \binom{4k+1}{2i+1} \binom{4k+2}{4k-2i+1}, \tag{19}$$

$$E(k,3) = \sum_{i=0}^{k} \binom{4k+2}{2i+1} \binom{4k+3}{4k-2i+2}. \tag{20}$$

Proof. Let

$$\sum_{i=1}^{n} s_i = S(\mathbf{s}) \quad \text{and} \quad \sum_{i=1}^{n} s'_i = S'(\mathbf{s}). \tag{21}$$

According to the value of j we consider four cases. Since $\mathbf{s}$ is an even sequence, therefore $S(\mathbf{s})$ is even in all cases.

1. If $j = 0$, then

$$S'(\mathbf{s}) = S(\mathbf{s}) + \sum_{i=0}^{4k-1} i = S(\mathbf{s}) + 2k(4k-1), \tag{22}$$

and so $S(\mathbf{s}')$ is also even, therefore it contains an even number of odd elements. The interval $[0, 8k-2]$ contains $8k-1$ elements and among them $4k$ even and $4k-1$ odd elements, so for $\mathbf{s}'$ we can choose $2i$ odd elements from $4k-1$ candidates and $4k-2i$ ($i = 0,\ 1,\ \ldots,\ 2k-1$) even elements from $4k+1$ candidates, so

$$E(k,0) = \sum_{i=0}^{2k-1} \binom{4k-1}{2i} \binom{4k}{4k-2i}. \tag{23}$$

2. If $j = 1,\ 2$ or $j = 3$, then the proof is similar to the proof in the first case.

□

For example let $n = 4$, then $k = 1$, $j = 0$ and

$$E(4) = E(1,0) = \sum_{i=0}^{1} \binom{3}{2i} \binom{4}{4-2i} = 1 \cdot 1 + 3 \cdot 6 = 19. \tag{24}$$

As another example let $n = 6$, then $k = 1$, $j = 2$ and

$$E(6) = \sum_{i=0}^{2} \binom{5}{1} \binom{6}{3} + \binom{5}{5} \binom{6}{1} = 530 + 200 + 6 = 236. \tag{25}$$

Let the number of zerofree even sequences denoted by $E_z(n)$. Let $\mathbf{q} = (q_1, \ldots, q_n)$ be a zerofree n-even sequence and let $\mathbf{q}' = (q'_1, \ldots, q'_n)$ be defined by $q'_i = q_i + n - i$ for $i = 1, \ldots, n$. Then the number of different possible sequences $\mathbf{q}$ is $E_z(n)$ and the number of different sequences $\mathbf{q}'$ is $R_z(n)$.

Theorem 7 *Let* $n = 4k + j$ *for* $k = 0, 1, \ldots$ *and* $j = 0, 1, 2, 3$*, further let* $E_z(n)$ *be denoted by* $E_z(k, j)$*. Then*

$$E_z(k, 0) = \sum_{i=0}^{2k-1} \binom{4k-1}{2i} \binom{4k-1}{4k-2i}, \tag{26}$$

$$E_z(k, 1) = \sum_{i=0}^{2k} \binom{4k}{2i} \binom{4k}{4k-2i+1}, \tag{27}$$

$$E_z(k, 2) = \sum_{i=0}^{2k} \binom{4k+1}{2i+1} \binom{4k+1}{4k-2i+1}, \tag{28}$$

$$E_z(k, 3) = \sum_{i=0}^{2k+1} \binom{4k+2}{2i+1} \binom{4k+2}{4k-2i+2}. \tag{29}$$

Proof. Let

$$\sum_{i=1}^{n} q_i = Q(\mathbf{q}) \quad \text{and} \quad \sum_{i=1}^{n} q'_i = Q'(\mathbf{q}). \tag{30}$$

According to the value of j we consider four cases. Since $\mathbf{q}$ is an even sequence, therefore $Q(\mathbf{q})$ is alwys even.

1. If $j = 0$, then

$$Q'(\mathbf{q}) = Q(\mathbf{q}) + \sum_{i=0}^{4k-1} i = Q(\mathbf{q}) + 2k(4k-1) \tag{31}$$

is even, therefore the number of odd elements of $\mathbf{q}'$ is also even. The interval $[1, 8k-2]$ contains $8k-2$ elements and among them $4k-1$ even and $4k-1$ odd elements, so for $\mathbf{q}'$ we can choose $2i$ odd elements from $4k-1$ candidates and $4k-2i$ ($i = 0, \ldots, 2k-1$) even elements from $4k-1$ candidates, so we get (26).

2. If $j = 1$, then

$$Q'(\mathbf{q}) = Q(\mathbf{q}) + \sum_{i=0}^{4k} i = Q(\mathbf{q}) + 2k(4k+1), \tag{32}$$

is even, therefore the number of odd elements of $\mathbf{q}'$ is also even. The interval $[1, 8k]$ contains $8k$ elements and among them $4k$ odd and $4k$ even elements, so for $\mathbf{q}'$ we can choose $2i$ odd elements from $4k$ candidates and $4k-2i+1$ ($i = 0, \ldots, 2k$) even elements from $4k-1$ candidates, so we get (27).

3. If $j = 2$, then

$$Q'(\mathbf{q}) = Q(\mathbf{q}) + \sum_{i=0}^{4k+1} i = Q(\mathbf{q}) + (2k+1)(4k+1) \tag{33}$$

is odd, therefore the number of odd elements of $\mathbf{q}'$ is also odd. The interval $[1, 8k+2]$ contains $8k+2$ elements and among them $4k+1$ even and $4k+1$ odd elements, so for $\mathbf{q}'$ we can choose $2i+1$ odd elements from $4k+2$ candidates and $4k-2i-1$ ($i = 0, \ldots, 2k-1$) even elements from $4k+1$ candidates, so we get (28).

4. If $j = 3$, then

$$Q'(\mathbf{q}) = Q(\mathbf{q}) + \sum_{i=0}^{4k+2} i = Q(\mathbf{q}) + (2k+1)(4k+3), \tag{34}$$

and so $Q(\mathbf{q}')$ is also odd, therefore $\mathbf{q}'$ contains an odd number of odd elements. The interval $[1, 8k+4]$ contains $8k+4$ elements and among them $4k+2$ even and $4k+2$ odd elements, so for $\mathbf{q}'$ we can choose $2i+1$ odd elements from $4k+2$ candidates and $4k-2i-1$ ($i = 0, \ldots, 2k-1$) even elements from $4k+2$ candidates, so

$$E_z(k, 3) = \sum_{i=0}^{2k+1} \binom{4k+2}{2i+1} \binom{4k+2}{4k-2i+2}. \tag{35}$$

□

Table 1 shows the values of $R(n)/R(n+1)$, $R_z(n)/R_z(n+1)$, $E(n)/R(n)$, $E(n)/E(n+1)$, $E_z(n)/E_z(n+1)$, and $E_z(n)/R_z(n)$ for $n = 1, \ldots, 32$.

n	$\frac{R(n)}{R(n+1)}$	$\frac{R_z(n)}{R_z(n+1)}$	$\frac{E(n)}{R(n)}$	$\frac{E(n)}{E(n+1)}$	$\frac{E_z(n)}{E_z(n+1)}$	$\frac{E_z(n)}{R_z(n)}$
1	0.333333	0.000000	1.00000000	0.000000	0.000000	− − −
2	0.300000	0.250000	0.66666667	0.500000	0.500000	1.000000
3	0.287714	0.266667	0.60000000	0.222220	0.222222	0.500000
4	0.277778	0.257857	0.487179	0.321427	0.321429	0.600000
5	0.270562	0.266667	0.523810	0.254545	0.254545	0.500000
6	0.269231	0.265151	0.510823	0.277778	0.277778	0.523810
7	0.266667	0.263736	0.505828	0.260698	0.260698	0.500000
8	0.264706	0.262500	0.502720	0.265559	0.265559	0.505828
9	0.263158	0.261437	0.501440	0.260687	0.260687	0.500000
10	0.261905	0.260526	0.500682	0.261276	0.261276	0.501440
11	0.260870	0.259740	0.500357	0.259555	0.259555	0.500000
12	0.260000	0.259058	0.500171	0.259243	0.259243	0.500357
13	0.259259	0.258461	0.500089	0.258415	0.258416	0.500000
14	0.258621	0.257937	0.500043	0.257982	0.257982	0.500089
15	0.258065	0.257471	0.500022	0.257460	0.257460	0.500000
16	0.257578	0.257056	0.500011	0.257068	0.257068	0.500022
17	0.257143	0.256684	0.500005	0.256682	0.256682	0.500000
18	0.256757	0.256349	0.500003	0.256352	0.256352	0.500006
19	0.256410	0.256046	0.500001	0.256045	0.256045	0.500000
20	0.256098	0.255769	0.500001	0.255770	0.255770	0.500001
21	0.255814	0.255517	0.50000034	0.255517	0.255517	0.500000
22	0.255556	0.255285	0.50000016	0.255286	0.255286	0.50000034
23	0.255319	0.255072	0.50000009	0.255072	0.255072	0.50000000
24	0.255102	0.254876	0.50000004	0.254876	0.254876	0.50000000
25	0.254902	0.254694	0.50000002	0.254694	0.254694	0.50000009
26	0.254717	0.254525	0.50000001	0.254525	0.254525	0.50000000
27	0.254545	0.254368	0.50000001	0.254368	0.254368	0.50000000
28	0.254386	0.254221	0.50000000	0.254221	0.254221	0.50000000
29	0.254237	0.254083	0.50000000	0.254083	0.254083	0.50000000
30	0.254098	0.253854	0.50000000	0.253955	0.253955	0.50000000
31	0.253968	0.253834	0.50000000	0.253834	0.253834	0.50000000
32	0.253846	0.253720	0.50000000	0.253720	0.253720	0.50000000

Table 1: The values of $R(n)/R(n+1)$, $R_z(n)/R_z(n+1)$, $E(n)/R(n)$, $E(n)/E(n+1)$, $E_z(n)/E_z(n+1)$, and $E_z(n)/R_z(n)$ for $n = 1, \ldots, 32$

It is remarkable that in $R(101)/R(102)$ and $R_z(101)/R_z(102)$ the first nine decimal digits are equal.

For example let $n = 4$, then $k = 1$, $j = 0$ and

$$E_z(4) = E_z(1,0) = \binom{3}{0}\binom{3}{4} + \binom{3}{2}\binom{3}{2} = 1 \cdot 0 + 3 \cdot 3 = 9. \tag{36}$$

If $n = 5$, then $k = 1$, $j = 1$ and

$$E_z(5) = \binom{4}{0}\binom{4}{5} + \binom{4}{2}\binom{4}{3} + \binom{4}{4}\binom{4}{1} = 0 + 24 + 4 = 28. \tag{37}$$

If $n = 6$, then $k = 1$, $j = 2$ and

$$E_z(6) = \binom{5}{1}\binom{5}{5} + \binom{5}{3}\binom{5}{3} + \binom{5}{5}\binom{5}{1} = 5 + 100 + 5 = 110. \tag{38}$$

If $n = 7$, then $k = 1$, $j = 3$ and

$$E_z(7) = \binom{6}{1}\binom{6}{6} + \binom{6}{3}\binom{6}{4} + \binom{6}{5}\binom{6}{2} = 6 + 300 + 90 = 396. \tag{39}$$

If $n = 8$, then $k = 2$, $j = 0$ and

$$E_z(8) = \binom{7}{0}\binom{7}{8} + \binom{7}{2}\binom{7}{6} + \binom{7}{4}\binom{7}{4} + \binom{7}{6}\binom{7}{2} = 1519. \tag{40}$$

Simulaton results in Table 1 show, that if $1 \le n \le 32$ and n is odd, then $E_z(n)/R_z(n) = 0.5$. This property is true for larger odd n's too.

Lemma 8 *If* $1 \le k \le 600$*, then*

$$\frac{E_z(2k-1)}{R_z(2k-1)} = 0.5. \tag{41}$$

Proof. See the computed values of $R_z(n)$ and $E_z(n)$ in [68]. □

Table 2 contains the ratios $E_z(n)/G_z(n)$ for $n = 23, \ldots, 29$ and the ratios $T(n)/G_z(n)$ for $n = 30$ and $n = 31$.

The data in Table 2 show that the function G_z/E_z) is decreasing. We suppose that this function tends monotonically decreasing to zero when n tends to infinity (in a similar way as the function $G(n)/E(n)$ tends to zero according to Corollary 23 [52, page 260].

Table 3 contains the values of $G_z(n)$, $T(n)$, and $G_z(n)/T(n)$ for $n = 30$ and $n = 31$: the ratio of the graphical and tested sequences is much higher and these ratios are increasing. These changes are dut to the fact that EGE2 jumps many nongraphical zerofree ebven sequences withous testing them.

3.2 New algorithmic results

Using the following Lemma 9 later we will further fasten EGE.

If $b = (b_1, \ldots, b_n)$ is a regular sequence, then $c = (c_1, \ldots, c_n)$ is called *lexicographically* i*-smaller, than* b if there exist indices i and j such that

$$1 \le i < j <= n, \tag{42}$$

n	$G_z(n)$	$E_z(n)$	$G_z(n)/E_z(n)$
17	130 038 230	282 861 360	0.459724
18	499 753 855	1 101 992 870	0.453500
19	1 924 912 894	4 298 748 300	0.447784
20	7 429 160 296	16 789 046 494	0.442500
21	28 723 877 732	65 641 204 200	0.437589
22	111 236 423 288	256 895 980 068	0.433002
23	431 403 470 222	1 006 308 200 040	0.428699
24	1 675 316 535 350	3 945 186 233 014	0.424648
25	6 513 837 679 610	15 478 849 767 888	0.420821
26	25 354 842 100 894	60 774 332 618 300	0.417197
27	98 794 053 269 694	238 775 589 937 976	0.413752
28	385 312 558 571 890	938 702 947 395 204	0.410473
29	1 504 105 116 253 904	3 692 471 324 505 040	0.407344
30	5 876 236 938 019 300	14 532 512 180 224 216	0.404351
31	22 974 847 474 172 100	57 224 797 531 384 560	0.400148

Table 2: The values of $G_z(n)$, $E_z(n)$, and $G_z(n)/E_z(n)$ for $n = 17, \ldots, 31$

n	$G_z(n)$	$T(n)$	$G_z(n)/T(n)$
31	5 876 236 938 019 300	6 790 865 476 867 340	86, 531487
32	22 974 847 471 172 100	26 507 499 250 791 700	86, 673010

Table 3: The values of $G_z(n)$, $T(n)$, and $G_z(n)/T(n)$ for $n = 30$ and $n = 31$

further

$$c_k = b_k \quad \text{for} \quad k = 1, \ldots, i, \tag{43}$$

and

$$\sum_{k=i+1}^{n} c_k \leq \sum_{k=i+1}^{n} b_k. \tag{44}$$

Lemma 9 *If* $b = (b_1, \ldots, b_n)$ *and is a nongraphical sequence and* $c = (c_1, \ldots, c_n)$ *is lexicographically* i*-smaller than* b *for some* i $(1 \leq i < n$*, then* c *is also graphical.*

Proof. See [54]. □

Using this lemma the running time of EGLJ decreased substantially. It was very useful when we enumerated the edge sequences of the simple graphs on

30 vertices between June 21 and 18 July of 2013 and on 31 vertices between 18 July and 24 August (the results see in Table 4).

Using the results of Tripathi and Vijay [52, Lemma 6 and Theorem 7] we can substantially decrease the average testing time of the zerofree even sequences. It is known that the expected number of checking points proposed by Tripathi and Vijay is about $n/2$ [52].

Algorithm EGE2 [51, pages 274–277] used in the enumerations for $n = 30$ and $n = 31$ vertices is based on Lemma 9.

We develop algorithm EGE3 for the enumeration of the degree sequences in the case of 32 vertices. EGE3 will be based on Lemmas 10 and 11.

Lemma 10 *If the investigated by EG3 sequence is graphical and has the form* $b = b_{i_1}^{c_1} b_{i_2}^{c_2}$ *and the upper bound* $c_1(c_1-1)$ *of the inner capacity of the* c_1*-length head of* b *covers* H_n*, that is if*

$$c_1(c_1 - 1) \geq H_n \tag{45}$$

then all sequences starting with $b_1^{c_1}$ *are graphical.*

Proof. See [54]. □

The next lemma allows to enumerate many graphical sequences without their time consuming testing.

Lemma 11 *If the investigated by EG3 graphical sequence has the form* $b = b_1^{c_1} b_2^{c_2} \dots b_p^{c_p} 1^{c_{p+1}}$*, where* $p \geq 1$*,* $b_1 > b_2 > \dots > b_p \geq 3$*,* $c_1, \dots, c_{p+1} \geq 1$*, then all zerofree even sequences starting with the prefix*

$$b_1^{c_1} b_2^{c_2} \dots b_{p-1}^{c_{p-1}} b_p^{c_p-1} (b_p - 1)$$

are also graphical.

Proof. See [54]. □

3.3 New simulation results

Table 4 contains the values of $G_z(n)$ and $G(n)$ for $n = 1, \dots, 31$. The values for $n = 1, \dots, 9$ were computed by E. Weisstein, for $n = 10, \dots, 20$ by G. Royle in 2006, for $n = 21, 22$ and $n = 23$ by F. Ruskey in 2006, for $n = 24, \dots, 29$ by T. Matuszka in January of 2013, for $n = 30$ by L. Lucz in July of 2013 and for $n = 31$ also by L. Lucz in September of 2013 [48, 50, 51, 52, 79].

Column 4 of Table 4 supports the following conjecture formulated by Gordon Royle in 2012.

Conjecture 12 (Royle, 2012). *If* n *tends to infinity, then* $G_z(n+1)/G_z(n)$ *tends to 4.*

We think, that the following conjecture is also true.

Conjecture 13 *If* n *tends to infinity, then* $G(n+1)/G(n)$ *tends to 4.*

We observed that when we enumerated these sequences, that in the case $n = 30$ vertices 85.40 percent, while in the case $n = 31$ vertices 86.67 percent of the investigated potential degree sequences was graphical. Therefore it is useful if we know without a linear time testing that a given tested sequence is graphical.

Figure 1 shows the number of the tested and the graphical sequences as the function of the index of the slices when $n = 30$.

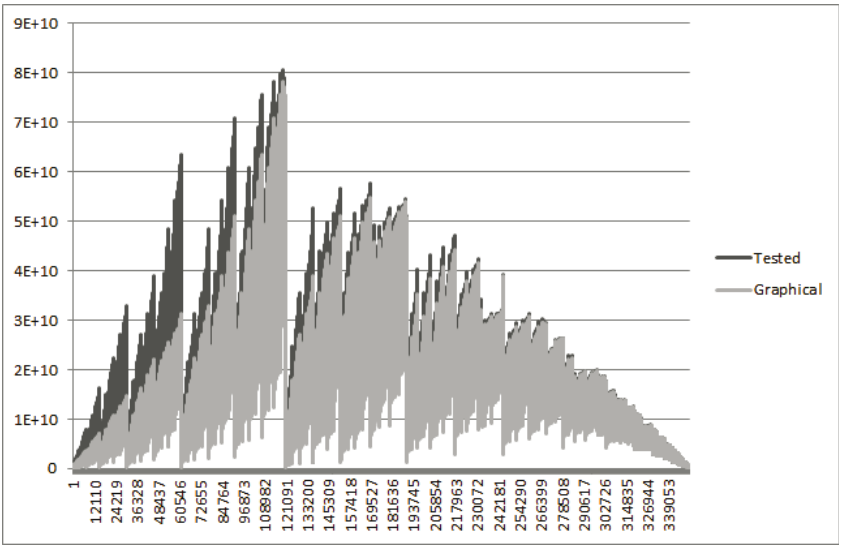

Figure 1: The number of tested (trimmed even) sequences and the number of graphical sequences as the function of the index of slices when $n = 30$

Figure 2 shows the similar data for $n = 31$.

We remark that on the site of the journal the Figures 1 and 2 are color (the graphical sequences are represented by red, while the tested sequences by blue color).

Table 5 contains the data of PC's used for the enumeration of $G_z(31)$, where Comp. alg. = Computer Algebra, Prog. lang. = Program languages, Core =

n	$G_z(n)$	$G(n)$	$\frac{G_z(n+1)}{G_z(n)}$	$\frac{G(n+1)}{G(n)}$
1	0	1	0.000000	0.500000
2	1	2	0.500000	0.500000
3	2	4	3.500000	3.750000
4	7	11	2.857143	2.818182
5	20	31	3.550000	3.290323
6	71	102	3.380282	3.352941
7	240	342	3.629167	3.546784
8	871	1 213	3.614237	3.595218
9	3 148	4 361	3.702351	3.672552
10	11 655	16 016	3.717889	3.705544
11	43 332	59 348	3.756323	3.742620
12	162 769	222 117	3.773434	3.786674
13	614 198	836 315	3.794439	3.802710
14	2 330 537	3 166 852	3.808465	3.817067
15	8 875 768	12 042 620	3.822189	3.828918
16	33 924 859	45 967 479	3.833125	3.839418
17	130 038 230	176 005 709	3.843130	3.848517
18	499 753 855	675 759 564	3.851172	3.856630
19	1 924 912 894	2 600 672 458	3.859479	3.863844
20	7 429 160 296	10 029 832 754	3.866369	3.870343
21	28 723 877 732	38 753 710 486	3.872612	3.876212
22	111 236 423 288	149 990 133 774	3.878257	3.881553
23	431 403 470 222	581 393 603 996	3.883410	3.886431
24	1 675 316 535 350	2 256 710 139 346	3.888124	3.890907
25	6 513 837, 679 610	8 770 547 818 956	3.894458	3.895031
26	25 354 842 100 894	34 125 389 919 850	3.895503	3.897978
27	98 794 053 269 694	132 919 443 189 544	3.900159	3.898843
28	385 312 558 571 890	518 232 001 761 434	3.903597	3.902238
29	1 504 105 116 253 904	2 022 337 118 015 338	3.906814	3.905666
30	5 876 236 938 019 300	7 898 574 056 034 638	3.909789	3.908734
31	22 974 847 474 172 100	30 873 429 530 206 738	− − −	− − −

Table 4: The number $G_z(n)$ of zerofree graphical sequences and the number $G(n)$ of graphical sequences for $n = 1, \ldots, 31$, further the ratios $G_z(n)/G_z(n+1)$ and $G(n)/G(n+1)$ for $n = 1, \ldots, 30$

Core(TM)Kása 1 = Z. Kása (Cluj), Kása 2 = Z. Kása (Tg.-Mureś), Kása 3 = Z. Kása // (Tg.-Mureś), Sp1 = Speed of a machine in GHz, Sp2 = Speed of the laboratory in GFLOPS, Intel (R) = Intel (R) Xeon (R).

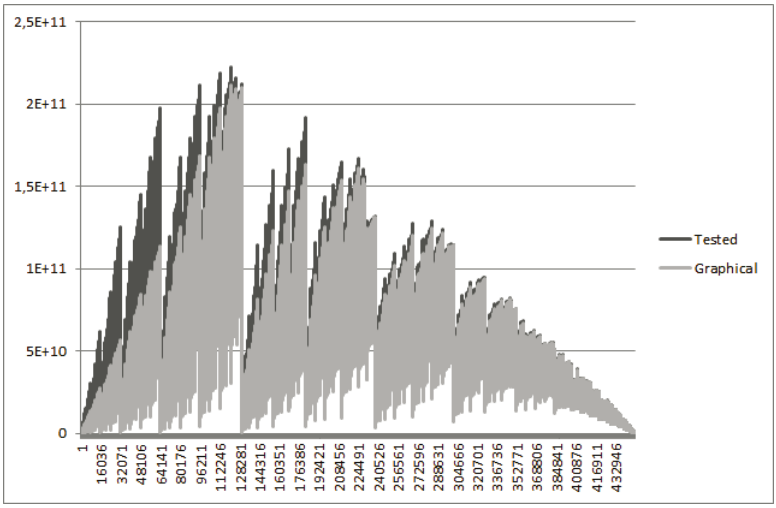

Figure 2: The number of tested (trimmed even) sequences and the number of graphical sequences as the function of the index of slices when $n = 31$

The total number of machines was 350.

Table 6 contains the algorithms, running times and number of jobs in the case $n = 25, \ldots, 31$.

3.4 The growth of the functions $R(n)$, $E(n)$, $R_z(n)$, $E_z(n)$, $G(n)$ and $G_z(n)$

In this subsection we present concrete values of the functions characterizing the sizes of the investigated sets of sequences.

The number $R(n)$ of the regular sequences is presented in Figure 1 of [52] for $n = 1, \ldots, 38$ and up to $n = 1200$ in [47].

The values of the zerofree regular $R_z(n)$ can be quickly computed using formula (22) in [47]. The values for $n = 1, \ldots, 1200$ can be found in [68].

The number $E(n)$ of even sequences is presented in Figure 1 of [52] for $n = 1, \ldots, 38$ and up to 1000 in [49] and up to $n = 1200$ in [68].

The number $E_z(n)$ of the zerofree even sequences is contained in Figure 3 of [52] for $n = 1, \ldots, 20$ (these data are the results of brute force simulation) and up to $n = 1200$ in [68].

The order of growth of these functions is $\Theta(4^n/n)$.

According to theorem of Burns [22, 52] the order of growth of $G(n)$ is smaller (see 12).

Laboratory	Number	Type	Sp1	Sp2
Central	87	Core 2 Duo	2.93	2041
Comp. algebra	13	Core 2 Duo	2.13	403
Data base	34	Core 2 Duo	3.25	1631
Graphical	16	Core 2 Quad	2.33	597
Prog. lang.	54	Core 4 Duo	3.25	6621
PC1	20	Core i5-2320	3.00	1920
PC3	28	Core i3-2100	3.10	1389
PC4	19	Core 2 Duo	2.93	446
PC5	19	Core 4 Duo	2.93	446
PC6	18	Core 2 Quad	2,33	672
PC7	18	Core 2 Quad	2.40	691
PC9	19	Core 2 Quad	2.66	810
Server	1	Core i5 650	3.20	26
Kása 1	1	AMD K7	0.75	8
Kása 2	1	Intel (R)	3.00	50
Kása 3	1	Intel (R)	2.13	23
P. Ősze	1	Core 4 Duo	2.20	37
Total	350			17811

Table 5: Names of laboratories, number of machines, type of machines, speed of machines in GHz, speed of laboratories in GLOPS, used in the case $\mathfrak{n} = 31$

The known values of $\mathsf{G}(\mathfrak{n})$ and $\mathsf{G}_z(\mathfrak{n})$ are summarized in Table 4.

3.5 Further plans

Our new program (EGE3) is able to jump the test of some part of zerofree graphical sequences [54]. Due to this property of the new program EGE3 the number of tested sequences is smaller than the number of zerofree graphical ones (see Table 7).

4 Summary

The log files and source codes of our programs can be found at

http://people.inf.elte.hu/lulsaai/Holzhacker .

n	Algorithm	Running time (in days)	Running time (in years)	Number of jobs
25	EGE	26	0.0712	435
26	EGE	70	0.1918	435
27	EGE	316	0.8657	435
28	EGE	1 130	3.0959	2 001
29	EGE	6 733	18.4466	15 119
30	EGE2	7 221	19.7835	351 155
31	EGE2	32 702	89.5954	448 957

Table 6: Number of vertices, used algorithm, total running time (in days and in years) and number of jobs

n	$T(n)$	$G_z(n)$
3	3	2
4	8	7
5	24	20
6	77	71
7	245	240
8	852	871
9	2991	3148
10	10807	11655
11	39407	43332
12	145673	162769
13	542531	614198
14	2036196	2330537
15	7684164	8875768
16	29143362	33924859
17	110973050	130038230
18	424055902	499753855
19	1625265958	1924912894
20	6245498873	7429160296

Table 7: Number of vertices (n), number of tested sequences ($T(n)$) and number of zerofree graphical sequences ($G_z(n)$)

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