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Studia Mathematica XVIII (2019)***Mirosław Ślosarski***Multi-invertible maps and their applications**

Abstract. In this article, we define multi-invertible, multivalued maps. These mappings are a natural generalization of r -maps (in particular, the singlevalued invertible maps). They have many interesting properties and applications. In this article, the multi-invertible maps are applied to the construction of morphisms and to the theory of coincidence.

1. Introduction

In mathematical literature multivalued admissible maps are known (see [4]). In [8] we defined multivalued locally admissible maps that are an essentially wider class of maps than admissible maps. In this article we define multivalued, multi-invertible maps in the context of locally admissible maps. These maps are a natural, essential generalization of r -maps (see [1]) in particular, the singlevalued invertible maps. The generalized Vietoris maps considered in the article (see [9]) are particularly multi-invertible. Multi-invertible maps have a lot of interesting properties. The composition of multi-invertible maps is multi-invertible. If a map is multi-invertible then there exists exactly one multi-inverse map. Multi-invertible maps have many applications. In this article they are applied to the construction of morphisms and to the theory of coincidence. The property of coincidence of multi-invertible maps can be applied to solving differential inclusions (see [3]). It is worth mentioning that multi-invertible maps can be also applied to the theory of multi-domination (see [9, 10]).

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2. Preliminaries

Throughout this paper all spaces are assumed to be Hausdorff topological spaces. A continuous mapping $f: X \rightarrow Y$ is called *perfect* if for each $y \in Y$ the set $f^{-1}(y)$ is non-empty and compact and f is a closed map. Let X and Y be two spaces and assume that for every $x \in X$ a non-empty subset $\varphi(x)$ of Y is given. In such a case we say that $\varphi: X \multimap Y$ is a *multivalued mapping*. For a multivalued mapping $\varphi: X \multimap Y$ and a subset $A \subset Y$ we let

$$\begin{aligned}\varphi_s^{-1}(A) &= \{x \in X : \varphi(x) \subset A\}, \\ \varphi_b^{-1}(A) &= \{x \in X : \varphi(x) \cap A \neq \emptyset\}.\end{aligned}$$

Let H_* be the Čech homology functor with compact carriers and coefficients in the field of rational numbers $\mathbb{Q}$ from the category of Hausdorff topological spaces and continuous maps to the category of a graded vector space and linear maps of degree zero. Thus $H_*(X) = \{H_q(X)\}$ is a graded vector space, $H_q(X)$ being a q -dimensional Čech homology group with compact carriers of X . For a continuous map $f: X \rightarrow Y$, $H_*(f)$ is the induced linear map $f_* = \{f_q\}$, where $f_q: H_q(X) \rightarrow H_q(Y)$ (see [4]).

A set X is *acyclic* if

- (i) X is non-empty,
- (ii) $H_q(X) = 0$ for every $q \geq 1$ and
- (iii) $H_0(X) \approx \mathbb{Q}$.

A perfect map $p: X \rightarrow Y$ is called *Vietoris* provided for every $y \in Y$ the set $p^{-1}(y)$ is acyclic. We recall that the composition of two Vietoris mappings is a Vietoris mapping and if $p: X \rightarrow Y$ is a Vietoris map then $p_*: H_*(X) \rightarrow H_*(Y)$ is an isomorphism (see [4]). The symbol $D(X, Y)$ will denote the set of all diagrams of the form

$$X \xleftarrow{p} Z \xrightarrow{q} Y,$$

where $p: Z \rightarrow X$ denotes a Vietoris map and $q: Z \rightarrow Y$ denotes a continuous map. Each such diagram will be denoted by (p, q) .

DEFINITION 2.1 (see [4])

Let $(p_1, q_1) \in D(X, Y)$ and $(p_2, q_2) \in D(Y, T)$. The composition of diagrams

$$X \xleftarrow{p_1} Z_1 \xrightarrow{q_1} Y \xleftarrow{p_2} Z_2 \xrightarrow{q_2} T,$$

is called a diagram $(p, q) \in D(X, T)$,

$$X \xleftarrow{p} Z_1 \triangle_{q_1 p_2} Z_2 \xrightarrow{q} T,$$

where

$$Z_1 \triangle_{q_1 p_2} Z_2 = \{(z_1, z_2) \in Z_1 \times Z_2 : q_1(z_1) = p_2(z_2)\},$$

$$p = p_1 \circ \pi_1, \quad q = q_2 \circ \pi_2,$$

$$Z_1 \xleftarrow{\pi_1} Z_1 \triangle_{q_1 p_2} Z_2 \xrightarrow{\pi_2} Z_2,$$

$$\pi_1(z_1, z_2) = z_1 \text{ (Vietoris map)}, \quad \pi_2(z_1, z_2) = z_2 \quad \text{for each } (z_1, z_2) \in Z.$$

It shall be written

$$(p, q) = (p_2, q_2) \circ (p_1, q_1).$$

Let $\varphi: X \multimap Y$ be a multivalued map. We recall that the map φ is *admissible* (resp. *s-admissible*) (see [4]) if there exist a Vietoris map $p: Z \rightarrow X$ and a continuous map $q: Z \rightarrow Y$ such that for each $x \in X$,

$$q(p^{-1}(x)) \subset \varphi(x) \quad (\text{resp. } q(p^{-1}(x)) = \varphi(x)),$$

to simplify the notation we will write $(p, q) \subset \varphi$ (resp. $(p, q) = \varphi$).

Let $\varphi: X \multimap Y$ be a map and let $A \subset X$ be a non-empty set. We denote by $\varphi_A: A \multimap X$ a map given by the formula $\varphi_A(x) = \varphi(x)$ for each $x \in A$.

DEFINITION 2.2

A multivalued map $\varphi: X \multimap Y$ is called *locally admissible* provided for any compact and non-empty set $K \subset X$ there exists an open set $V \subset X$ such that $K \subset V$ and $\varphi_V: V \multimap X$ is admissible.

PROPOSITION 2.3 ([8])

Let $\varphi: X \multimap Y$ and $\psi: Y \multimap Z$ be locally admissible maps. Then $\psi \circ \varphi: X \multimap Z$ is locally admissible.

It is obvious that if a space X is compact then $\varphi: X \multimap Y$ is locally admissible if and only if $\varphi: X \multimap Y$ is admissible.

DEFINITION 2.4

A topological vector space is called *Klee admissible* provided for every compact $K \subset E$ and for every open neighborhood of zero V in E there exists a continuous map $\pi_V: K \rightarrow E$ such that

$$(2.4.1) \quad (x - \pi_V(x)) \in V \text{ for every } x \in K,$$

$$(2.4.2) \quad \text{there exists a natural number } n = n_K \text{ such that } \pi_V(K) \subset E^n, \text{ where } E^n \text{ is an } n\text{-dimensional subspace of } E.$$

It is well known that any locally convex space is Klee admissible. We will write that a space $X \in A_C$ ($X \in NA_C$) if there exists a Klee admissible space E and a closed embedding $h: X \rightarrow E$ such that $h(X)$ is a retract of E ($h(X)$ is a retract of some open set $U \subset E$ such that $h(X) \subset U$).

PROPOSITION 2.5 ([4, 6])

Let $X \in NA_C$ and let $U \subset X$ be an open set. Then $U \in NA_C$.

THEOREM 2.6 ([4, 6])

Let $X \in NA_C$. Consider a diagram

$$X \xleftarrow{p} Z \xrightarrow{q} X,$$

where p is Vietoris and q is compact ($\overline{q(Z)} \subset X$ is compact). Then $q_* \circ p_*^{-1}$ is a Leray endomorphism and $\Lambda(q_* \circ p_*^{-1}) \neq 0$ implies that p and q have a coincidence point, that is, there is a point $z \in Z$ such that $p(z) = q(z)$.

3. Multi-invertible maps

DEFINITION 3.1

We say that a multivalued map $\varphi: X \multimap Y$ is *multi-invertible* if there exists a multivalued map $\overleftarrow{\varphi}: Y \multimap X$ (multi-inverse map) such that the following conditions are satisfied

$$(3.1.1) \text{ for each } x \in X \text{ and for each } y \in Y \ (y \in \varphi(x) \Rightarrow x \in \overleftarrow{\varphi}(y)),$$

$$(3.1.2) \text{ for each } x \in X \text{ and for each } y \in Y \ (x \in \overleftarrow{\varphi}(y) \Rightarrow y \in \varphi(x)).$$

We observe that if $\varphi: X \multimap Y$ is multi-invertible then there exists exactly one multi-inverse map $\overleftarrow{\varphi}: Y \multimap X$ given by the formula

$$\overleftarrow{\varphi}(y) = \varphi_b^{-1}(y) = \{x \in X : y \in \varphi(x)\}.$$

It is clear that a multivalued map $\varphi: X \multimap Y$ is multi-invertible if and only if $\varphi(X) = Y$ and if a singlevalued map $f: X \rightarrow Y$ is invertible then $\overleftarrow{f} = f^{-1}$. Let $\varphi: X \multimap Y$ be a multivalued map. If φ has compact images and for every open $U \subset Y$ the set $\varphi_s^{-1}(U)$ ($\varphi_b^{-1}(Y \setminus U)$) is open (is closed), then φ is called an *upper semi-continuous mapping*; we shall write that φ is u.s.c. We will say that an u.s.c. multivalued map $\varphi: X \multimap Y$ is *perfect*, if for each non-empty and compact set $A \subset Y$ the set $\varphi_b^{-1}(A)$ is non-empty and compact and φ is a closed map.

PROPOSITION 3.2

A map $\varphi: X \multimap Y$ is perfect if and only if φ is multi-invertible and $\overleftarrow{\varphi}$ is a perfect map.

Proof. It is obvious that

$$\varphi(A) = \overleftarrow{\varphi}_b^{-1}(A) \quad \text{for each non-empty set } A \subset X$$

and

$$\overleftarrow{\varphi}(B) = \varphi_b^{-1}(B) \quad \text{for each non-empty set } B \subset Y.$$

From Proposition 3.2 we get

PROPOSITION 3.3

Let X and Y be compact spaces and let $\varphi: X \multimap Y$ such that $\varphi(X) = Y$. A multivalued map $\varphi: X \multimap Y$ is u.s.c. if and only if $\overleftarrow{\varphi}: Y \multimap X$ is u.s.c..

We will give a few examples. Let $\mathbb{K}^n$ be a closed ball in euclidean space $\mathbb{R}^n$ with the center of 0 and radius 1 and let $\mathbb{S}^n \subset \mathbb{K}^{n+1}$ be a sphere. We denote by $\odot$ a scalar product in $\mathbb{R}^n$ and let $I = [0, 1]$.

EXAMPLE 3.4

Let $\varphi: \mathbb{S}^n \multimap \mathbb{S}^n$ be a multivalued map given by the formula

$$\varphi(x) = \{y \in \mathbb{S}^n : x \odot y = 0\} \quad \text{for each } x \in \mathbb{S}^n.$$

We observe that φ is multi-invertible and $\overleftarrow{\varphi} = \varphi$. For $n = 2k$, φ is not admissible. Indeed, assume the contrary, i.e. that φ is admissible. Then there exist a Vietoris

map $p: Z \rightarrow \mathbb{S}^n$ and a continuous map $q: Z \rightarrow \mathbb{S}^n$ such that $(p, q) \subset \varphi$. Hence, for each $z \in Z$,

$$p(z) \odot q(z) = 0, \quad \text{so } p \text{ and } q \text{ are homotopic.}$$

We have

$$\Lambda(q_* \circ p_*^{-1}) = \Lambda(p_* \circ p_*^{-1}) = \Lambda(Id_{H_*(\mathbb{S}^n)}) = 2.$$

From Theorem 2.6 there exists a point $z \in Z$ such that $p(z) = q(z)$, but it is a contradiction. From the mathematical literature we know that for $n = 2k - 1$ there exists a continuous map $f: \mathbb{S}^n \rightarrow \mathbb{S}^n$ such that, for each $x \in \mathbb{S}^n$, $x \odot f(x) = 0$. Hence, φ is admissible, because $(Id_{\mathbb{S}^n}, f) \subset \varphi$.

EXAMPLE 3.5

Let $\varphi: \mathbb{S}^n \multimap \mathbb{K}^{n+1}$ be a multivalued map given by the formula

$$\varphi(x) = \{t \cdot x : t \in I\}.$$

The map φ is u.s.c. and for each $x \in \mathbb{S}^n$ the set $\varphi(x)$ is compact and convex, so φ is s-admissible (in particular, admissible). A multivalued map $\psi: \mathbb{K}^{n+1} \multimap \mathbb{S}^n$ given by the formula

$$\psi(x) = \begin{cases} x/\|x\| & \text{for } x \neq 0, \\ \mathbb{S}^n & \text{for } x = 0 \end{cases}$$

is multi-inverse to φ . We observe that ψ it is not an admissible map. Indeed, assume the contrary, i.e. that φ is admissible. Then there exist a Vietoris map $p: Z \rightarrow \mathbb{K}^{n+1}$ and a continuous map $q: Z \rightarrow \mathbb{S}^n$ such that $(p, q) \subset \psi$. Let $j: \mathbb{S}^n \rightarrow \mathbb{K}^{n+1}$ be an inclusion. We have the following diagram

$$H_*(\mathbb{S}^n) \xrightarrow{j_*} H_*(\mathbb{K}^{n+1}) \xleftarrow{p_*} H_*(Z) \xrightarrow{q_*} H_*(\mathbb{S}^n).$$

Hence, it result that

$$(q_* \circ p_*^{-1}) \circ j_* = Id_{H_*(\mathbb{S}^n)},$$

but it is not possible.

DEFINITION 3.6

A locally admissible map $\varphi: X \multimap Y$ is *multi-invertible* if there exists a multi-inverse and locally admissible map $\overleftarrow{\varphi}: Y \multimap X$.

PROPOSITION 3.7

Let $f: X \rightarrow Y$ be a continuous map. The map f is multi-invertible if and only if for each compact set $K \subset Y$ there exist an open neighborhood U of K and a continuous map $g: U \rightarrow X$ such that $f \circ g: U \rightarrow Y$ is a Vietoris map.

Proof. Let $f: X \rightarrow Y$ be a continuous and multi-invertible map. From Definition 3.6 the multi-inverse map $\overleftarrow{f}: Y \multimap X$ is locally admissible. Let $K \subset Y$ be a compact set. There exists an open neighborhood $U \subset Y$ of K such that $\overleftarrow{f}|_U: U \multimap X$ is an admissible map, that is, there exists a diagram $(p, q) \in D(U, X)$ such that $(p, q) \subset \overleftarrow{f}$. Hence we have

$$q(p^{-1}(y)) \subset f^{-1}(y) \quad \text{for each } y \in U,$$

so $f \circ q = p$ is a Vietoris map. The proof in the opposite direction is obvious.

In particular, if a continuous map $f: X \rightarrow Y$ is an r -map, that is, there exists $g: Y \rightarrow X$ such that $f \circ g = Id_Y$, then it is multi-invertible.

PROPOSITION 3.8

Let $\varphi: X \multimap Y$ be an admissible map. Assume that there exist Vietoris maps $p_1: Z \rightarrow X$ and $p_2: Z \rightarrow Y$ such that $(p_1, p_2) \subset \varphi$. Then φ is multi-invertible.

Proof. We observe that $\varphi(X) = Y$ and $((p_1, p_2) \subset \varphi) \Leftrightarrow ((p_2, p_1) \subset \overleftarrow{\varphi})$.

We will say that $\varphi: X \multimap Y$ is relatively proper if for each non-empty and compact set $K \subset Y$ the set $\overline{\varphi_b^{-1}(K)}$ is non-empty and compact.

PROPOSITION 3.9

Let $\varphi: X \multimap Y$ be relatively proper. Assume that for each compact set $K \subset X$ there exist an open neighborhood $U \subset X$ of K , an open neighborhood $V \subset Y$ of $\varphi(K)$ and Vietoris maps $p_1: Z \rightarrow U$ and $p_2: Z \rightarrow V$ such that $(p_1, p_2) \subset \varphi_U$. Then φ is multi-invertible and $\overleftarrow{\varphi}$ is locally admissible.

Proof. It is easy to observe that φ is locally admissible. We show that the multi-inverse map $\overleftarrow{\varphi}: Y \multimap X$ is locally admissible. Let $K \subset Y$ be a compact set. We denote by $K_1 = \overline{\varphi_b^{-1}(K)} = \overline{\overleftarrow{\varphi}(K)}$. From the assumption the set K_1 is compact, so there exist an open neighborhood $U \subset X$ of K_1 , an open neighborhood $V \subset Y$ of $\varphi(K_1)$ and Vietoris maps $p_1: Z \rightarrow U$ and $p_2: Z \rightarrow V$ such that $(p_1, p_2) \subset \varphi_U$. We have

$$K \subset \varphi(\overleftarrow{\varphi}(K)) \subset \varphi(\overline{\overleftarrow{\varphi}(K)}) \subset V.$$

Hence $(p_2, p_1) \subset \overleftarrow{\varphi}_V$ and the proof is complete.

A few obvious properties of multi-invertible mappings will follow, which do not require proof.

PROPOSITION 3.10

Let $\varphi, \psi: X \multimap Y$, $\eta: Y \multimap Z$ and $\theta: T \multimap S$ be multi-invertible maps. Then we have

$$(3.10.1) \text{ for each } x \in X \text{ and } y \in Y \text{ } y \in \varphi(\overleftarrow{\varphi}(y)) \text{ and } x \in \overleftarrow{\varphi}(\varphi(x)),$$

$$(3.10.2) \text{ } \overleftarrow{\eta \circ \varphi} = \overleftarrow{\varphi} \circ \overleftarrow{\eta},$$

$$(3.10.3) \text{ } \overleftarrow{(\overleftarrow{\varphi})} = \varphi,$$

$$(3.10.4) \text{ } \overleftarrow{\varphi \times \theta} = \overleftarrow{\varphi} \times \overleftarrow{\theta}, \text{ where the map } \varphi \times \theta: X \times T \multimap Y \times S \text{ given by the formula}$$

$$(\varphi \times \theta)(x, t) = \varphi(x) \times \theta(t) \quad \text{for each } (x, t) \in (X \times T),$$

$$(3.10.5) \text{ if } \varphi(x) \cap \psi(x) \neq \emptyset \text{ for each } x \in X \text{ and if } \Theta: X \multimap Y \text{ is a map given by the formula}$$

$$\Theta(x) = \varphi(x) \cap \psi(x) \quad \text{for each } x \in X,$$

then Θ is multi-invertible and $\overleftarrow{\Theta}: Y \multimap X$ is given by the formula

$$\overleftarrow{\Theta}(y) = \overleftarrow{\varphi}(y) \cap \overleftarrow{\psi}(y) \quad \text{for each } y \in Y.$$

REMARK 3.11

Let

$$\begin{array}{ccc}
X & \xrightarrow{f_1} & T \\
\downarrow \Delta_f & & \downarrow Id_T \\
Y & \xrightarrow{f_2} & T
\end{array}$$

be a commutative diagram, where f_1 and f_2 are continuous maps such that $f_1(X) = f_2(Y)$ and $\Delta_f: X \multimap Y$ is a multivalued map given by the formula

$$\Delta_f(x) = f_2^{-1}(f_1(x)) \quad \text{for each } x \in X. \quad (1)$$

We observe that the map Δ_f has a closed graph. Furthermore the map Δ_f is multi-invertible and $\overleftarrow{\Delta}_f: Y \multimap X$ is given by the formula

$$\overleftarrow{\Delta}_f(y) = f_1^{-1}(f_2(y)) \quad \text{for each } y \in Y. \quad (2)$$

Moreover, if f_1 and f_2 are perfect maps then Δ_f is u.s.c.. Let $(p_1, q_1), (p_2, q_2) \in D(X, Y)$ be such that $q_1 \circ \overleftarrow{p}_1 = q_2 \circ \overleftarrow{p}_2$ and let

$$\begin{array}{ccccc}
X & \xleftarrow{p_1} & Z_1 & \xrightarrow{q_1} & Y \\
\downarrow Id_X & & \downarrow \Delta_{pq} & & \downarrow Id_Y \\
X & \xleftarrow{p_2} & Z_2 & \xrightarrow{q_2} & Y
\end{array}$$

be commutative diagrams, where $\Delta_{pq}: Z_1 \multimap Z_2$ is a multivalued map given by the formula

$$\Delta_{pq}(x) = \Delta_p(x) \cap \Delta_q(x) \quad \text{for each } x \in X$$

(see (1)). It is easy to show that Δ_{pq} is well defined. From Proposition 3.10 (see (3.10.5)) the map Δ_{pq} is multi-invertible (see (2)) and u.s.c (see [4, 2]). It is clear that

$$p_2 \circ \Delta_{pq} = p_1 \quad \text{and} \quad q_2 \circ \Delta_{pq} = q_1.$$

We observe that if $\varphi: Z_1 \multimap Z_2$ is a multivalued map such that

$$p_2 \circ \varphi = p_1 \quad \text{and} \quad q_2 \circ \varphi = q_1,$$

then $\varphi(x) \subset \Delta_{pq}(x)$ for each $x \in X$.

(i) If there exists a homeomorphism $h: Z_1 \rightarrow Z_2$ such that

$$p_2 \circ h = p_1 \quad \text{and} \quad q_2 \circ h = q_1,$$

then $(p_1, q_1) \approx_{K1} (p_2, q_2)$ (in the sense of Kryszewski (see [7])) and $(Id_{Z_1}, h) \subset \Delta_{pq}$ and $(Id_{Z_2}, h^{-1}) \subset \overleftarrow{\Delta}_{pq}$, where $h^{-1}: Z_2 \rightarrow Z_1$ is an inverse homeomorphism to h .

(ii) If there exist continuous maps $f: Z_1 \rightarrow Z_2$ and $g: Z_2 \rightarrow Z_1$ such that

$$p_2 \circ f = p_1, \quad q_2 \circ f = q_1, \quad p_1 \circ g = p_2 \quad \text{and} \quad q_1 \circ g = q_2,$$

then $(p_1, q_1) \approx_G (p_2, q_2)$ (in the sense of Górniewicz (see [5])) and $(Id_{Z_1}, f) \subset \Delta_{pq}$ and $(Id_{Z_2}, g) \subset \overleftarrow{\Delta}_{pq}$.

(iii) If there exist Vietoris maps $v_1: Z \rightarrow Z_1$ and $v_2: Z \rightarrow Z_2$ such that

$$p_2 \circ v_2 = p_1 \circ v_1 \quad \text{and} \quad q_2 \circ v_2 = q_1 \circ v_1, \quad (3)$$

then $(p_1, q_1) \approx_{K2} (p_2, q_2)$ (in the sense of Kryszewski (see [7, 11])) and $(v_1, v_2) \subset \Delta_{pq}$ and $(v_2, v_1) \subset \overleftarrow{\Delta}_{pq}$.

4. Admissible morphisms

Remark 3.11 justify the following definition.

DEFINITION 4.1

Let $(p_1, q_1), (p_2, q_2) \in D(X, Y)$ and let

$$X \xleftarrow{p_1} Z_1 \xrightarrow{q_1} Y, \quad X \xleftarrow{p_2} Z_2 \xrightarrow{q_2} Y.$$

We will say that the diagrams (p_1, q_1) and (p_2, q_2) are in relation in the set $D(X, Y)$ (we will write $(p_1, q_1) \approx_{ad} (p_2, q_2)$) if there exists an admissible and multi-invertible map $\varphi: Z_1 \multimap Z_2$ such that $\overleftarrow{\varphi}$ is admissible and the following diagram is commutative

$$\begin{array}{ccccc} X & \xleftarrow{p_1} & Z_1 & \xrightarrow{q_1} & Y \\ \downarrow Id_X & & \downarrow \varphi & & \downarrow Id_Y \\ X & \xleftarrow{p_2} & Z_2 & \xrightarrow{q_2} & Y \end{array}$$

that is

$$p_2 \circ \varphi = p_1 \quad \text{and} \quad q_2 \circ \varphi = q_1.$$

PROPOSITION 4.2

The relation in the set $D(X, Y)$ introduced in Definition 4.1 is an equivalence relation.

Proof. In the proof of reflexivity of the relation, it is enough to assume that $Z_1 = Z_2$ and $\varphi = Id_{Z_1}$. We observe that if

$$p_2 \circ \varphi = p_1 \quad \text{and} \quad q_2 \circ \varphi = q_1$$

then

$$p_1 \circ \overleftarrow{\varphi} = p_2 \quad \text{and} \quad q_1 \circ \overleftarrow{\varphi} = q_2, \quad (4)$$

where $\varphi: Z_1 \multimap Z_2$ is a multi-invertible and admissible map and $\overleftarrow{\varphi}$ is admissible. Hence, the relation is symmetric.

It shall be now proven that the relation is transitive. Suppose that $(p_1, q_1) \approx_{ad} (p_2, q_2)$ and $(p_2, q_2) \approx_{ad} (p_3, q_3)$. Then from the assumption we have the following commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{p_1} & Z_1 & \xrightarrow{q_1} & Y \\ \downarrow Id & & \downarrow \varphi_1 & & \downarrow Id \\ X & \xleftarrow{p_2} & Z_2 & \xrightarrow{q_2} & Y \\ \downarrow Id & & \downarrow \varphi_2 & & \downarrow Id \\ X & \xleftarrow{p_3} & Z_3 & \xrightarrow{q_3} & Y \end{array}$$

that is

$$p_2 \circ \varphi_1 = p_1, \quad q_2 \circ \varphi_1 = q_1, \quad p_3 \circ \varphi_2 = p_2, \quad q_3 \circ \varphi_2 = q_2,$$

where φ_1 and φ_2 are admissible and multi-invertible maps. Let $\varphi = \varphi_2 \circ \varphi_1$. By Proposition 3.10 (see (3.10.2)) φ is an admissible and multi-invertible map. We have

$$p_3 \circ \varphi = p_3 \circ (\varphi_2 \circ \varphi_1) = p_1 \quad \text{and} \quad q_3 \circ \varphi = q_3 \circ (\varphi_2 \circ \varphi_1) = q_1$$

and the proof is complete.

PROPOSITION 4.3

Let $(p_1, q_1), (p_2, q_2) \in D(X, Y)$. The relation $\approx_{ad}$ in the set $D(X, Y)$ satisfies the following conditions

$$(4.3.1) \quad ((p_1, q_1) \approx_{ad} (p_2, q_2)) \Rightarrow (q_1 \circ \overleftarrow{p_1} = q_2 \circ \overleftarrow{p_2}),$$

$$(4.3.2) \quad ((p_1, q_1) \approx_{ad} (p_2, q_2)) \Rightarrow (q_{1*} \circ p_{1*}^{-1} = q_{2*} \circ p_{2*}^{-1}),$$

$$(4.3.3) \quad \text{let } (p_3, q_3), (p_4, q_4) \in D(Y, T), \text{ then}$$

$$\begin{aligned} & ((p_1, q_1) \approx_{ad} (p_2, q_2) \text{ and } (p_3, q_3) \approx_{ad} (p_4, q_4)) \\ & \Rightarrow (((p_3, q_3) \circ (p_1, q_1)) \approx_{ad} ((p_4, q_4) \circ (p_2, q_2))). \end{aligned}$$

Proof. (4.3.1). Let $(p_1, q_1) \approx_{ad} (p_2, q_2)$. Then there exists a multi-invertible and admissible map φ such that $p_2 \circ \varphi = p_1$ and $q_2 \circ \varphi = q_1$. We observe that if $p_2 \circ \varphi = p_1$ then $\overleftarrow{\varphi} \circ \overleftarrow{p_2} = \overleftarrow{p_1}$. Hence we have

$$q_1 \circ \overleftarrow{p_1} = q_1 \circ (\overleftarrow{\varphi} \circ \overleftarrow{p_2}) = (q_1 \circ \overleftarrow{\varphi}) \circ \overleftarrow{p_2} = q_2 \circ \overleftarrow{p_2} \quad (\text{see (4)}).$$

(4.3.2). Let $(p_1, q_1) \approx_{ad} (p_2, q_2)$. Then there exists a multi-invertible and admissible map φ such that $p_2 \circ \varphi = p_1$ and $q_2 \circ \varphi = q_1$. Let $(r, s) \subset \overleftarrow{\varphi}$ then (see (4)),

$$q_1 \circ s = q_2 \circ r \quad \text{and} \quad p_1 \circ s = p_2 \circ r.$$

Hence

$$q_{1*} \circ s_* = q_{2*} \circ r_* \quad \text{and} \quad p_{1*} \circ s_* = p_{2*} \circ r_*.$$

We observe that s_* is an isomorphism, so

$$p_{1*} = p_{2*} \circ r_* \circ s_*^{-1}.$$

We have

$$\begin{aligned} q_{1*} \circ p_{1*}^{-1} &= q_{1*} \circ (p_{2*} \circ r_* \circ s_*^{-1})^{-1} = q_{1*} \circ (s_* \circ r_*^{-1} \circ p_{2*}^{-1}) \\ &= (q_{1*} \circ s_*) \circ r_*^{-1} \circ p_{2*}^{-1} = q_{2*} \circ p_{2*}^{-1}. \end{aligned}$$

(4.3.3). We have the following commutative diagrams

$$\begin{array}{ccccccccc} X & \xleftarrow{p_1} & Z_1 & \xrightarrow{q_1} & Y & \xleftarrow{p_3} & Z_3 & \xrightarrow{q_3} & T \\ \downarrow Id_X & & \downarrow \varphi_1 & & \downarrow Id_Y & & \downarrow \varphi_2 & & \downarrow Id_T \\ X & \xleftarrow{p_2} & Z_2 & \xrightarrow{q_2} & Y & \xleftarrow{p_4} & Z_4 & \xrightarrow{q_4} & T \end{array}$$

that is

$$p_2 \circ \varphi_1 = p_1, \quad q_2 \circ \varphi_1 = q_1, \quad p_4 \circ \varphi_2 = p_3, \quad q_4 \circ \varphi_2 = q_3.$$

We make the following diagram (see Definition 2.1)

$$\begin{array}{ccccc} X & \xleftarrow{p} & Z_1 \triangle_{q_1 p_3} Z_3 & \xrightarrow{q} & T \\ \downarrow Id_X & & \downarrow \psi & & \downarrow Id_T, \\ X & \xleftarrow{p'} & Z_2 \triangle_{q_2 p_4} Z_4 & \xrightarrow{q'} & T \end{array}$$

where $(p, q) = (p_3, q_3) \circ (p_1, q_1)$, $(p', q') = (p_4, q_4) \circ (p_2, q_2)$ and $\psi(z_1, z_3) = \varphi_1(z_1) \times \varphi_2(z_3)$ for each $(z_1, z_3) \in Z_1 \triangle_{q_1 p_3} Z_3$. First we need to prove that the map ψ is well defined. Let $(z_1, z_3) \in Z_1 \triangle_{q_1 p_3} Z_3$ and $(z_2, z_4) \in \varphi_1(z_1) \times \varphi_2(z_3)$. We have

$$q_2(z_2) = q_1(z_1) = p_3(z_3) = p_4(z_4).$$

It is clear that (see (3.10.4)) the map ψ is multi-invertible, admissible and $\overleftarrow{\psi}$ given by the formula

$$\overleftarrow{\psi}(z_2, z_4) = \overleftarrow{\varphi_2}(z_2) \times \overleftarrow{\varphi_4}(z_4)$$

is admissible. We will show now that the above diagram is commutative. Let $f_1: Z_1 \triangle_{q_1 p_3} Z_3 \rightarrow Z_1$, $f_3: Z_1 \triangle_{q_1 p_3} Z_3 \rightarrow Z_3$, $f_2: Z_2 \triangle_{q_2 p_4} Z_4 \rightarrow Z_2$, $f_4: Z_2 \triangle_{q_2 p_4} Z_4 \rightarrow Z_4$ be projections (see Definition 2.1). Note that f_1 and f_2 are Vietoris mappings. We recall that by Definition 2.1 we have

$$p = p_1 \circ f_1 \quad q = q_3 \circ f_3, \quad p' = p_2 \circ f_2, \quad q' = q_4 \circ f_4.$$

Let $(z_1, z_3) \in Z_1 \triangle_{q_1 p_3} Z_3$ and $(z_2, z_4) \in \varphi_1(z_1) \times \varphi_2(z_3)$. Thus

$$p'(z_2, z_4) = p_2(f_2(z_2, z_4)) = p_2(z_2) = p_1(z_1) = p_1(f_1(z_1, z_3)) = p(z_1, z_3)$$

and similarly

$$q'(z_2, z_4) = q_4(f_4(z_2, z_4)) = q_4(z_4) = q_3(z_3) = q_3(f_3(z_1, z_3)) = q(z_1, z_3)$$

and the proof is complete.

The set of the class of the abstraction of the relation $\approx_{ad}$ will be denoted by the symbol

$$M_{ad}(X, Y) = D(X, Y) / \approx_{ad}.$$

The elements of the set $M_{ad}(X, Y)$ will be called admissible morphisms and denoted by $\varphi_{ad}, \psi_{ad}, \dots$. The following denotation is assumed

$$\varphi_{ad} = [(p, q)]_{ad} \quad (\text{we write } (p, q) \in \varphi_{ad}),$$

where the diagram (p, q) is representative of the class of the abstraction $[(p, q)]_{ad}$ in the relation $\approx_{ad}$. We recall that a multivalued u.s.c. map $\varphi: X \multimap Y$ is *acyclic* if for each $x \in X$ the set $\varphi(x)$ is acyclic. The acyclic map φ is determined by an admissible morphism $\varphi_{ad} = [(p_\varphi, q_\varphi)]_{ad} \in M_{ad}(X, Y)$, where

$$X \xleftarrow{p_\varphi} \Gamma_\varphi \xrightarrow{q_\varphi} Y,$$

$\Gamma_\varphi = \{(x, y) \in X \times Y : y \in \varphi(x)\}$, $p_\varphi(x, y) = x$ (Vietoris map), $q_\varphi(x, y) = y$ for each $(x, y) \in \Gamma_\varphi$ such that for each $x \in X$,

$$q_\varphi(p_\varphi^{-1}(x)) = \varphi(x).$$

For singlevalued mappings, there is the following fact (see [11]).

PROPOSITION 4.4

Let $f: X \rightarrow Y$ be a continuous mapping and let $(p, q) \in D(X, Y)$, where

$$X \xleftarrow{p} Z \xrightarrow{q} Y.$$

Then the following conditions are equivalent

$$(4.4.1) \quad q = f \circ p,$$

$$(4.4.2) \quad (p, q) \approx_{ad} (Id, f),$$

$$(4.4.3) \quad q(p^{-1}(x)) = f(x) \text{ for each } x \in X.$$

Proof. (4.4.1) $\Rightarrow$ (4.4.2). There is the following commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{p} & Z & \xrightarrow{q} & Y \\ \downarrow Id_X & & \downarrow p & & \downarrow Id_Y \\ X & \xleftarrow{Id_X} & X & \xrightarrow{f} & Y \end{array}$$

Let $\varphi = p$. Then φ is a multi-invertible and admissible map and $\overleftarrow{\varphi}$ is admissible. Hence $(p, q) \approx_{ad} (Id, f)$.

(4.4.2) $\Rightarrow$ (4.4.3). This implication is the result of Proposition 4.3 (see 4.3.1).

(4.4.3) $\Rightarrow$ (4.4.1). Let $(p, q) \in D(X, Y)$ such that for each $x \in X$ $q(p^{-1}(x)) = f(x)$ and let $z \in Z$. Then there exists a point $x_1 \in X$ such that $z \in p^{-1}(x_1)$. Hence we get

$$q(z) = f(x_1) = f(p(z)),$$

and the proof is complete.

Let $(p_1, q_1), (p_2, q_2) \in D(X, Y)$ and let $(p_1, q_1) \approx_{K2} (p_2, q_2)$ (see (3)). If $\varphi = v_2 \circ \overleftarrow{v_1}$ then $\overleftarrow{\varphi} = v_1 \circ \overleftarrow{v_2}$ and

$$p_2 \circ \varphi = p_1 \quad \text{and} \quad q_2 \circ \varphi = q_1.$$

Hence $(p_1, q_1) \approx_{ad} (p_2, q_2)$.

EXAMPLE 4.5

Let $J = [-1, 1]$.

$$\begin{array}{ccccc} I & \xleftarrow{p_1} & I \times J & \xrightarrow{q_1} & I \\ \downarrow Id_I & & \downarrow \varphi & & \downarrow Id_I, \\ I & \xleftarrow{p_2} & I \times I & \xrightarrow{q_2} & I \end{array}$$

where $p_1(x, y) = x$, $q_1(x, y) = y^2$ for each $(x, y) \in I \times J$, $p_2(x, y) = x$, $q_2(x, y) = y$ for each $(x, y) \in I \times I$ and $\varphi(x, y) = (x, y^2)$ for each $(x, y) \in I \times J$. It is clear that

$$(p_1, q_1) \approx_{ad} (p_2, q_2).$$

Assume that $(p_1, q_1) \approx_{K2} (p_2, q_2)$. Then there exist Vietoris maps $v_1: Z \rightarrow I \times J$ and $v_2: Z \rightarrow I \times I$ such that $q_1 \circ v_1 = q_2 \circ v_2$. Let $y \in I$. We have

$$q_1^{-1}(y) \xleftarrow{v_1} v_1^{-1}(q_1^{-1}(y)) = v_2^{-1}(q_2^{-1}(y)) \xrightarrow{v_2} q_2^{-1}(y).$$

Hence, $H_*(q_1^{-1}(y)) \cong H_*(q_2^{-1}(y))$ for each $y \in I$, but this is not possible.

DEFINITION 4.6

For any $\varphi_{ad} \in M_{ad}(X, Y)$, the set $\varphi(x) = q(p^{-1}(x))$ where $\varphi_{ad} = [(p, q)]_{ad}$ is called an *image of point x in an admissible morphism φ_{ad}* .

We denote by

$$\varphi: X \rightarrow_{ad} Y$$

a multivalued map (see, Definition 4.6) determined by an admissible morphism $\varphi_{ad} = [(p, q)]_{ad} \in M_{ad}(X, Y)$.

Let **TOP** denote categories in which Hausdorff topological spaces are objects and continuous mappings are category mappings. Let **TOP_{ad}** denote categories in which Hausdorff topological spaces are objects and multivalued maps determined by admissible morphisms are category mappings. According to Proposition 4.3, (4.4.3) the category of **TOP_{ad}** is well defined and **TOP** $\subset$ **TOP_{ad}**. Let **VECT_G** denote categories in which linear graded vector spaces are objects and linear mappings of degree zero are category mappings.

THEOREM 4.7 (see [12])

The mapping $\widetilde{\mathbf{H}}_*: \mathbf{TOP}_{ad} \rightarrow \mathbf{VECT}_G$ given by the formula

$$\widetilde{\mathbf{H}}_*(\varphi) = q_* \circ p_*^{-1},$$

where φ is a multivalued map determined by $\varphi_{ad} = [(p, q)]_{ad}$ is a functor and the extension of the functor of the Čech homology $\mathbf{H}_*: \mathbf{TOP} \rightarrow \mathbf{VECT}_G$.

5. The strongly acyclic spaces

We will say that a space X is strongly acyclic if for each compact set $K \subset X$ there exists a compact and acyclic set $A \subset X$ such that $K \subset A$. We observe that from the construction of the Čech homology with compact carriers, we get:

PROPOSITION 5.1

If X is a strongly acyclic space then X is an acyclic space.

Proof. Let

$$C(X) = \{K \subset X : K \text{ is compact}\}$$

and let

$$CA(X) = \{A \subset X : A \text{ is compact and acyclic}\}.$$

From the assumption the set $CA(X)$ is cofinal in the set $C(X)$ and the proof is complete.

The following fact results from the Mazur's Lemma.

PROPOSITION 5.2

If E is a Banach space then it is a strongly acyclic space.

PROPOSITION 5.3

Let $\{X_t, \pi_s^t, \Sigma\}$ be an inverse system, where Σ is a directed set and for each $t \in \Sigma$ a space X_t is strongly acyclic. Assume that for each $t \in \Sigma$ and for any compact set $K_t \subset X_t$ there exists an acyclic set $A_t \subset X_t$ such that $K_t \subset A_t$ and $\{A_t, (\pi_A)_s^t, \Sigma\}$ is an inverse system, where for $s \leq t$ the map $(\pi_A)_s^t$ is a restriction of π_s^t . Then a space

$$Y = \lim_{\leftarrow} \{X_t, \pi_s^t, \Sigma\}$$

is strongly acyclic.

Proof. Let $K \subset Y$ be a compact set and let $t \in \Sigma$. We denote by $f_t: Y \rightarrow X_t$ a restriction of projection map. The set $K_t = f_t(K) \subset X_t$ is compact. From the mathematical literature (see [2]) we know that

$$K = \lim_{\leftarrow} \{K_t, (\pi_K)_s^t, \Sigma\},$$

where for $s \leq t$ the map $(\pi_K)_s^t$ is a restriction of π_s^t . In turn, from the assumption there exists an acyclic and compact set $A_t \subset X_t$ such that $K_t \subset A_t$ and $\{A_t, (\pi_A)_s^t, \Sigma\}$ is an inverse system, where for $s \leq t$ the map $(\pi_A)_s^t$ is a restriction of π_s^t . The set

$$A = \lim_{\leftarrow} \{A_t, (\pi_A)_s^t, \Sigma\} \subset Y$$

is compact and from the continuity of the Čech homology it results that A is acyclic. It is clear that $K \subset A$ and the proof is complete.

PROPOSITION 5.4

If $X_1, X_2, \dots, X_n$ are strongly acyclic spaces then $X_1 \times X_2 \times \dots \times X_n$ is a strongly acyclic space.

Proof. Let $X_1, X_2, \dots, X_n$ be strongly acyclic spaces and let $K \subset X_1 \times X_2 \times \dots \times X_n$ be a compact set. We denote by $K_i = \pi_i(K)$, where $\pi_i: X_1 \times X_2 \times \dots \times X_n \rightarrow X_i$ is a projection, $i = 1, 2, \dots, n$. The set K_i is compact, so there exists an acyclic and compact set $A_i \subset X_i$ such that $K_i \subset A_i$, $i = 1, 2, \dots, n$. Let $A = A_1 \times A_2 \times \dots \times A_n$. Then A is compact and acyclic (see [4]), $K \subset A$ and the proof is complete.

From Proposition 5.3 and Proposition 5.4 we get the following fact.

PROPOSITION 5.5

Let S be a non-empty set and let for each $s \in S$ a space X_s be strongly acyclic. Then the cartesian product

$$X = \prod_{s \in S} X_s$$

is a strongly acyclic space.

Proof. Let $\Sigma = \{\xi \subset S : \xi \text{ is a finite set}\}$. Then $(\Sigma, \leq)$ is a directed set, where $\leq$ is an inclusion. From the mathematical literature we know that

$$X = \lim_{\leftarrow} \{Y_\xi, \pi_\zeta^\xi, \Sigma\},$$

where $Y_\xi = X_{s_1} \times X_{s_2} \times \dots \times X_{s_n}$, $\xi = \{s_1, s_2, \dots, s_n\} \subset S$ and for each $\zeta \leq \xi$, $\pi_\zeta^\xi: Y_\xi \rightarrow Y_\zeta$ is a projection. From Proposition 5.4 the space Y_ξ for each $\xi \in \Sigma$ is strongly acyclic. We observe that the inverse system satisfies the assumption of Proposition 5.3 (see proof of Proposition 5.4) and the proof is complete.

PROPOSITION 5.6

Let Σ be a non-empty, directed set and let E_t be a Banach space for each $t \in \Sigma$. Let $\{E_t, \pi_s^t, \Sigma\}$ be an inverse system. Assume that

$$E = \lim_{\leftarrow} \{E_t, \pi_s^t, \Sigma\}$$

is a linear space. Then E is a strongly acyclic space.

Proof. Let $K \subset E$ be a compact set and let $t \in \Sigma$. We denote by $f_t: E \rightarrow E_t$ a restriction of a projection map. The set $K_t = f_t(K) \subset E_t$ is compact. We have (see proof of Proposition 5.3),

$$K = \lim_{\leftarrow} \{K_t, (\pi_K)_s^t, \Sigma\},$$

where for $s \leq t$ the map $(\pi_K)_s^t$ is a restriction of π_s^t . From the assumption E_t is a Banach space, so the set $\overline{\text{conv}}(K_t) \subset E_t$ is compact and convex. Let

$$A = \prod_{t \in \Sigma} \overline{\text{conv}}(K_t).$$

Then A is compact and convex. The space E is a closed subset in the cartesian product $\prod_{t \in \Sigma} E_t$, so the set $A \cap E \subset E$ is compact and, from the assumption, is convex. Hence the set $f_t(A \cap E) = A_t \subset E_t$ is compact and convex (in particular, acyclic), $K_t \subset A_t$ and $\{A_t, (\pi_A)_s^t, \Sigma\}$ is an inverse system, where for $s \leq t$ the map $(\pi_A)_s^t$ is a restriction of π_s^t . From Proposition 5.3 it results that E is strongly acyclic and the proof is complete.

The next fact is obvious.

PROPOSITION 5.7

Let X and Y be homeomorphic spaces. The space X is strongly acyclic if and only if the space Y is strongly acyclic.

We will give the following important example.

EXAMPLE 5.8

By $C^k([0, m], \mathbb{R}^n)$, where $m \in \mathbb{N}$, $k = 0, 1, \dots$, we denote the Banach space of all C^k -functions with the usual maximum norm

$$\|x\|_m = \sum_{i=0}^k \max\{\|x^{(i)}(t)\|, t \in [0, m]\}.$$

Here $x^{(k)}$ denotes the k -th derivative of x and we also put $x^{(1)} = x'$, $x^{(0)} = x$. Let $C^k([0, \infty], \mathbb{R}^n)$ be a Fréchet space of all C^k -functions with the metric

$$d(x, y) = \sum_{m=1}^{\infty} 2^{-m} \frac{\|x - y\|_m}{1 + \|x - y\|_m}.$$

Let $\{C_m^k, \pi_m^p, \mathbb{N}\}$ be an inverse system, where $\pi_m^p = x|_{[0, m]}$ for every $x \in C_p^k$. One can easily check that

$$E = \varprojlim \{C_m^k, \pi_m^p, \mathbb{N}\} \text{ is homeomorphic to } C^k([0, \infty], \mathbb{R}^n)$$

and E is a linear space. From Proposition 5.6 and Proposition 5.7 the space $C^k([0, \infty], \mathbb{R}^n)$ is strongly acyclic.

6. The points of coincidence

We observe that a map $\varphi: X \multimap Y$ is admissible if and only if there exists a map $\Delta_\varphi: X \rightarrow_{ad} Y$ such that $\Delta_\varphi(x) \subset \varphi(x)$ (we write $\Delta_\varphi \subset \varphi$) for each $x \in X$. We say that a map φ is compact if $\overline{\varphi(X)} \subset Y$ is a compact set. Let $\varphi: X \rightarrow_{ad} X$. By the symbol $\Lambda(\varphi)$ we will denote a generalized Lefschetz number of φ (see [4]), that is

$$\Lambda(\varphi) = \Lambda(\varphi_*) = \Lambda(q_* \circ p_*^{-1}) \quad (\text{provided that it is well defined}),$$

where $(p, q) \in \varphi_{ad}$ (see Proposition 4.3). Let $\varphi, \psi: X \multimap Y$ be multivalued maps. We recall that the maps φ and ψ have a coincidence point if there exists a point $x \in X$ such that

$$\varphi(x) \cap \psi(x) \neq \emptyset.$$

Let $\varphi: X \rightarrow_{ad} X$ be a multivalued map given by $\varphi_{ad} = [(p, q)]_{ad} \in M_{ad}(X, Y)$. It is easy to see that $p, q: Z \rightarrow X$ have a coincidence point if and only if there exists a fixed point of φ , that is, there exists $x_0 \in X$ such that $x_0 \in \varphi(x_0)$.

THEOREM 6.1

Let $\varphi: X \multimap Y$ be a multi-invertible and locally admissible map and let $\psi: X \multimap Y$ be a compact and locally admissible map. Let $X \in NA_C$ ($Y \in NA_C$). Then there exists an open set $U \subset X$ ($U \subset Y$) and $\Delta: U \rightarrow_{ad} U$, $\Delta \subset (\overleftarrow{\varphi} \circ \psi)_U$ ($\Delta \subset (\psi \circ \overleftarrow{\varphi})_U$) such that $\Lambda(\Delta)$ is well defined and if $\Lambda(\Delta) \neq 0$ then φ and ψ have a coincidence point.

Proof. Let $X \in NA_C$. From the assumption the map $\overleftarrow{\varphi}: Y \multimap X$ is locally admissible. Let $\psi: X \multimap Y$ be a compact and locally admissible map. The set $K_1 = \psi(\overline{X}) \subset Y$ is compact, so there exists an open neighborhood $V \subset Y$ of K_1 such that $\overleftarrow{\varphi}_V: V \multimap X$ is admissible. Hence, there exists a multivaueled map $\Phi_V: V \rightarrow_{ad} X$ such that $\Phi_V \subset \overleftarrow{\varphi}_V$. Let $K = \Phi_V(K_1) \subset X$. It is clear that K is compact. There exists an open neighborhood $U \subset X$ of K such that $\psi_U: U \multimap V \subset Y$ is admissible. Let $\Psi_U: U \rightarrow_{ad} V$ be a map such that $\Psi_U \subset \psi_U$. We have the following diagram

$$U \xrightarrow{\Psi_U} V \xrightarrow{\Phi_V} X.$$

Let $\Delta = \Phi_V \circ \Psi_U$. We observe that Δ is compact, $\overline{\Delta(U)} \subset U$ and $\Delta \subset (\overleftarrow{\varphi} \circ \psi)_U$. Hence and from Proposition 2.5 and Theorem 2.6 $\Lambda(\Delta)$ is well defined. Assume that $\Lambda(\Delta) \neq 0$ then there exists a point $x \in U$ such that

$$x \in \Delta(x) \subset \overleftarrow{\varphi}(\psi(x)).$$

There exists a point $y \in \psi(x)$ such that $x \in \overleftarrow{\varphi}(y)$. Hence, $y \in \varphi(x)$, so

$$\varphi(x) \cap \psi(x) \neq \emptyset.$$

Let $Y \in NA_C$ and let $\psi: X \multimap Y$ be a compact and locally admissible map. From Proposition 2.3 the map $\psi \circ \overleftarrow{\varphi}: Y \multimap Y$ is locally admissible. From the assumption the set $K = \psi(\overleftarrow{\varphi}(Y))$ is compact, so there exists an open neighborhood $U \subset Y$ of K such that $(\psi \circ \overleftarrow{\varphi})_U: U \multimap U \subset Y$ is admissible. Hence, there exists $\Delta: U \rightarrow_{ad} U$ such that $\Delta \subset (\psi \circ \overleftarrow{\varphi})_U$. It is obvious that Δ is compact and $\Lambda(\Delta)$ is well defined. Assume that $\Lambda(\Delta) \neq 0$ then there exists a point $y \in U$ such that

$$y \in \Delta(y) \subset \psi(\overleftarrow{\varphi}(y)).$$

There exists a point $x \in \overleftarrow{\varphi}(y)$ such that $y \in \psi(x)$. Hence, $y \in \varphi(x)$, so

$$\varphi(x) \cap \psi(x) \neq \emptyset$$

and the proof is complete.

The following fact results from Theorem 6.1.

PROPOSITION 6.2

Let $\varphi: X \multimap Y$ be a multi-invertible and locally admissible map and let $\psi: X \multimap Y$ be a compact and locally admissible map. Assume that $X \in NA_C$ is strongly acyclic or $Y \in NA_C$ is strongly acyclic, then φ and ψ have a coincidence point.

Proof. In the proof of Theorem 6.1 it is enough to replace the set K with the set of acyclic and compact A such that $K \subset A$. Then $\Lambda(\Delta) = 1$ and the proof is complete.

From the last fact and Proposition 5.2, we get

PROPOSITION 6.3

Let E be a Banach space. Let $\varphi: X \multimap E$ be a multi-invertible and locally admissible map and let $\psi: X \multimap E$ be a compact and locally admissible map. Then φ and ψ have a coincidence point.

The next fact is the simple conclusion of Proposition 6.2.

PROPOSITION 6.4

Let $\varphi: X \multimap Y$ be a multi-invertible and admissible map and let $\psi: X \multimap Y$ be a compact and admissible map. Furthermore, assume that an inverse map $\overleftarrow{\varphi}$ is admissible. If $X \in A_C$ or $Y \in A_C$ then the maps φ and ψ have a coincidence point.

7. Conclusion

In the third paragraph we have proposed the definition of a multivalued invertible mapping. In the context of such a definition if a mapping is multi-invertible then there exists exactly one multi-inverse mapping. Moreover, if a singlevalued mapping $f: X \rightarrow Y$ is invertible, then it is multi-invertible and $\overleftarrow{f} = f^{-1}$. Multi-invertible mappings constitute a wide class of mappings and have many interesting applications. In paragraph four we have applied multi-invertible mappings for the construction of morphisms. Then, in paragraph six, it was proven that multi-invertible mappings have coincidence properties.

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