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Stability of a generalization of the Fréchet functional equation

Abstract. We prove some stability and hyperstability results for a generalization of the well known Fréchet functional equation, stemming from one of the characterizations of the inner product spaces. As the main tool we use a fixed point theorem for some function spaces. We end the paper with some new inequalities characterizing the inner product spaces.

1. Introduction

The following theorem has been proved in [2] (N and Z stand, as usual, for the sets of all positive integers and integers, respectively; moreover, $Z_0 := Z \setminus \{0\}$).

Theorem 1
Let $(X, +)$ be a commutative group, $X_0 := X \setminus \{0\}$, $Y$ be a Banach space, and $f: X \to Y$, $c: Z_0 \to [0, \infty)$ and $L: X_0^3 \to [0, \infty)$ satisfy the following three conditions:

$$\mathcal{M} := \{m \in Z_0 : c(-2m) + 2c(m + 1) + 2c(-m) + c(2m + 1) < 1\} \neq \emptyset,$$

$$L(kx, ky, kz) \leq c(k)L(x, y, z), \quad x, y, z \in X_0, m \in \mathcal{M},$$

$$k \in \{-2m, m + 1, -m, 2m + 1\}, \quad (1)$$

$$\|f(x + y + z) + f(x) + f(y) + f(z) - f(x + y) - f(x + z) - f(y + z)\| \leq L(x, y, z),$$

for all $x, y, z \in X_0$. Then there is a unique function $F: X \to Y$ satisfying

$$F(x + y + z) + F(x) + F(y) + F(z) = F(x + y) + F(x + z) + F(y + z) \quad (2)$$

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for all $x, y, z \in X$ and such that

$$\|f(x) - F(x)\| \leq \rho_L(x), \quad x \in X_0,$$

where

$$\rho_L(x) := \inf_{m \in M} L((2m + 1)x, -mx, -mx) \frac{1}{1 - c(-2m) - 2c(m + 1) - 2c(-m) - c(2m + 1)}, \quad x \in X_0.$$  

Equation (2) is sometimes called the Fréchet functional equation. The reason for this is that M. Fréchet [12] used it to characterize the inner product spaces in a similar way as Jordan and von Neumann [15] did using the parallelogram law. Namely, he proved that a normed space $(X, \| \cdot \|)$ is an inner product space if and only if, for all $x, y, z \in X$,

$$\|x + y + z\|^2 + \|x\|^2 + \|y\|^2 + \|z\|^2 = \|x + y\|^2 + \|x + z\|^2 + \|y + z\|^2. \quad (3)$$

For more information we refer to [1, 10, 21, 22, 23].

Theorem 1 yields the subsequent characterization of inner product spaces (see [2, Corollary 5(i)]).

**Corollary 2**

Let $X$ be a normed space and $X_0 := X \setminus \{0\}$. Write

$$D(x, y, z) := \|x + y + z\|^2 + \|x\|^2 + \|y\|^2 + \|z\|^2 - \|x + y\|^2 - \|x + z\|^2 - \|y + z\|^2$$

for $x, y, z \in X$. Assume that there exist $w_0, \alpha_i, s_i \in \mathbb{R}$ such that $\alpha_i > 0$ and $w_0s_i < 0$ for $i = 1, 2, 3$ and

$$\sup_{x, y, z \in X_0} \frac{D(x, y, z)}{(\alpha_1\|x\|^{s_1} + \alpha_2\|y\|^{s_2} + \alpha_3\|z\|^{s_3})^{w_0}} < \infty.$$  

Then $X$ is an inner product space.

Actually it is assumed in [2, Corollary 5(i)] that $w_0s_i > 0$ for $i = 1, 2, 3$, but it is a mistake; the inequality should be as in Corollary [2].

Equation (2) can also be written in the form

$$\Delta_{x, y, z}f(0) = 0 \quad \text{and} \quad f(0) = 0, \quad (4)$$

where $\Delta$ denotes the Fréchet difference operator defined by

$$\Delta_y f(x) = \Delta^1_y f(x) := f(x + y) - f(x), \quad x, y \in S,$$

$$\Delta_{t, z} := \Delta_t \circ \Delta_z, \quad \Delta^2_t := \Delta_{t, t}, \quad t, z \in S,$$

$$\Delta_{t, u, z} := \Delta_t \circ \Delta_u \circ \Delta_z, \quad \Delta^3_t := \Delta_{t, t, t}, \quad t, u, z \in S$$

for functions mapping a commutative group $(S, +)$ into a group (see [2]). Moreover, [2] can be written as

$$C^2 f(x, y, z) = 0,$$
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where

\[ C^2 f(x, y, z) = C f(x, y + z) - C f(x, y) - C f(x, z) \]

and

\[ C f(x, y) = f(x + y) - f(x) - f(y), \]

i.e. \( C^2 f \) is the Cauchy difference of \( f \) of the second order.

It is known (see \[2, 18\]) that every solution \( f \) of (2), mapping a commutative group \((G, +)\) into a real linear space \( X \), has the form \( f = a + q \) with an additive function \( a: G \to X \) and a quadratic function \( q: G \to X \).

In this paper we show that results analogous to Theorem 1 can be proved for the following more general functional equation

\[ A_1 f(x + y + z) + A_2 f(x) + A_3 f(y) + A_4 f(z) = A_5 f(x + y) + A_6 f(x + z) + A_7 f(y + z), \]

(5)

in the class of functions mapping a commutative group \( X \) into a Banach space \( Y \) over a field \( K \in \{\mathbb{R}, \mathbb{C}\} \), where \( A_1, \ldots, A_7 \in K \) are fixed (\( \mathbb{R} \) and \( \mathbb{C} \) denote the sets of real and complex numbers, respectively). It is easily seen that (5) becomes (2) with \( A_1 = \ldots = A_7 = 1 \).

The results we prove correspond also to some outcomes in \[8, 11, 16, 19, 25\].

The results in \[2\] as well as our main theorem have been motivated by the notion of hyperstability of functional equations (see, e.g., \[3, 4, 5, 13, 20\]), introduced in connection with the issue of stability of functional equations (for more details see, e.g., \[14, 17\]).

2. Auxiliary fixed point result

We need the subsequent fixed point theorem proved for function spaces in \[6\]; it will be the main tool in the proof of our main theorem (\( \mathbb{R}_+ \) stands for the set of nonnegative reals and \( A^B \) denotes the family of all functions mapping a set \( B \neq \emptyset \) into a set \( A \neq \emptyset \)). For related outcomes we refer to \[7, 9\]; a similar approach to stability of functional equations has already been applied in \[3, 24\].

**Theorem 3**

Let the following three hypotheses be valid.

(H1) \( S \) is a nonempty set, \( E \) is a Banach space, and functions \( f_1, \ldots, f_k: S \to S \) and \( L_1, \ldots, L_k: S \to \mathbb{R}_+ \) are given.

(H2) \( T: E^S \to E^S \) is an operator satisfying the inequality

\[ \|T\xi(x) - T\mu(x)\| \leq \sum_{i=1}^{k} L_i(x)\|\xi(f_i(x)) - \mu(f_i(x))\|, \quad \xi, \mu \in E^S, \quad x \in S. \]

(H3) \( \Lambda: \mathbb{R}_+^S \to \mathbb{R}_+^S \) is defined by

\[ \Lambda \delta(x) := \sum_{i=1}^{k} L_i(x)\delta(f_i(x)), \quad \delta \in \mathbb{R}_+^S, \quad x \in S. \]
Assume that functions $\varepsilon: S \to \mathbb{R}_+$ and $\varphi: S \to E$ fulfill the following two conditions

$$\|T\varphi(x) - \varphi(x)\| \leq \varepsilon(x), \quad x \in S,$$

$$\varepsilon^*(x) := \sum_{n=0}^{\infty} \Lambda^n \varepsilon(x) < \infty, \quad x \in S.$$

Then there exists a unique fixed point $\psi$ of $T$ with

$$\|\varphi(x) - \psi(x)\| \leq \varepsilon^*(x), \quad x \in S.$$

Moreover,

$$\psi(x) := \lim_{n \to \infty} T^n \varphi(x), \quad x \in S.$$

### 3. The main result

The next theorem is the main result of the paper.

**Theorem 4**

Let $(X, +)$ be a commutative group, $\hat{X} := X^3 \setminus \{(0,0,0)\}$, $Y$ be a Banach space, and $A_1, \ldots, A_7 \in K \in \{\mathbb{R}, \mathbb{C}\}$ such that $A_1 \neq 0$ and

$$A_2 + A_3 + A_4 = A_5 + A_6 + A_7. \tag{6}$$

Assume that $f: X \to Y$, $c: \mathbb{Z}_0 \to [0, \infty)$ and $L: \hat{X} \to [0, \infty)$ satisfy \([1]\) and the following two conditions:

$$\mathcal{M} := \{ m \in \mathbb{Z}_0 : |A_7| c(-2m) + |A_5 + A_6| c(m + 1) + |A_3 + A_4| c(-m) + |A_2| c(2m + 1) < |A_1| \} \neq \emptyset, \tag{7}$$

$$\|A_1 f(x + y + z) + A_2 f(x) + A_3 f(y) + A_4 f(z) - A_5 f(x + y) - A_6 f(x + z) - A_7 f(y + z)\| \leq L(x, y, x), \quad (x, y, z) \in \hat{X}. \tag{8}$$

Then there exists a unique function $F: X \to Y$ satisfying \([5]\) for all $x, y, z \in X$ and such that

$$\|f(x) - F(x)\| \leq \rho_L(x), \quad x \in X_0 := X \setminus \{0\}, \tag{9}$$

where

$$\rho_L(x) := \inf_{m \in \mathcal{M}} \frac{L((2m + 1)x, -mx, -mx)}{|A_1| - \beta_m}, \tag{10}$$

$$\beta_m := |A_7| c(-2m) + |A_5 + A_6| c(m + 1) + |A_3 + A_4| c(-m) + |A_2| c(2m + 1).$$
Moreover, for every the form described in (H3) with Stability of a generalization of the Fréchet functional equation [73] \( \eta \) for \( y \in X_0 \), \( m \in \mathbb{Z}_0 \).

Next put

\[ T_m \xi(x) := A_7 \xi(-2mx) + (A_5 + A_6) \xi((m+1)x) \]

\[ - (A_3 + A_4) \xi(-mx) - A_2 \xi((2m+1)x), \quad \xi \in Y^X, \ x \in X, \ m \in \mathbb{Z}_0. \]

It is easy to notice that, by (6),

\[ T^n_m f(0) = 0, \quad n \in \mathbb{N}, \ m \in \mathbb{Z}_0, \]

and inequality (11) can be written as

\[ \|T_m f(x) - f(x)\| \leq \varepsilon_m(x), \quad x \in X_0, \ m \in \mathbb{Z}_0. \]

Define an operator \( \Lambda_m : \mathbb{R}^X_+ \to \mathbb{R}^X_+ \) for \( m \in \mathbb{Z}_0 \) by

\[ \Lambda_m \eta(x) := |A_7|\eta(-2mx) + |A_5 + A_6|\eta((m+1)x) \]

\[ + |A_3 + A_4|\eta(-mx) + |A_2|\eta((2m+1)x) \]

for \( \eta \in \mathbb{R}^X_+ \) and \( x \in X_0 \). Notice that, for each \( m \in \mathbb{Z}_0 \), the operator \( \Lambda := \Lambda_m \) has the form described in (H3) with \( k = 4, S = X_0, E = Y \) and

\[ f_1(x) = -2mx, \quad f_2(x) = (m+1)x, \quad f_3(x) = -mx, \quad f_4(x) = (2m+1)x, \]

\[ L_1(x) = |A_7|, \quad L_2(x) = |A_5 + A_6|, \quad L_3(x) = |A_3 + A_4|, \quad L_4(x) = |A_2|, \quad x \in X_0. \]

Moreover, for every \( \xi, \mu \in Y^X_0, x \in X_0, \ m \in \mathbb{Z}_0, \)

\[ \|T_m \xi(x) - T_m \mu(x)\| \]

\[ = \|A_7 \xi(-2mx) + (A_5 + A_6) \xi((m+1)x) - (A_3 + A_4) \xi(-mx) \]

\[ - A_2 \xi((2m+1)x) - A_7 \mu(-2mx) - (A_5 + A_6) \mu((m+1)x) \]

\[ + (A_3 + A_4) \mu(-mx) + A_2 \mu((2m+1)x)\| \]

\[ \leq |A_7|\|(\xi - \mu)(-2mx)\| + |A_5 + A_6|\|(\xi - \mu)((m+1)x)\| \]

\[ + |A_3 + A_4|\|(\xi - \mu)(-mx)\| + |A_2|\|(\xi - \mu)((2m+1)x)\| \]

\[ = \sum_{i=1}^4 L_i(x)\|(\xi - \mu)(f_i(x))\|, \]

where

\[ (\xi - \mu)(y) := \xi(y) - \mu(y), \quad y \in X_0. \]

Note that, in view of (11), we have

\[ \Lambda_m \varepsilon_k(x) \leq \beta_m \varepsilon_k(x), \quad k, m \in \mathbb{Z}_0, \ x \in X_0. \]
By induction it is easy to show that the linearity of \( \Lambda_m \) implies
\[
\Lambda_m^n \varepsilon_k(x) \leq (\beta_m)^n \varepsilon_k(x)
\]
for \( x \in X_0, \, k, n \in \mathbb{N} \). So, we receive the following estimation
\[
\varepsilon_m^*(x) := \sum_{n=0}^{\infty} \Lambda_m^n \varepsilon_m(x) \leq \sum_{n=0}^{\infty} (\beta_m)^n \varepsilon_m(x) = \frac{\varepsilon_m(x)}{1 - \beta_m}, \quad m \in \mathcal{M}, \, x \in X_0.
\]

By Theorem 3 (with \( S = X_0 \) and \( E = Y \), for each \( m \in \mathcal{M} \) there exists a function \( F'_m : X_0 \to Y \) such that
\[
F'_m(x) = A_7 F'_m(-2mx) + (A_5 + A_6) F'_m(m + 1)x
- (A_3 + A_4) F'_m(-mx) - A_2 F'_m((2m + 1)x), \quad x \in X_0
\]
and
\[
\|f(x) - F'_m(x)\| \leq \frac{\varepsilon_m(x)}{1 - \beta_m}, \quad x \in X_0.
\]
Moreover,
\[
F'_m(x) = \lim_{n \to \infty} T^n_m f(x), \quad x \in X_0, \, m \in \mathcal{M}.
\]

Now, define \( F_m : X \to Y \) by
\[
F_m(0) = 0, \quad F_m(x) := F'_m(x), \quad x \in X_0, \, m \in \mathcal{M}.
\]
Then it is easily seen that, by (12),
\[
F_m(x) = \lim_{n \to \infty} T^n_m f(x), \quad x \in X, \, m \in \mathcal{M}.
\]

Next, by induction we show that
\[
\left\| T^n_m f(x + y + z) + A_2 T^{n+1}_m f(x) + A_3 T^n_m f(y) + A_4 T^n_m f(z)
- A_5 T^{n+1}_m f(x + y) - A_6 T^n_m f(x + z) - A_7 T^n_m f(y + z) \right\| \
\leq (\beta_m)^n L(x, y, z)
\]
for every \( (x, y, z) \in \hat{X} \), \( n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \, m \in \mathcal{M} \).

Fix \( m \in \mathcal{M} \). For \( n = 0 \) condition (15) becomes (3). So, take \( l \in \mathbb{N}_0 \) and suppose that (15) holds for \( n = l \) and \( (x, y, z) \in \hat{X} \). Then we have
\[
\left\| T^{l+1}_m f(x + y + z) + A_2 T^{l+1}_m f(x) + A_3 T^{l+1}_m f(y) + A_4 T^{l+1}_m f(z)
- A_5 T^{l+1}_m f(x + y) - A_6 T^{l+1}_m f(x + z) - A_7 T^{l+1}_m f(y + z) \right\|
= \left\| A_7 T^l_m f(-2m(x + y + z)) + (A_5 + A_6) T^l_m f((m + 1)(x + y + z))
- (A_3 + A_4) T^l_m f(-m(x + y + z)) - A_2 T^l_m f((2m + 1)(x + y + z))
+ A_2 A_7 T^l_m f(-2mx) + (A_5 + A_6) A_2 T^l_m f((m + 1)x)
- (A_3 + A_4) A_2 T^l_m f(-mx) - A_2 A_7 T^l_m f((2m + 1)x)
\right\|
\leq (\beta_m)^n L(x, y, z)
\]
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\[ + A_7 A_5 T_{m} f(-2my) + (A_5 + A_6) A_3 T_{m} f((m + 1)y) \\
- (A_3 + A_4) A_3 T_{m} f(-my) - A_2 A_3 T_{m} f((2m + 1)y) \\
+ A_7 A_4 T_{m} f(-2mz) + (A_5 + A_6) A_4 T_{m} f((m + 1)z) \\
- (A_3 + A_4) A_4 T_{m} f(-mz) - A_2 A_4 T_{m} f((2m + 1)z) \\
- A_7 A_5 T_{m} f(-2(m + x + y)) - (A_5 + A_6) A_5 T_{m} f((m + 1)(x + y)) \\
+ (A_3 + A_4) A_5 T_{m} f(-m(x + y)) + A_2 A_5 T_{m} f((2m + 1)(x + y)) \\
- A_7 A_6 T_{m} f(-2m(x + z)) - (A_5 + A_6) A_6 T_{m} f((m + 1)z) \\
+ (A_3 + A_4) A_6 T_{m} f(-m(x + z)) + A_2 A_6 T_{m} f((2m + 1)(x + z)) \\
- A_7 A_7 T_{m} f(-2m(y + z)) - (A_5 + A_6) A_7 T_{m} f((m + 1)(y + z)) \\
+ (A_3 + 1) A_7 T_{m} f(-m(y + z)) + A_2 A_7 T_{m} f((2m + 1)(y + z)) \]

\[ \leq (\beta_m)^{l+1}\|L(x, y, z)\| \]

for every \((x, y, z) \in \hat{X}\), which ends the proof of (15).

Letting \(n \to \infty\) in (13), we obtain

\[ F_m(x + y + z) + A_2 F_m(x) + A_3 F_m(y) + A_4 F_m(z) \]

\[ = A_5 F_m(x + y) + A_6 F_m(x + z) + A_7 F_m(y + z), \quad (x, y, z) \in \hat{X}. \]  \(\text{(16)}\)

So, we have proved that, for each \(m \in M\) there exists a function \(F_m: X \to Y\) satisfying the equation (5) for \((x, y, z) \in \hat{X}\) and such that

\[ \|f(x) - F_m(x)\| \leq \frac{\varepsilon_m(x)}{1 - \beta_m}, \quad x \in X_0. \]  \(\text{(17)}\)

Next, we show that \(F_m = F_k\) for all \(m, k \in M\). So, fix \(m, k \in M\). Note that \(F_k\) satisfies (10) with \(m\) replaced by \(k\). Hence, replacing \(x\) by \((2m+1)x\) and taking \(y = z = -mx\) in (16), we obtain that \(T_m F_j = F_j\) for \(j = m, k\) and

\[ \|F_m(x) - F_k(x)\| \leq \frac{\varepsilon_m(x)}{1 - \beta_m} + \frac{\varepsilon_k(x)}{1 - \beta_k}, \quad x \in X_0, \]

whence, by the linearity of \(\Lambda\) and (14),

\[ \|F_m(x) - F_k(x)\| = \|T_m^n F_m(x) - T_m^n F_k(x)\| \leq \frac{\Lambda_m^n \varepsilon_m(x)}{1 - \beta_m} + \frac{\Lambda_k^n \varepsilon_k(x)}{1 - \beta_k} \]

\[ \leq (\beta_m)^n \varepsilon_m(x) \frac{1}{1 - \beta_m} + (\beta_k)^n \varepsilon_k(x) \frac{1}{1 - \beta_k} \]

for every \(x \in X_0\) and \(n \in \mathbb{N}\). Therefore, letting \(n \to \infty\) we get \(F_m = F_k \Rightarrow F\).

Thus, in view of (17), we have proved that

\[ \|f(x) - F(x)\| \leq \frac{\varepsilon_m(x)}{1 - \beta_m}, \quad x \in X, \ x \neq 0, \ m \in M, \]

whence we derive (9).
Since (in view of (16)) it is easy to notice that $F$ is a solution to (5) (i.e. (5) holds for all $x,y,z \in X$), it remains to prove the statement concerning the uniqueness of $F$. So, let $G: X \to Y$ be also a solution of (5) and $\|f(x) - G(x)\| \leq \rho_L(x)$ for $x \in X$, $x \neq 0$. Then

$$\|G(x) - F(x)\| \leq 2\rho_L(x), \quad x \in X, x \neq 0.$$  

(18)

Further, $T_m G = G$ for each $m \in \mathbb{Z}_0$. Hence, with a fixed $m \in \mathcal{M}$, by (14) we get

$$\|G(x) - F(x)\| = \|T_m^n G(x) - T_m^n F(x)\| \leq 2\Lambda_m^n \rho_L(x) \leq \frac{2\Lambda_m^n \varepsilon_m(x)}{1 - \beta_m} \leq \frac{2(\beta_m)^n \varepsilon_m(x)}{1 - \beta_m}$$

for $x \in X_0$ and $n \in \mathbb{N}$. Consequently, letting $n \to \infty$ we obtain that $G = F$, and that ends the proof in the case $A_1 = 1$.

If $A_1 \neq 1$, then (8) can be rewritten in the form

$$\|f(x + y + z) + A^*_2 f(x) + A'_3 f(y) + A'_4 f(z) - A'_5 f(x + y) - A'_6 f(x + z) - A'_7 f(y + z)\| \leq L'(x,y,z), \quad (x,y,z) \in \tilde{X},$$

(19)

where

$$A'_i := \frac{A_i}{A_1}, \quad i = 2, \ldots, 7, \quad L'(x,y,z) := \frac{L(x,y,z)}{|A_1|}, \quad (x,y,z) \in \tilde{X},$$

and it is easily seen that the statement can be easily deduced from the case $A_1 = 1$.

The following hyperstability result can be deduced from Theorem 4. It corresponds to the recent hyperstability outcomes in [5, 24] and some classical stability results concerning the Cauchy equation (see, e.g., [3, p.3], [14, p.15,16] and [17, p.2]).

**Corollary 5**

Let $(X, +)$ be a commutative group, $\tilde{X} := X^3 \setminus \{(0,0,0)\}$, $Y$ be a Banach space over $K \in \{\mathbb{R}, \mathbb{C}\}$, $A_1, \ldots, A_7 \in K$, $A_1 \neq 0$ and (6) be valid. Assume that $f: X \to Y$, $c: \mathbb{Z}_0 \to [0, \infty)$ and $L: \tilde{X} \to [0, \infty)$ satisfy conditions (1), (7), (8) and

$$\sup_{m \in \mathcal{M}} \beta_m < |A_1|, \quad \inf_{m \in \mathcal{M}} L((2m + 1)x, -mx, -mx) = 0 \quad x \in X, x \neq 0.$$

Then there exists a function $F: X \to Y$ satisfying (5) for all $x,y,z \in X_0$ such that $f(x) = F(x)$ for $x \in X_0$.

**Proof.** It is easily seen that $\rho_L(x) = 0$ for each $x \in X_0$, where $\rho_L$ is defined by (10). Hence Theorem 4 implies the statement.
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Remark 6
If, in Theorem 4,

\[ L(x, y, z) = (\alpha_1||x||^p + \alpha_2||y||^p + \alpha_3||z||^p)^w, \quad (x, y, z) \in \hat{X}, \]  

(20)

with some \( \alpha_i, p, w \in \mathbb{R} \) such that \( \alpha_i > 0 \) for \( i = 1, 2, 3 \), \( p > 0 \) and \( w < 0 \), then it is easily seen that the function \( c \) can, for instance, have the form \( c(m) = |m|^{pw} \).

The next corollary generalizes Corollary 2 to some extent; it shows possible application of the main result of this paper.

Corollary 7
Let \( X \) be a normed space over \( K \in \{\mathbb{R, C}\} \), \( X_0 := X \setminus \{0\} \), \( A_1, \ldots, A_7 \in K \), \( A_1 \neq 0 \), and \( L \) be valid. Write

\[ \hat{D}(x, y, z) := |A_1||x + y + z||^2 + A_2||x||^2 + A_3||y||^2 + A_4||z||^2 - A_5||x + y||^2 - A_6||x + z||^2 - A_7||y + z||^2| \]

for \( x, y, z \in X \). Assume that there exist \( \alpha_i, w, p \in \mathbb{R} \) such that \( p > 0 \), \( w < 0 \), \( \alpha_i > 0 \) for \( i = 1, 2, 3 \) and

\[ \sup_{(x, y, z) \in \hat{X}} \left( \frac{\hat{D}(x, y, z)}{(\alpha_1||x||^p + \alpha_2||y||^p + \alpha_3||z||^p)^w} \right) < \infty. \]

Then \( X \) is an inner product space.

Proof. Write \( f(x) = ||x||^2 \) for \( x \in X \). Then, with \( L \) and \( c \) of the forms described in Remark 6 from Corollary 3 and \( (2) \), we easily derive that \( f \) is a solution to equation (3).

We show that \( A_1 = \ldots = A_7 \). Replacing \( x \) by \( \alpha x \), \( y \) by \( \beta x \) and \( z \) by \( \gamma x \) in (3), where \( \alpha, \beta, \gamma \in K \), we obtain

\[ (A_1(\alpha + \beta + \gamma)^2 + A_2\alpha^2 + A_3\beta^2 + A_4\gamma^2)||x||^2 \]
\[ = (A_5(\alpha + \beta)^2 + A_6(\alpha + \gamma)^2 + A_7(\beta + \gamma)^2)||x||^2, \quad x \in X, \]

whence

\[ A_1(\alpha + \beta + \gamma)^2 + A_2\alpha^2 + A_3\beta^2 + A_4\gamma^2 \]
\[ = A_5(\alpha + \beta)^2 + A_6(\alpha + \gamma)^2 + A_7(\beta + \gamma)^2, \quad \alpha, \beta, \gamma \in K. \]  

(21)

Taking \( \alpha = 1, \beta = \gamma = 0 \) in (21) we have \( A_1 + A_2 = A_5 + A_6 \), and next, with \( \beta = -\alpha = 1 \) and \( \gamma = 0 \) in (21) we obtain the equality \( A_2 + A_3 = A_6 + A_7 \) and consequently

\[ A_1 - A_3 = A_5 - A_7. \]  

(22)

Analogously, with \( \beta = 1, \alpha = \gamma = 0 \) and \( \beta = -\gamma = 1, \alpha = 0 \) we obtain

\[ A_1 - A_4 = A_7 - A_6, \]

(23)
and $\gamma = 1$, $\alpha = \beta = 0$ and $\alpha = -\gamma = 1$, $\beta = 0$ gives

$$A_1 - A_2 = A_6 - A_5. \quad (24)$$

Further, inserting, $1 = \alpha = -\beta = -\gamma$, $1 = -\alpha = \beta = -\gamma$ and $1 = -\alpha = -\beta = \gamma$ into (21), we respectively get

$$A_1 + A_2 + A_3 + A_4 = 4A_7,$$
$$A_1 + A_2 + A_3 + A_4 = 4A_6,$$
$$A_1 + A_2 + A_3 + A_4 = 4A_5,$$

whence $A_5 = A_6 = A_7$ and consequently, by (22)–(24), $A_1 = A_2 = A_3 = A_4$. This and (6) finally yield $A_1 = \ldots = A_7$. Thus we have proved that (3) holds, which implies the statement.

References


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