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Adaptive algorithm for solving the SCFPP of demicontractive operators without a priori knowledge of operator norms

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Abstract

In this paper, we study the split common fixed point problem in Hilbert spaces. We find a common solution of the split common fixed point problem for two demicontractive operators without *a priori* knowledge of operator norms. A strong convergence theorem is obtained under some additional conditions and numerical examples are included to illustrate the applications in signal compressed sensing and image restoration.

1 Introduction

Let H be a real Hilbert space. The convex feasibility problem (shortly, **(CFP)**) is defined as follows:

$$\text{find } \ x^* \in H \ \text{ such that } \ x^* \in \bigcap_{i=1}^{m} C_i,$$

where $m \geq 1$ is an integer and each C_i is a nonempty closed convex subset of H.

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A special case of the problem **(CFP)** is the following *split feasibility problem* (shortly, **(SFP)**):

$$\text{find } x^* \in C \text{ such that } Ax^* \in Q,$$

where C and Q are two closed convex subsets of two Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \to H_2$ is a bounded linear operator.

Byrne [2] introduced a very popular algorithm $\{x_n\}$ that solves the problem **(SFP)**:

$$x_{n+1} = P_C(x_n - \gamma A^*(I - P_Q)Ax_n)$$

for each $n \geq 0$, where P_C and P_Q are metric projections onto C and Q, respectively, A^* denotes the adjoint of the mapping $A : H_1 \to H_2$ and $\gamma \in (0, \frac{2}{\lambda})$ with λ being the spectral radius of the mapping A^*A.

If $C = F(T)$ and $Q = F(S)$, then, from the problem **(SFP)**, we have the split common fixed point problem (shortly, **(SCFPP)**) which is defined as follows:

$$\text{find a point } x^* \in F(T) \text{ such that } Ax^* \in F(S),$$

where $F(T)$, $F(S)$ stand for the fixed point sets of the mappings $T : H_1 \to H_1$, $S : H_2 \to H_2$, respectively, and $A : H_1 \to H_2$ is a bounded linear operator. We denote the set of solutions of the problem **(SCFPP)** by

$$\Gamma := \{y^* \in C : Ay^* \in Q\} = C \cap A^{-1}(Q).$$

Let $T : H_1 \to H_1$ and $S : H_2 \to H_2$ be two mappings such that

$$C := F(T) = \{x^* \in H_1 : Tx^* = x^*\} \neq \emptyset$$

and

$$Q := F(S) = \{x^* \in H_2 : Sx^* = x^*\} \neq \emptyset.$$

In this paper, we prove a result on the existence of solutions of the split common fixed point problem **(SCFPP)** for two demicontractive mappings $T : H_1 \to H_1$ and $S : H_2 \to H_2$ with $C := F(T) \neq \emptyset$ and $Q := F(S) \neq \emptyset$ and obtain the solution by a new algorithm $\{x_n\}$.

Censor and Segal [3] introduced, in finite-dimensional spaces, the following algorithm $\{x_n\}$ for solving the problem **(SCFPP)**:

$$x_{n+1} = T(x_n + \tau A^t(S - I)Ax_n) \tag{1.1}$$

for each $n \geq 1$, where $\tau \in (0, \frac{2}{\gamma})$ with γ being the largest eigenvalue of the matrix A^tA (A^t is matrix transposition).

Moudafi [18] proved some weak convergence theorems in Hilbert spaces when two mappings T and S are quasi-nonexpansive mappings by using the following relaxed algorithm $\{x_n\}$:

$$\begin{cases} y_n = x_n + \tau A^*(S - I)Ax_n, \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n T y_n \end{cases}$$

for each $n \geq 1$, where $\alpha_n \in (0,1)$ and $\tau \in (0, \frac{1}{\beta\gamma})$ with γ being the spectral radius of the operator A^*A and $\beta \in (0,1)$.

Moudafi [17] also proposed an iterative algorithm $\{x_n\}$ to solve the problem **(SCFPP)**, where S and T are demicontractive mappings as follows:

$$\begin{cases} u_n = x_n + \tau A^*(S - I)Ax_n, \\ x_{n+1} = (1-\alpha_n)u_n + \alpha_n T u_n \end{cases}$$

for each $n \geq 1$, where $\alpha_n \in (0,1)$ and $\tau \in \left(0, \frac{1-\mu}{\gamma}\right)$ with γ being the spectral radius of the operator A^*A and $\beta \in (0,1)$.

In particular, it was noted that the problem **(SCFPP)** is equivalent to solving the following fixed point problem

$$x = x - \tau((x - Tx) + A^*(I - S)A)x, \tag{1.2}$$

where $\tau > 0$ is a constant and T and S are directed operators.

Based on the fixed point equation approach, Wang [24] suggested the following algorithm $\{x_n\}$:

$$x_{n+1} = x_n - \tau((x_n - Tx_n) + A^*(I - S)Ax_n),$$

where $T : \mathbb{R}^n \to \mathbb{R}^n$ and $S : \mathbb{R}^n \to \mathbb{R}^n$ are two directed operators and the step size τ is in the interval $\left(0, \dfrac{1}{\max\{0, \|A\|^2\}}\right)$ and proved some weak convergence theorems of the sequence $\{x_n\}$ to a solution of the problem **(SCFPP)**.

In 2006, Marino and Xu [15] introduced a new iterative which combines the viscosity approximation method and is defined as follows:

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n)$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying suitable conditions. They proved that $\{x_n\}$ converges strongly to a fixed point x of T which solves the variational inequality

$$\langle (A - \gamma f)x, x - z\rangle \leq 0, \quad z \in F(T).$$

Equivalently, $P_{F(T)}(I - A + \gamma f)x = x$.
Inspired by the work mentioned above, we propose a new self-adaptive algorithm for solving the **(SCFPP)** with two demicontractive mappings in Hilbert spaces. We prove a strong convergence theorem for our proposed algorithm and present some numerical examples to illustrate our main results and their applications.

2 Preliminaries

Let $T : H \to H$ be a mapping. A point $x \in H$ is said to be a fixed point of T provided that $Tx = x$. In this paper, we denote by $F(T)$ the fixed point set of T. The symbols $\to$ and $\rightharpoonup$ denote the strong convergence and the weak convergence, respectively. The mapping $T : H \to H$ is said to be:
a) *quasi-nonexpansive* if

$$\|Tx - Tp\| \le \|x - p\|, \text{ for all } x \in H \text{ and } p \in F(T).$$

b) *strictly pseudocontractive* if there exists $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|(x - y) - (Tx - Ty)\|^2, \text{ for all } x \in H.$$

c) *pseudocontractive* if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \|(x - y) - (Tx - Ty)\|^2, \text{ for all } x \in H.$$

d) *demicontractive* (or *k-demicontractive*) if there exists $k < 1$ such that

$$\|Tx - Tp\|^2 \le \|x - p\|^2 + k\|x - Tx\|^2, \text{ for all } x \in H \text{ and } p \in F(T). \quad (2.1)$$

Remark 2.1. It is clear that, in a real Hilbert space H, (2.1) is equivalent to

$$\langle x - p, x - Tx\rangle \ge \frac{1-k}{2}\|x - Tx\|^2, \text{ for all } x \in H \text{ and } p \in F(T). \quad (2.2)$$

Now, we give some definitions and lemmas needed to prove our main results.

Definition 2.2. A mapping $T : H \to H$ is said to be *demiclosed* at 0 if, for each sequence $\{x_n\}$ in H, the conditions that the sequence $\{x_n\}$ converges weakly to y and the sequence $\{Tx_n\}$ converges strongly to 0 imply $Ty = 0$.

Lemma 2.3. Let H be a real Hilbert space. Then the following results hold:

(1) $\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$, for all $x, y \in H$.

(2) $\|x-y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2$, for all $x, y \in H$.

(3) $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$, for all $x, y \in H$ and $\alpha \in \mathbb{R}$.

Lemma 2.4. [25] Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n$$

for each $n \geq 0$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that

(a) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(b) $\lim_{n\to\infty} \sigma_n \leq 0$ or $\sum_{n=0}^{\infty} |\sigma_n\alpha_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5. [16] Let $q > 1$. Then the following inequality holds:

$$ab \leq \frac{1}{q}a^q + \frac{q-1}{q}b^{\frac{q}{q-1}}$$

for arbitrary positive real number a and b.

3 The Main Results

In this section, we first construct an iterative algorithm for solving the **SCFPP** under the following hypotheses.

$(A1)$ H_1 and H_2 are two real Hilbert spaces;

$(A2)$ $A : H_1 \to H_2$ is a bounded linear operator with its adjoint operator A^*.

$(A3)$ $D : H_1 \to H_1$ is a strongly positive bounded linear operator with coefficient $r > 0$.

$(A4)$ $f : H_1 \to H_1$ is a k-contraction;

$(A5)$ $S : H_1 \to H_1$ and $T : H_2 \to H_2$ are two demicontractive operators with coefficients $\beta \in [0, 1)$ and $\mu \in [0, 1)$, respectively;

$(A6)$ $S : H_1 \to H_1$ and $T : H_2 \to H_2$ are Lipschitz continuous with Lipschitz constant $L > 1$.

We use Ω to denote the solution set of problem **SCFPP**, that is,

$$\Omega := \{u^* : u^* \in F(S) \quad \text{and} \quad Au^* \in F(T)\}.$$

Algorithm 3.1. Choose an arbitrary initial guess x_0. Assume x_n has been constructed. If

$$\|x_n - Sx_n + A^*(I-T)Ax_n\| = 0,$$

then stop; otherwise, continue and construct x_{n+1} via the formula:

$$\begin{cases} y_n = x_n - \rho_n[x_n - Sx_n + A^*(I-T)Ax_n], \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n D)y_n, \end{cases}$$

where $\gamma \in (0, \min\{1-\beta, 1-\mu\})$ is a positive constant and $\rho_n \subset (0,\infty)$ is chosen self-adaptively as

$$\rho_n = \sigma_n \frac{\|x_n - Sx_n\|^2 + \|Ax_n - TAx_n\|^2}{\|x_n - Sx_n + A^*(I-T)Ax_n\|^2}.$$

We need two lemmas to complete the convergence analysis of our proposed algorithm. The first lemma shows that the proposed algorithm is well defined.

Lemma 3.2. If the equality

$$\|x_n - Sx_n + A^*(I-T)Ax_n\| = 0,$$

holds for some $n \geq 0$, then x_n is a solution of problem **(SCFPP)**.

Proof. For any $z \in S$, we have

$$\begin{aligned} &\|x_n - Sx_n + A^*(I-T)Ax_n\|\|x_n - z\| \\ &\geq \langle x_n - Sx_n + A^*(I-T)Ax_n\rangle\langle x_n - z\rangle \\ &= \langle x_n - Sx_n, x_n - z\rangle + \langle A^*(I-T)Ax_n, x_n - z\rangle \\ &= \langle x_n - Sx_n, x_n - z\rangle + \langle (I-T)Ax_n, Ax_n - Az\rangle \\ &\geq \frac{1-\beta}{2}\|x_n - Sx_n\|^2 + \frac{1-\mu}{2}\|(I-T)Ax_n\|^2. \end{aligned}$$

Since $\beta, \mu \in [0,1)$, we deduce $x_n \in F(S)$ and $Ax_n \in F(T)$. □

Lemma 3.3. If the sequence $\{x_n\}$ satisfies

$$\lim_{n\to\infty} \frac{(\|x_n - Sx_n\|^2 + \|Ax_n - TAx_n\|^2)^2}{\|x_n - Sx_n + A^*(I-T)Ax_n\|^2} = 0$$

then

$$\lim_{n\to\infty} \|x_n - Sx_n\| = \lim_{n\to\infty} (I-T)Ax_n = 0.$$

Proof. By our hypotheses, we have

$$\begin{aligned}
&\frac{(\|x_n - Sx_n\|^2 + \|Ax_n - TAx_n\|^2)^2}{\|x_n - Sx_n + A^*(I-T)Ax_n\|^2} \\
&\geq \frac{(\|x_n - Sx_n\|^2 + \|Ax_n - TAx_n\|^2)^2}{2(\|x_n - Sx_n\|^2 + \|A\|^2\|(I-T)Ax_n\|^2)} \\
&\geq \frac{(\|x_n - Sx_n\|^2 + \|Ax_n - TAx_n\|^2)^2}{2\max(1, \|A\|^2)(\|x_n - Sx_n\|^2 + \|Ax_n - TAx_n\|^2)} \\
&= \frac{\|x_n - Sx_n\|^2 + \|Ax_n - TAx_n\|^2}{2\max(1, \|A\|^2)}.
\end{aligned}$$

Taking $n \to \infty$, we have

$$\lim_{n\to\infty} \|x_n - Sx_n\| = \lim_{n\to\infty} (I-T)Ax_n = 0.$$

□

Theorem 3.4. Assume the following conditions are satisfied

(*i*) $\sum_{n=1}^{\infty} \rho_n = \infty$ and $\sum_{n=1}^{\infty} \rho_n^2 < \infty$;

(*ii*) $0 < r < \dfrac{1}{\alpha_n}, 0 < \gamma < \dfrac{r}{k}$.

Then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to a solution u^* of problem **SCFPP**, where $u^* = P_\Omega(I - D + \gamma f)u^*$.

Proof. Setting $w_n = x_n - Sx_n + A^*(I-T)Ax_n$. Analogously, we have

$$\begin{aligned}
\langle w_n, x_n - u^*\rangle &= \langle x_n - Sx_n + A^*(I-T)Ax_n, x_n - u^*\rangle \\
&= \langle x_n - Sx_n, x_n - u^*\rangle + \langle A^*(I-T)Ax_n, x_n - u^*\rangle \\
&= \langle x_n - Sx_n, x_n - u^*\rangle + \langle (I-T)Ax_n, Ax_n - Au^*\rangle \\
&\geq \frac{1-\beta}{2}\|x_n - Sx_n\|^2 + \frac{1-\mu}{2}\|(I-T)Ax_n\|^2 \\
&\geq \frac{1}{2}\min\{1-\beta, 1-\mu\}(\|x_n - Sx_n\|^2 + \|(I-T)Ax_n\|^2).
\end{aligned} \tag{3.1}$$

From $y_n = x_n - \rho_n[x_n - Sx_n + A^*(I-T)Ax_n]$ and (3.1), we have

$$\begin{aligned}
\|y_n - u^*\|^2 &= \|x_n - \rho_n w_n - u^*\|^2 \\
&\le \|x_n - u^*\|^2 - 2\rho_n \langle w_n, x_n - u^* \rangle + \rho_n^2 \|w_n\|^2 \\
&\le \|x_n - u^*\|^2 + \rho_n^2 \|x_n - Sx_n + A^*(I-T)Ax_n\|^2 \\
&\quad - 2\rho_n \frac{1}{2} \min\{1-\beta, 1-\mu\}(\|x_n - Sx_n\|^2 + \|(I-T)Ax_n\|^2) \\
&\le \|x_n - u^*\|^2 - \frac{(\|x_n - Sx_n\|^2 + \|(I-T)Ax_n\|^2)^2}{\|x_n - Sx_n + A^*(I-T)Ax_n\|^2}.
\end{aligned} \tag{3.2}$$

In particular, we have $\|y_n - u^*\| \le \|x_n - u^*\|$. In what follows, we divide the proof into four steps.

Step 1. Show that $\{x_n\}$ is bounded. To see this, we observe

$$\begin{aligned}
\|x_{n+1} - u^*\| &= \|\alpha_n \gamma f(x_n) + (I - \alpha_n D)y_n - u^*\| \\
&= \|\alpha_n \gamma (f(x_n) - Du^*) + (I - \alpha_n D)(y_n - u^*)\| \\
&\le \alpha_n(\gamma \|f(x_n) - f(u^*)\| + \|\gamma f(u^*) - Du^*\|) + (1 - \alpha_n r)\|y_n - u^*\| \\
&\le \alpha_n \gamma k \|x_n - u^*\| + \alpha_n \|\gamma f(u^*) - Du^*\| + (1 - \alpha_n r)\|y_n - u^*\| \\
&\le (1 - \alpha_n(r - k\gamma))\|x_n - u^*\| + \alpha_n \|\gamma f(u^*) - Du^*\| \\
&\le (1 - \alpha_n(r - k\gamma))\|x_n - u^*\| + \alpha_n (r - k\gamma) \frac{\|\gamma f(u^*) - Du^*\|}{r - k\gamma} \\
&\le \max\left\{ \|x_n - u^*\|, \frac{\|\gamma f(u^*) - Du^*\|}{r - k\gamma} \right\} \\
&\ \ \vdots \\
&\le \max\left\{ \|x_0 - u^*\|, \frac{\|\gamma f(u^*) - Du^*\|}{r - k\gamma} \right\}.
\end{aligned} \tag{3.3}$$

Therefore $\{x_n\}$ is a bounded sequence. Furthermore, $\{y_n\}$ and $\{f(x_n)\}$ are also bounded sequences.

Step 2. Show that the following inequality holds:

$$a_{n+1} \le (1 - \alpha_n)a_n + \alpha_n b_n \tag{3.4}$$

where we define $a_n := \|x_n - u^*\|^2$ and

$$\begin{aligned} b_n :=& \frac{2(1-\alpha_n\gamma k)}{r-\gamma k}\langle \gamma f(u^*) - Du^*, x_{n+1} - u^*\rangle \\ &- \frac{1-\alpha_n r}{\alpha_n(r-\gamma k)} \frac{(\|x_n - Sx_n\|^2 + \|(I-T)Ax_n\|^2)^2}{\|x_n - Sx_n + A^*(I-T)Ax_n\|^2}. \end{aligned} \tag{3.5}$$

Indeed, it follows that

$$\begin{aligned} \|x_{n+1} - u^*\|^2 &= \|\alpha_n \gamma f(y_n) + (I - \alpha_n D)y_n - u^*\|^2 \\ &= \|\alpha_n\gamma(f(y_n) - Du^*) + (I-\alpha_n D)(y_n - u^*)\|^2 \\ &\leq \|(I-\alpha_n D)(y_n - u^*)\|^2 + 2\alpha_n\langle \gamma f(y_n) - Du^*, x_{n+1} - u^*\rangle \\ &\leq \|I - \alpha_n D\|^2\|y_n - u^*\|^2 + 2\alpha_n\langle \gamma f(y_n) - Du^*, x_{n+1} - u^*\rangle \\ &\leq (1-\alpha_n r)^2\|y_n - u^*\|^2 + 2\alpha_n\langle \gamma f(y_n) - Du^*, x_{n+1} - u^*\rangle \\ &\leq (1-\alpha_n r)\|y_n - u^*\|^2 + 2\alpha_n\langle \gamma f(y_n) - Du^*, x_{n+1} - u^*\rangle. \end{aligned} \tag{3.6}$$

From Lemma 2.5, we have

$$\begin{aligned} \langle \gamma f(y_n) - Du^*, x_{n+1} - u^*\rangle &= \langle \gamma f(y_n) - \gamma f(u^*), x_{n+1} - u^*\rangle \\ &\quad + \langle \gamma f(u^*) - Du^*, x_{n+1} - u^*\rangle \\ &\leq \gamma k\|y_n - u^*\|\|x_{n+1} - u^*\| \\ &\quad + \langle \gamma f(u^*) - Du^*, x_{n+1} - u^*\rangle \\ &\leq \gamma k\Big(\frac{1}{2}\|y_n - u^*\|^2 + \frac{1}{2}\|x_{n+1} - u^*\|^2\Big) \\ &\quad + \langle \gamma f(u^*) - Du^*, x_{n+1} - u^*\rangle. \end{aligned} \tag{3.7}$$

Substitute (3.7) into (3.6), we have

$$\begin{aligned} \|x_{n+1} - u^*\|^2 &\leq (1-\alpha_n r)\|y_n - u^*\|^2 + 2\alpha_n\Big(\gamma k\Big(\frac{1}{2}\|y_n - u^*\|^2 + \frac{1}{2}\|x_{n+1} - u^*\|^2\Big) \\ &\quad + \langle \gamma f(u^*) - Du^*, x_{n+1} - u^*\rangle\Big) \\ &\leq (1-\alpha_n r)\|y_n - u^*\|^2 + \alpha_n\gamma k\|y_n - u^*\|^2 + \alpha_n\gamma k\|x_{n+1} - u^*\|^2 \\ &\quad + 2\alpha_n\langle \gamma f(u^*) - Du^*, x_{n+1} - u^*\rangle \\ &\leq \Big(1 - \frac{\alpha_n(r-\gamma k)}{1-\alpha_n\gamma k}\Big)\|y_n - u^*\|^2 \\ &\quad + \frac{\alpha_n(r-\gamma k)}{1-\alpha_n\gamma k}\frac{2(1-\alpha_n\gamma k)}{r-\gamma k}\langle \gamma f(u^*) - Du^*, x_{n+1} - u^*\rangle. \end{aligned} \tag{3.8}$$

By inequality (3.2), this yields

$$
\begin{aligned}
\|x_{n+1}-u^*\|^2 &\le \Big(1-\frac{\alpha_n(r-\gamma k)}{1-\alpha_n\gamma k}\Big)\|x_n-u^*\|^2 \\
&\quad + \frac{\alpha_n(r-\gamma k)}{1-\alpha_n\gamma k}\frac{2(1-\alpha_n\gamma k)}{r-\gamma k}\langle \gamma f(u^*)-Du^*, x_{n+1}-u^*\rangle \\
&\le \Big(1-\frac{\alpha_n(r-\gamma k)}{1-\alpha_n\gamma k}\Big)\|x_n-u^*\|^2 \\
&\quad + \frac{\alpha_n(r-\gamma k)}{1-\alpha_n\gamma k}\Big[\frac{2(1-\alpha_n\gamma k)}{r-\gamma k}\langle \gamma f(u^*)-Du^*, x_{n+1}-u^*\rangle \\
&\quad - \frac{1-\alpha_n r}{\alpha_n(r-\gamma k)}\frac{(\|x_n-Sx_n\|^2+\|(I-T)Ax_n\|^2)^2}{\|x_n-Sx_n+A^*(I-T)Ax_n\|^2}\Big].
\end{aligned}
\tag{3.9}
$$

Hence, the desired inequality at once follows. **Step 3.** Show that $-\delta \le \overline{\lim}_{n\to\infty} b_n < +\infty$ for some $\delta > 0$, which indicate that $\overline{\lim}_{n\to\infty} b_n$ is finite. Since $\{x_n\}$ is bounded, we have

$$
\begin{aligned}
b_n &\le \frac{2(1-\alpha_n\gamma k)}{r-\gamma k}\langle \gamma f(u^*)-Du^*, x_{n+1}-u^*\rangle \\
&\le \frac{2(1-\alpha_n\gamma k)}{r-\gamma k}\|\gamma f(u^*)-Du^*\|\|x_{n+1}-u^*\| \\
&< +\infty
\end{aligned}
\tag{3.10}
$$

so that this implies $-\delta \le \overline{\lim}_{n\to\infty} b_n < +\infty$. We next prove $\overline{\lim}_{n\to\infty} b_n \ge -\delta$. To this aim, we processed by contradiction. Assume that $\overline{\lim}_{n\to\infty} b_n < -\delta$, which implies that there exists $n_0 \in \mathbb{N}$ such that $b_n \le -\delta$ for all $n \ge n_0$, it follows from that

$$
\begin{aligned}
a_{n+1} &\le (1-\alpha_n)a_n + \alpha_n b_n \\
&\le (1-\alpha_n)a_n - \alpha_n\delta \\
&= a_n - \alpha_n(a_n+\delta) \\
&\le a_n - \frac{\alpha_n(1-\gamma k)\delta}{1-\alpha_n\gamma k}.
\end{aligned}
\tag{3.11}
$$

for all $n \ge n_0$. By induction, we have

$$
a_{n+1} \le a_{n_0} - \Big(\frac{(1-\gamma k)\delta\sum_{i=n_0}^{n}\alpha_i}{1-\gamma k\sum_{i=n_0}^{n}\alpha_i}\Big). \tag{3.12}
$$

Hence, taking $\overline{\lim}$ as $n \to \infty$ in the last inequality, we have

$$
\overline{\lim}_{n\to\infty} a_n \le \overline{\lim}_{n\to\infty}\Big[a_{n_0} - \Big(\frac{(1-\gamma k)\delta\sum_{i=n_0}^{n}\alpha_i}{1-\gamma k\sum_{i=n_0}^{n}\alpha_i}\Big)\Big] = -\infty \tag{3.13}
$$

which clearly contradicts the fact that $\{a_n\}$ is a nonnegative real sequence. Thus, $\overline{\lim}_{n\to\infty} b_n \geq -\delta$ and it is finite.

Step 4. Show that $\{x_n\}$ converges to z. Since $\overline{\lim}_{n\to\infty} b_n$ is finite, we can take a subsequence $\{n_k\}$ such that

$$\begin{aligned}\overline{\lim}_{n\to\infty} b_n &= \lim_{k\to\infty} b_{n_k} \\ &= \lim_{k\to\infty} \Big[\frac{2(1-\alpha_n\gamma k)}{r-\gamma k}\langle \gamma f(u^*) - Du^*, x_{n_k+1} - u^*\rangle \\ &\quad - \frac{1-\alpha_n r}{\alpha_n(r-\gamma k)} \frac{(\|x_{n_k} - Sx_{n_k}\|^2 + \|(I-T)Ax_{n_k}\|^2)^2}{\|x_{n_k} - Sx_{n_k} + A^*(I-T)Ax_{n_k}\|^2}\Big].\end{aligned} \tag{3.14}$$

Since $\langle \gamma f(u^*) - Du^*, x_{n_k+1} - u^*\rangle$ is a bounded real sequence, without loss of generality, we may assume there exists the limit:

$$\lim_{k\to\infty} \langle \gamma f(u^*) - Du^*, x_{n_k+1} - u^*\rangle. \tag{3.15}$$

Consequently, from (3.14), the following limit also exists:

$$\lim_{k\to\infty} \frac{1-\alpha_n r}{\alpha_n(r-\gamma k)} \frac{(\|x_{n_k} - Sx_{n_k}\|^2 + \|(I-T)Ax_{n_k}\|^2)^2}{\|x_{n_k} - Sx_{n_k} + A^*(I-T)Ax_{n_k}\|^2} \tag{3.16}$$

which implies that the sequence

$$\frac{1}{\alpha_n(r-\gamma k)} \frac{(\|x_{n_k} - Sx_{n_k}\|^2 + \|(I-T)Ax_{n_k}\|^2)^2}{\|x_{n_k} - Sx_{n_k} + A^*(I-T)Ax_{n_k}\|^2} \tag{3.17}$$

is bounded. So, by condition $\alpha_n \to 0$, we have

$$\lim_{k\to\infty} \frac{(\|x_{n_k} - Sx_{n_k}\|^2 + \|(I-T)Ax_{n_k}\|^2)^2}{\|x_{n_k} - Sx_{n_k} + A^*(I-T)Ax_{n_k}\|^2} = 0. \tag{3.18}$$

By Lemma 3.3, we have

$$\lim_{n_k\to\infty} \|x_{n_k} - Sx_{n_k}\| = \lim_{n_k\to\infty} \|Ax_{n_k} - TAx_{n_k}\| = 0. \tag{3.19}$$

By the definition of x_{n_k+1}, we deduce that

$$\begin{aligned}\lim_{k\to\infty} \|x_{n_k} - y_{n_k}\| &= \lim_{k\to\infty} \rho_{n_k} \|x_{n_k} - Sx_{n_k} + A^*(I-T)Ax_{n_k}\| \\ &= \lim_{k\to\infty} \frac{(\|x_{n_k} - Sx_{n_k}\|^2 + \|(I-T)Ax_{n_k}\|^2)^2}{\|x_{n_k} - Sx_{n_k} + A^*(I-T)Ax_{n_k}\|^2} = 0\end{aligned} \tag{3.20}$$

which further implies

$$\begin{aligned}\|x_{n_k+1} - x_{n_k}\| &= \|\alpha_{n_k}\gamma f(x_{n_k}) + (I - \alpha_{n_k}B)y_{n_k} - x_{n_k}\| \\ &\leq \alpha_{n_k}\|\gamma f(x_{n_k}) - Dx_{n_k}\| + (1 - \alpha_{n_k}r)\|y_{n_k} - x_{n_k}\| \\ &\to 0.\end{aligned} \tag{3.21}$$

Since we have shown that the sequence $\{x_n\}$ is bounded. This implies that any weak cluster point of $\{x_{n_k+1}\}$ also belongs to Ω. Without loss of generality, we assume that $\{x_{n_k+1}\}$ converges weakly to $x \in \Omega$. Now by (3.14), we infer that

$$\begin{aligned}\overline{\lim}_{n\to\infty} b_n &= \lim_{k\to\infty} \frac{2(1 - \alpha_n\gamma k)}{r - \gamma k}\langle \gamma f(u^*) - Du^*, x_{n_k+1} - u^*\rangle \\ &= \frac{2(1 - \alpha_n\gamma k)}{r - \gamma k}\langle \gamma f(u^*) - Du^*, x - u^*\rangle \leq 0\end{aligned} \tag{3.22}$$

due to the fact that $u^* = P_\Omega(I - D + \gamma f)u^*$ and (2.3). Finally, applying Lemma 2.4 to (3.4), we arrive at $\|x_n - u^*\| \to 0$, which ends the proof. □

In the cases $f(x_n) = u$, we have the algorithm as follows

Algorithm 3.5. Choose an arbitrary initial guess u, x_0. Assume x_n has been constructed. If

$$\|x_n - Sx_n + A^*(I - T)Ax_n\| = 0,$$

then stop; otherwise, continue and construct x_{n+1} via the formula:

$$\begin{cases} y_n = x_n - \rho_n[x_n - Sx_n + A^*(I - T)Ax_n], \\ x_{n+1} = \alpha_n\gamma u + (I - \alpha_n D)y_n, \end{cases}$$

where $\gamma \in (0, \min\{1 - \beta, 1 - \mu\})$ is a positive constant and $\rho_n \subset (0, \infty)$ is chosen self-adaptively as

$$\rho_n = \sigma_n \frac{\|x_n - Sx_n\|^2 + \|Ax_n - TAx_n\|^2}{\|x_n - Sx_n + A^*(I - T)Ax_n\|^2}.$$

Corollary 3.6. Assume the following conditions are satisfied

(*i*) $\sum_{n=1}^\infty \rho_n = \infty$ and $\sum_{n=1}^\infty \rho_n^2 < \infty$;

(*ii*) $0 < r < \dfrac{1}{\alpha_n}, 0 < \gamma < r.$

Then the sequence $\{x_n\}$ generated by Algorithm 3.5 converges strongly to a solution u^* of problem **SCFPP**, where $u^* = P_\Omega(u^* - Du^* + u)$.

4 Numerical experiments

In this section, we construct a numerical example to illustrate the algorithm (3.1) and convergence analysis of the sequences of our main result. All codes were written in Matlab 2018a and run on Dell i-5 Core laptop.

Example 4.1. Let $H_1 = H_2 = (\mathbb{R}^2, \|\cdot\|_2)$. Define the mappings $S : \mathbb{R}^2 \to \mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$S(x_1, y_1) = (-3x_1, y_1) \quad \text{and} \quad T(x_1, y_1) = -5(x_1, y_1), \quad \forall\, x_1, y_1 \in \mathbb{R}.$$

First, we show that S is a $\frac{1}{2}-$demicontractive mapping. if $x = (x_1, y_1) \in \mathbb{R}^2$ and $p_1 = (0, a) \in F(S)$, then

$$\begin{aligned}\|Sx - p_1\|_2^2 &= \|(-3x_1, y_1) - (0, a)\|_2^2 \\ &= (-3)^2|x_1|^2 + |y_1 - a|^2 \\ &= 9|x_1|^2 + |y_1 - p_1|^2 \\ &= |x_1|^2 + |y_1 - a|^2 + 8|x_1|^2 \\ &= \|x - p_1\|_2^2 + \frac{8}{16}(16|x_1|^2) \\ &= \|x - p_1\|_2^2 + \frac{1}{2}\|x - Sx\|_2^2.\end{aligned}$$

Thus, S is a $\frac{1}{2}$-demicontractive mapping.
Second, we show that T is a $\frac{2}{3}-$demicontractive mapping. if $x = (x_1, y_1) \in \mathbb{R}^2$ and $p_2 = (0, 0) \in F(T)$, then

$$\begin{aligned}\|Tx - p_2\|_2^2 &= \|-5(x_1, y_1) - (0, 0)\|_2^2 \\ &= 25|x_1|^2 + 25|y_1|^2 \\ &= |x_1|^2 + |y_1|^2 + 24(|x_1|^2 + |y_1|^2) \\ &= \|x - p_2\|_2^2 + \frac{24}{36}(36(|x_1|^2 + |y_1|^2)) \\ &= \|x - p_2\|_2^2 + \frac{2}{3}\|x - Tx\|_2^2.\end{aligned}$$

Thus, T is a $\frac{2}{3}-$demicontractive mapping. Next, we define the mappings $f : \mathbb{R}^2 \to \mathbb{R}^2$, $A : \mathbb{R}^2 \to \mathbb{R}^2$ and $D : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(x_1, y_1) = (\frac{x_1}{8}, \frac{y_1}{8}),\ A(x_1, y_1) = 8(x_1, y_1),\ D(x_1, y_1) = (\frac{x_1}{2}, \frac{y_1}{2}),\ \forall\, x_1, y_1 \in \mathbb{R}.$$

Then f is a $\frac{1}{8}-$contraction, A is a bounded linear operator on $\mathbb{R}$ with adjoint operator A^* and D is a strongly positive bounded linear operator with coefficient $\xi = \frac{1}{2}$. In Algorithm (3.1), we set $\sigma_n = 0.15$, $\gamma = 0.1$ and $\alpha_n = \frac{1}{10n+100}$.

We compare with Moudafi in [27], by set $\gamma = 0.0018$ and $\alpha_n = \frac{1}{10n+100}$. Then, we have the results in Table 4.1 and Figure 4.1.

Table 4.1: Result of Example 4.1.

(x_1, y_1)		Algorithm 3.1	Moudafi
	No. of Iter.	6	15
$(-9, 9)$	Approximation	(-0.000000, 0.000000)	(-0.000000, 0.000000)
	$\|x_{n+1} - x_n\|_2$	0.000000	0.000000
	Time	0.036015	0.025440
	No. of Iter.	6	15
$(20, 10)$	Approximatio	(0.000000, 0.000000)	(0.000000, 0.000000)
	$\|x_{n+1} - x_n\|_2$	0.000000	0.000001
	Time	0.027071	0.015959
	No. of Iter.	6	14
$(-6, -2)$	Approximatio	(0.000000, -0.000000)	(-0.000000, -0.000000)
	$\|x_{n+1} - x_n\|_2$	0.000000	0.000001
	Time	0.036619	0.015571

4.1 Compressed sensing

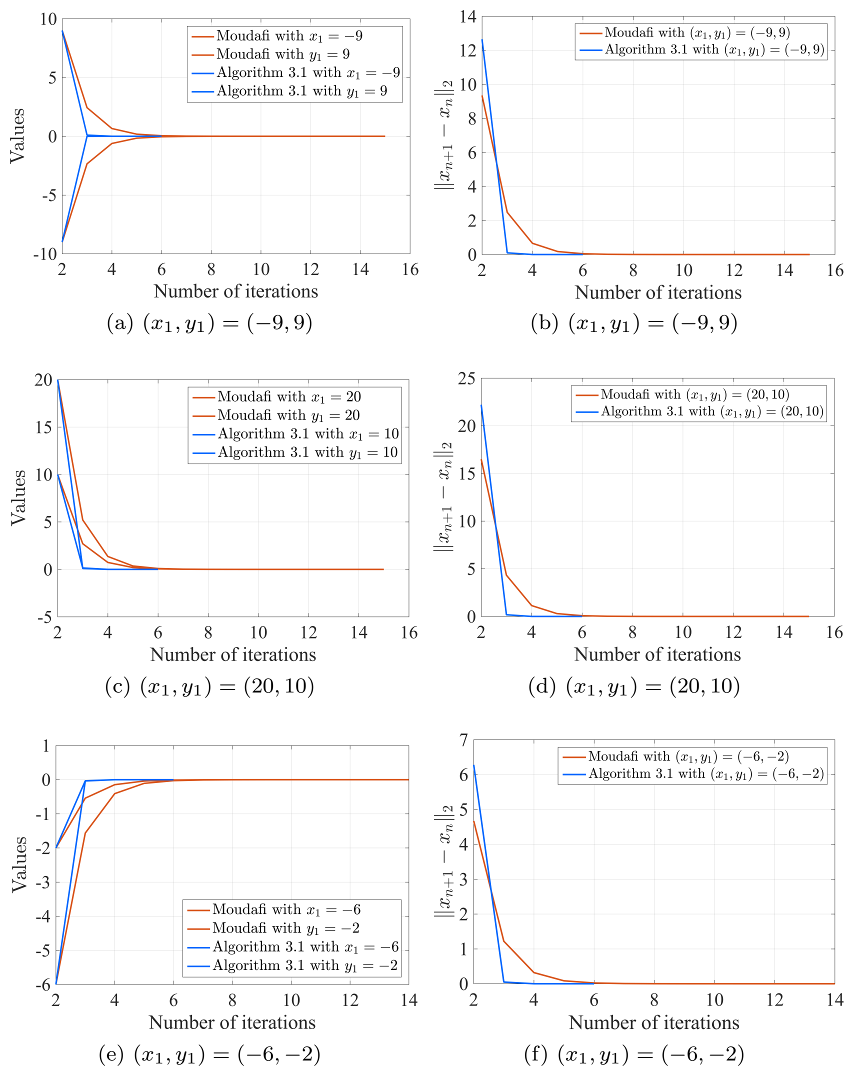
Compressed sensing is a very active domain of research and applications, based on the fact that an K-sample signal x with $M \leq N < K$. The sampling matrix $A \in \mathbb{R}^{M\times N}$ $(M < N)$ is stimulated by standard Gaussian distribution and vector $z = Ax_0 + b$, where b is additive noise. The most common form of disorder technique is l_1 regularization as:

$$\min_{x\in\mathbb{R}^N} \{\frac{1}{2}\|Ax - z\|^2 + \alpha\|x\|_1\}, \tag{4.1}$$

where is α a positive parameter and $\|\cdot\|_1$ denotes the sum of the absolute values of the components. By means of convex analysis, one is able to show that a solution to the constrained least squares problem:

$$\min_{x\in\mathbb{R}^N} \{\frac{1}{2}\|Ax - z\|^2\} \quad \text{subjet to} \quad \|x\|_1 \leq t, \tag{4.2}$$

for any nonnegative real number t, is a minimizer of (4.1) (see [9]). Clearly problem (4.2) is a particular case of problem (**SFP**) where $C = \{x \in \mathbb{R}^N :$



(a) $(x_1, y_1) = (-9, 9)$ (b) $(x_1, y_1) = (-9, 9)$

(c) $(x_1, y_1) = (20, 10)$ (d) $(x_1, y_1) = (20, 10)$

(e) $(x_1, y_1) = (-6, -2)$ (f) $(x_1, y_1) = (-6, -2)$

Figure 4.1: Result of Example 4.1.

$\|x\|_1 \leq t\}$ and $Q = \{z\}$. and thus can be solved by the proposed algorithm. In this case, P_C is the projection onto the closed l_1-ball in $\mathbb{R}^n$ (see [7]).

Algorithm 4.2. Choose an arbitrary initial guess x_0. Assume x_n has been constructed. If

$$\|x_n - P_C x_n + A^*(Ax_n - z)\| = 0,$$

then stop; otherwise, continue and construct x_{n+1} via the formula:

$$\begin{cases} y_n = x_n - \rho_n[x_n - P_C x_n + A^*(Ax_n - z)], \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n D) y_n, \end{cases}$$

where $\gamma \in (0,1)$ is a positive constant, $\sigma_n \in (0,1)$ and $\rho_n \in (0,1)$ is chosen self-adaptively as

$$\rho_n = \sigma_n \frac{\|x_n - P_C x_n\|^2 + \|Ax_n - z\|^2}{\|x_n - P_C x_n + A^*(Ax_n - z)\|^2}.$$

Theorem 4.3. Let $\{x_n\}$ be the sequence generated by Algorithm 4.2. If the sequence $\{\rho_n\}$ satisfies $\sum_{n=1}^{\infty} \rho_n = \infty$ and $\sum_{n=1}^{\infty} \rho_n^2 < \infty$, then $\{x_n\}$ converges weakly to a solution u^* of split feasibility problem .

Proof. Take $S = P_C$ and $T = P_Q$ in Theorem 3.4. □

In our experiment, we set the hits of a signal $x \in \mathbb{R}^N$ is $N = 2^{12}$. There exist $K = 50$ spikes with amplitude ± 1 distributed in the whole domain randomly. Then we set the observation dimension $M = 2^{10}$ with white Gaussian noise of variance $\epsilon^2 = 10^{-4}$. The process is started with initial signal $x_0 = A^* z$ and finishes with 400 iterations. The restoration accuracy is measured by means of the mean squared error: $\text{MSE} = \frac{\|x^* - x\|^2}{N}$, where x^* is an estimated signal of x. All codes were written in Matlab 2018a and run on Dell i-5 Core laptop. We compare the performances of Algorithm 4.2 by $f(x) = \frac{1}{8}x$, $D(x) = \frac{1}{2}x$, $\sigma_n = 0.15$, $\gamma = 0.1$ and $\alpha_n = \frac{1}{10n+100}$. with Byrne' s algorithm [2] by $\gamma = 0.1$ are reported in Figure 4.2.

4.2 Image restoration

We apply the algorithm 4.2 in the paper to image restoration. The observation model can also be described as (4.1), we wish to estimate an original image x from an observation z, while matrix A represents the blur operator ('motion',15,60), and b is random noise. The signal to noise ratio (SNR) is used to measure the quality of the restored images. They are defined as follows:

$$\text{SNR} = 20 \log \frac{\|x\|}{\|x - x_n\|}$$

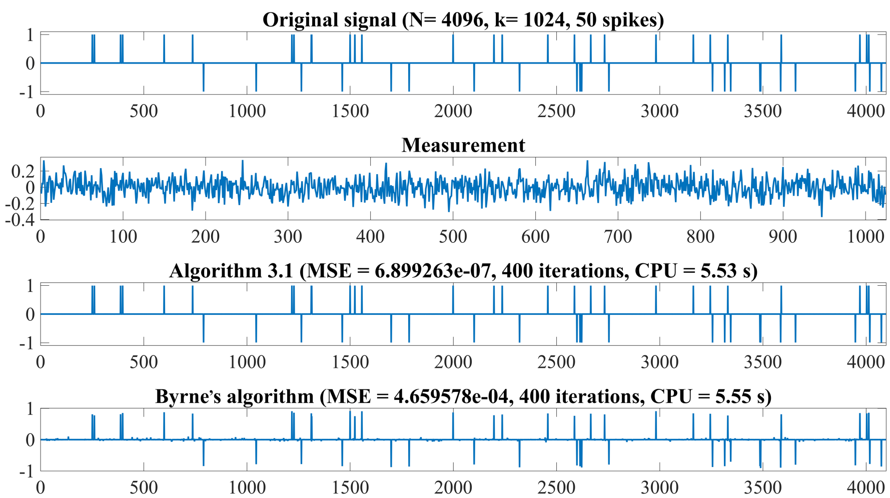


Figure 4.2: From top to bottom: original signal, reconstruction signals respectively by Algorithm 4.2 and Byrne' s algorithm

where x, z and x_n are the original image, the observed image and estimated image at iteration n, respectively. The process is started with initial signal $x_0 = Ax + b$ and finishes with 100 iterations. All codes were written in Matlab 2018a and run on Dell i-5 Core laptop.
We compare the performances of Algorithm 4.2 by $f(x) = \frac{1}{8}x$, $D(x) = \frac{1}{2}x$, $\sigma_n = 0.2$, $\gamma = 0.9$ and $\alpha_n = \frac{1}{10n+100}$ with Byrne's algorithm [2] by $\gamma = 0.9$ are reported in Figure 4.3 and Figure 4.4.

5 Conclusions

In this article, we proposed a new iterative scheme for finding common solutions of demicontractive operators. Under some suitable conditions imposed on parameters, we proved some strong convergence theorems of the proposed algorithm and, finally, we presented some numerical results to show that our algorithm performs better than some existing methods.

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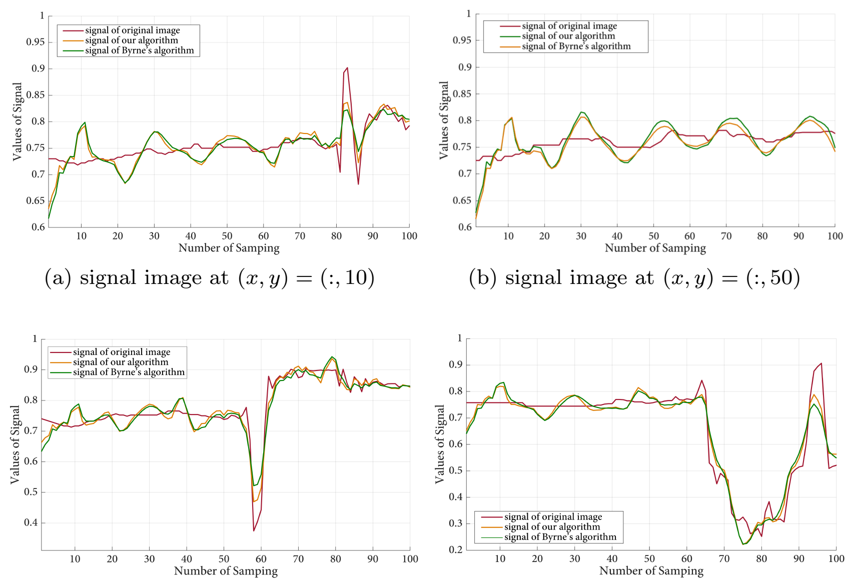
(a) original image

(b) blur and noisy image

(c) Algorithm 4.2 (SNR=50.30, 100 iter, CPU=8.63s)

(d) Byrne's algorithm (SNR=46.35, 100 iter, CPU=5.01s)

Figure 4.3: Result of restoration image size 192×256: (a) original image, (b) blur and noisy image, (c) restoration by Algorithm 4.2 and (d) restoration by Byrne' s algorithm



(a) signal image at $(x, y) = (:, 10)$ (b) signal image at $(x, y) = (:, 50)$

(c) signal image at $(x, y) = (:, 100)$ (d) signal image at $(x, y) = (:, 150)$

Figure 4.4: Signal of restoration: (a) signal of position $(x, y) = (:, 10)$, (b) signal of position $(x, y) = (:, 50)$, (c) signal of position $(x, y) = (:, 100)$ and (d) signal of position $(x, y) = (:, 150)$

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