Annales Universitatis Paedagogicae Cracoviensis Studia Mathematica XVIII (2019)

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New results for almost increasing sequences

Abstract. In the present paper, two theorems of absolute summability have been proved by using the definition of almost increasing sequence.

1. Introduction

Let $\sum a_n$ be a given infinite series with its partial sums (s_n) . Let (φ_n) be a sequence of positive real numbers. The series $\sum a_n$ is said to be *summable* $\varphi - |\tilde{N}, p_n; \delta|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [14])

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k + k - 1} |\gamma_n - \gamma_{n-1}|^k < \infty,$$

where (p_n) is a sequence of positive numbers such that

$$P_n = \sum_{v=0}^n p_v \rightarrow \infty \quad \text{as } n \rightarrow \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1),$$

and the sequence-to-sequence transformation

$$\gamma_n = \frac{1}{P_n} \sum_{v=0}^n p_v s_v$$

defines the sequence (γ_n) of the Riesz mean of the sequence (s_n) , generated by the sequence of coefficients (p_n) (see [7]).

AMS (2010) Subject Classification: 26D15, 40D15, 40F05, 40G99.

Keywords and phrases: almost increasing sequences, Hölder inequality, infinite series, Minkowski inequality, Riesz mean, summability factor.

For $\varphi_n = \frac{P_n}{p_n}$, $\varphi - |\bar{N}, p_n; \delta|_k$ summability reduces to $|\bar{N}, p_n; \delta|_k$ summability (see [3]). Also, for $\delta = 0$ and $\varphi_n = \frac{P_n}{p_n}$, $\varphi - |\bar{N}, p_n; \delta|_k$ summability reduces to $|\bar{N}, p_n|_k$ summability (see [2]). Some different applications of absolute summability can be find in [4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17].

A positive sequence (d_n) is said to be *almost increasing* if there is a positive increasing sequence (c_n) and two positive constants M and N such that

$$Mc_n \leq d_n \leq Nc_n$$

(see [1]). In [10], the following theorems of absolute summability have been proved by means of this sequence.

THEOREM 1 ([10])

Let (X_n) be an almost increasing sequence and let there be sequences (β_n) and (λ_n) such that

$$|\Delta\lambda_n| \leq \beta_n, \quad (1)$$

$$\beta_n \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (2)$$

$$\sum_{n=1}^{\infty} n|\Delta\beta_n|X_n < \infty, \quad (3)$$

$$|\lambda_n|X_n = O(1) \quad \text{as } n \rightarrow \infty, \quad (4)$$

where $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$. If

$$\begin{aligned} \sum_{v=1}^n \frac{|s_v|^k}{v} &= O(X_n) \quad \text{as } n \rightarrow \infty, \\ \sum_{v=1}^n \frac{p_v}{P_v} |s_v|^k &= O(X_n) \quad \text{as } n \rightarrow \infty, \end{aligned} \quad (5)$$

then the series $\sum a_n \lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

THEOREM 2 ([10])

Let (X_n) be an almost increasing sequence. If conditions (1)–(4), (5) of Theorem 1 and conditions

$$\begin{aligned} \sum_{n=1}^{\infty} P_n X_n |\Delta\beta_n| &< \infty, \\ \sum_{n=1}^m \frac{|s_n|^k}{P_n} &= O(X_m) \quad \text{as } m \rightarrow \infty, \end{aligned} \quad (6)$$

are satisfied, then the series $\sum a_n \lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

2. Main Results

In this section, two general theorems will be proved.

THEOREM 3

Let (X_n) be an almost increasing sequence and $\varphi_n p_n = O(P_n)$. If conditions (1)–(4) of Theorem 1 and

$$\sum_{v=1}^n \varphi_v^{\delta k} \frac{1}{v} |s_v|^k = O(X_n) \quad \text{as } n \rightarrow \infty, \quad (7)$$

$$\sum_{v=1}^n \varphi_v^{\delta k-1} |s_v|^k = O(X_n) \quad \text{as } n \rightarrow \infty, \quad (8)$$

$$\sum_{n=v+1}^{m+1} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}} = O\left(\varphi_v^{\delta k} \frac{1}{P_v}\right) \quad \text{as } m \rightarrow \infty, \quad (9)$$

are satisfied, then the series $\sum a_n \lambda_n$ is summable $\varphi - |\bar{N}, p_n; \delta|_k$, $k \geq 1$ and $0 \leq \delta < 1/k$.

THEOREM 4

Let (X_n) be an almost increasing sequence and $\varphi_n p_n = O(P_n)$. If conditions (1)–(4), (6), (8)–(9) and

$$\sum_{n=1}^m \varphi_n^{\delta k} \frac{|s_n|^k}{P_n} = O(X_m) \quad \text{as } m \rightarrow \infty, \quad (10)$$

are satisfied, then the series $\sum a_n \lambda_n$ is summable $\varphi - |\bar{N}, p_n; \delta|_k$, $k \geq 1$ and $0 \leq \delta < 1/k$.

For $\delta = 0$ and $\varphi_n = \frac{P_n}{p_n}$, Theorem 3 and Theorem 4 reduce to Theorem 1 and Theorem 2, respectively.

LEMMA 1 ([10])

If (X_n) is an almost increasing sequence, then under conditions (2)–(3), we have

$$n X_n \beta_n = O(1) \quad \text{as } n \rightarrow \infty, \quad (11)$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \quad (12)$$

LEMMA 2 ([10])

If (X_n) is an almost increasing sequence, then under conditions (2) and (6), we have

$$P_n X_n \beta_n = O(1) \quad \text{as } n \rightarrow \infty, \quad (13)$$

$$\sum_{n=1}^{\infty} p_n X_n \beta_n < \infty. \quad (14)$$

Proof of Theorem 3. Let (I_n) be the sequence of $(\bar{N}, p_n)$ mean of the series $\sum a_n \lambda_n$. Then, we have

$$I_n = \frac{1}{P_n} \sum_{v=0}^n p_v \sum_{r=0}^v a_r \lambda_r = \frac{1}{P_n} \sum_{v=0}^n (P_n - P_{v-1}) a_v \lambda_v.$$

For $n \geq 1$, we get

$$I_n - I_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^n P_{v-1} a_v \lambda_v.$$

From Abel's transformation, we obtain

$$\begin{aligned} I_n - I_{n-1} &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \Delta(P_{v-1} \lambda_v) s_v + \frac{p_n s_n \lambda_n}{P_n} \\ &= \frac{p_n s_n \lambda_n}{P_n} - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v s_v \lambda_v + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v s_v \Delta \lambda_v \\ &= I_{n,1} + I_{n,2} + I_{n,3}. \end{aligned}$$

In order to prove that $\sum a_n \lambda_n$ is summable $\varphi - |\bar{N}, p_n; \delta|_k$, we will show

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k + k - 1} |I_{n,r}|^k < \infty \quad \text{for } r = 1, 2, 3.$$

First, using condition (4) and the fact that (X_n) is an almost increasing sequence, we obtain $|\lambda_n|^{k-1} = O(1)$. Moreover, using the fact that $\varphi_n p_n = O(P_n)$, we have

$$\sum_{n=1}^m \varphi_n^{\delta k + k - 1} |I_{n,1}|^k = O(1) \sum_{n=1}^m \varphi_n^{\delta k - 1} |\lambda_n| |s_n|^k.$$

By Abel's transformation,

$$\begin{aligned} \sum_{n=1}^m \varphi_n^{\delta k + k - 1} |I_{n,1}|^k &= O(1) \sum_{n=1}^{m-1} \Delta |\lambda_n| \sum_{r=1}^n \varphi_r^{\delta k - 1} |s_r|^k + O(1) |\lambda_m| \sum_{n=1}^m \varphi_n^{\delta k - 1} |s_n|^k \\ &= O(1) \sum_{n=1}^{m-1} \beta_n X_n + O(1) |\lambda_m| X_m = O(1) \quad \text{as } m \rightarrow \infty, \end{aligned}$$

by virtue of (1), (8), (12) and (4).

Now, by means of Hölder's inequality, using the fact that $\varphi_n p_n = O(P_n)$ and conditions (4) and (9), we get

$$\begin{aligned} \sum_{n=2}^{m+1} \varphi_n^{\delta k + k - 1} |I_{n,2}|^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k - 1} \frac{1}{P_{n-1}^k} \left(\sum_{v=1}^{n-1} p_v |s_v| |\lambda_v| \right)^k \\ &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k - 1} \frac{1}{P_{n-1}^k} \sum_{v=1}^{n-1} p_v |\lambda_v|^k |s_v|^k \left(\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right)^{k-1} \end{aligned}$$

$$\begin{aligned}
&= O(1) \sum_{v=1}^m p_v |\lambda_v| |s_v|^k \sum_{n=v+1}^{m+1} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}} \\
&= O(1) \sum_{v=1}^m \varphi_v^{\delta k-1} |\lambda_v| |s_v|^k = O(1) \quad \text{as } m \rightarrow \infty,
\end{aligned}$$

as in $I_{n,1}$.

Finally, again using the fact that $\varphi_n p_n = O(P_n)$, Hölder's inequality and condition (1), we obtain

$$\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} |I_{n,3}|^k = O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v |s_v|^k \beta_v \left(\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v \beta_v \right)^{k-1}.$$

Here, (12) yields

$$\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} |I_{n,3}|^k = O(1) \sum_{v=1}^m P_v |s_v|^k \beta_v \sum_{n=v+1}^{m+1} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}}.$$

Now, from (9), we get

$$\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} |I_{n,3}|^k = O(1) \sum_{v=1}^m \varphi_v^{\delta k} v \beta_v \frac{|s_v|^k}{v}.$$

Then,

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} |I_{n,3}|^k &= O(1) \sum_{v=1}^{m-1} \Delta(v \beta_v) \sum_{r=1}^v \varphi_r^{\delta k} \frac{1}{r} |s_r|^k + O(1) m \beta_m \sum_{v=1}^m \varphi_v^{\delta k} \frac{1}{v} |s_v|^k \\
&= O(1) \sum_{v=1}^{m-1} v |\Delta \beta_v| X_v + O(1) \sum_{v=1}^{m-1} \beta_v X_v + O(1) m \beta_m X_m \\
&= O(1) \quad \text{as } m \rightarrow \infty,
\end{aligned}$$

by using Abel's transformation, (7), (3), (12) and (11). Therefore, the proof of Theorem 3 is completed.

Proof of Theorem 4. For $r = 1$ and $r = 2$, the proof of Theorem 4 as in the proof of Theorem 3. Thus, they can be omitted. Now, we will show

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} |I_{n,r}|^k < \infty$$

only for $r = 3$, by using the hypotheses of Theorem 4, Lemma 1 and Lemma 2.

For $r = 3$, we get

$$\begin{aligned} \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} |I_{n,3}|^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v |s_v|^k \beta_v \left(\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v \beta_v \right)^{k-1} \\ &= O(1) \sum_{v=1}^m P_v |s_v|^k \beta_v \sum_{n=v+1}^{m+1} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}}. \end{aligned}$$

Here, from (9), we get

$$\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} |I_{n,3}|^k = O(1) \sum_{v=1}^m \varphi_v^{\delta k} P_v \beta_v \frac{|s_v|^k}{P_v}.$$

Thus, we have

$$\begin{aligned} \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} |I_{n,3}|^k &= O(1) \sum_{v=1}^{m-1} \Delta(P_v \beta_v) \sum_{r=1}^v \varphi_r^{\delta k} \frac{|s_r|^k}{P_r} + O(1) P_m \beta_m \sum_{v=1}^m \varphi_v^{\delta k} \frac{|s_v|^k}{P_v} \\ &= O(1) \sum_{v=1}^{m-1} P_v |\Delta \beta_v| X_v + O(1) \sum_{v=1}^{m-1} p_v \beta_v X_v + O(1) P_m \beta_m X_m \\ &= O(1) \quad \text{as } m \rightarrow \infty, \end{aligned}$$

by using Abel's transformation, (10), (6), (14) and (13). Hence, the proof of Theorem 4 is completed.

Acknowledgement. This work was supported by Research Fund of the Erciyes University, project number: FDK-2017-6945.

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Received: February 7, 2019; final version: May 9, 2019;
available online: May 16, 2019.