Depth and Stanley depth of the edge ideals of the powers of paths and cycles

Zahid Iqbal and Muhammad Ishaq

Abstract

Let k be a positive integer. We compute depth and Stanley depth of the quotient ring of the edge ideal associated to the k^{th} power of a path on n vertices. We show that both depth and Stanley depth have the same values and can be given in terms of k and n . If $n \equiv 0, k+1, k+2, \dots, 2k \pmod{2k+1}$, then we give values of depth and Stanley depth of the quotient ring of the edge ideal associated to the k^{th} power of a cycle on n vertices and tight bounds otherwise, in terms of n and k . We also compute lower bounds for the Stanley depth of the edge ideals associated to the k^{th} power of a path and a cycle and prove a conjecture of Herzog for these ideals.

1 Introduction

Let K be a field and $S := K[x_1, \dots, x_n]$ the polynomial ring over K . Let M be a finitely generated $\mathbb{Z}^n$ -graded S -module. A Stanley decomposition of M is a presentation of the K -vector space M as a finite direct sum $\mathcal{D} : M = \bigoplus_{i=1}^s v_i K[W_i]$, where $v_i \in M$, $W_i \subseteq \{x_1, \dots, x_n\}$, and $v_i K[W_i]$ denotes the K -subspace of M , which is generated by all elements $v_i w$, where w is a monomial in $K[W_i]$. The $\mathbb{Z}^n$ -graded K -subspace $v_i K[W_i] \subset M$ is called a Stanley space of dimension $|W_i|$, if $v_i K[W_i]$ is a free $K[W_i]$ -module, where $|W_i|$ denotes the cardinality of W_i . Define $\text{sdepth}(\mathcal{D}) = \min\{|W_i| : i = 1, \dots, s\}$,

Key Words: Monomial ideal, edge ideal, depth, Stanley decomposition, Stanley depth.
2010 Mathematics Subject Classification: Primary 13C15; Secondary 13F20; 05C38; 05E99.

Received: 06.11.2018

Accepted: 11.01.2019

and $\text{sdepth}(M) = \max\{\text{sdepth}(\mathcal{D}) : \mathcal{D} \text{ is a Stanley decomposition of } M\}$. The number $\text{sdepth}(\mathcal{D})$ is called the Stanley depth of decomposition $\mathcal{D}$ and $\text{sdepth}(M)$ is called the Stanley depth of M . Stanley conjectured in [24] that $\text{sdepth}(M) \geq \text{depth}(M)$ for any $\mathbb{Z}^n$ -graded S -module M . This conjecture was disproved by Duval et al. in [8] as was expected due to different nature of these two invariants. However, the relation between Stanley depth and some other invariants has already been established; see [11, 12, 21, 26]. In [11], Herzog, Vladioiu and Zheng proved that the Stanley depth of M can be computed in a finite number of steps, if $M = J/I$, where $I \subset J \subset S$ are monomial ideals. But practically it is too hard to compute Stanley depth by using this method; see for instance, [2, 5, 15, 16]. For computing Stanley depth for some classes of modules we refer the reader to [14, 20, 22, 23]. In this paper we attempt to find values and reasonable bounds for depth and Stanley depth of I and S/I , where I is the edge ideal of a power of a path or a cycle. We also compare the values of $\text{sdepth}(I)$ and $\text{sdepth}(S/I)$ and give positive answers to the following conjecture of Herzog.

Conjecture 1.1. [9] *Let $I \subset S$ be a monomial ideal then $\text{sdepth}(I) \geq \text{sdepth}(S/I)$.*

The above conjecture is proved in some other cases; see [13, 16, 20, 23]. The paper is organized as follows: First two sections are devoted to introduction, definitions, notation, and discussion of some known results. In third section, we compute depth and Stanley depth of $S/I(P_n^k)$, where $I(P_n^k)$ denotes the edge ideal of the k^{th} power of a path P_n on n vertices. Let for $q \in \mathbb{Q}$, $\lceil q \rceil$ denotes the smallest integer greater than or equal to q . Then in Theorems 3.8 and 3.14 we prove that

$$\text{depth}(S/I(P_n^k)) = \text{sdepth}(S/I(P_n^k)) = \lceil \frac{n}{2k+1} \rceil.$$

Let $I(C_n^k)$ be the edge ideal of the k^{th} power of a cycle C_n on n vertices. In fourth section we give some lower bounds for depth and Stanley depth of $S/I(C_n^k)$; see Theorems 4.5 and 4.7. If $n \geq 2k+2$, then by Corollaries 4.6 and 4.8 we prove that if $n \equiv 0, k+1, \dots, 2k \pmod{2k+1}$ then $\text{depth}(S/I(C_n^k)) = \text{sdepth}(S/I(C_n^k)) = \lceil \frac{n}{2k+1} \rceil$. Otherwise,

$$\lceil \frac{n}{2k+1} \rceil - 1 \leq \text{depth}(S/I(C_n^k)), \text{sdepth}(S/I(C_n^k)) \leq \lceil \frac{n}{2k+1} \rceil.$$

Last section is devoted to Conjecture 1.1 for $I(P_n^k)$ and $I(C_n^k)$. By our Theorem 5.2 we have

$$\text{sdepth}(I(P_n^k)) \geq \lceil \frac{n}{2k+1} \rceil + 1,$$

which shows that $I(P_n^k)$ satisfies Conjecture 1.1. Let $n \geq 2k + 1$. Proposition 5.3 gives a lower bound for $I(C_n^k)/I(P_n^k)$ that is

$$\text{sdepth}(I(C_n^k)/I(P_n^k)) \geq \lceil \frac{n+k+1}{2k+1} \rceil.$$

Corollary 5.5 of this paper proves that $I(C_n^k)$ satisfies Conjecture 1.1.

2 Definitions and notation

Throughout this paper $\mathfrak{m}$ denotes the unique maximal graded ideal $(x_1, \dots, x_n)$ of S . We set $S_m := K[x_1, x_2, \dots, x_m]$, $\text{supp}(v) := \{i : x_i | v\}$ and $\text{supp}(I) := \{i : x_i | u, \text{ for some } u \in \mathcal{G}(I)\}$, where $\mathcal{G}(I)$ denotes the unique minimal set of monomial generators of the monomial ideal I . Let $I \subset S$ be an ideal. Then we write I instead of IS . Thus every ideal will be considered an ideal of S unless otherwise stated. Let I and J be monomial ideals of S , then for $I + J$ we write (I, J) .

We review some notation and refer the reader to [3] for further details. Let G be a simple graph. For a positive integer k , the k^{th} power of graph G is another graph G^k on the same set of vertices, such that two vertices are adjacent in G^k when their distance in G is at most k . In the whole paper we label the vertices of the graph G by $1, 2, \dots, n$. We denote the set of vertices of G by $[n] := \{1, 2, \dots, n\}$ and its edge set by $E(G)$. We assume that all graphs and their powers are simple graphs. We also assume that all graphs have at least two vertices and a non-empty edge set. For a graph G , the edge ideal $I(G)$ associated to G is defined as $I(G) := (x_i x_j : \{i, j\} \in E(G))$. For $n \geq 2$, a graph G is called a path if $E(G) = \{\{i, i+1\} : i \in [n-1]\}$. A path on n vertices is denoted by P_n . For $n \geq 3$, a graph G is called a cycle if $E(G) = \{\{i, i+1\} : i \in [n-1]\} \cup \{1, n\}$. A cycle on n vertices is denoted by C_n . For $n \geq 2$, the k^{th} power of a path, denoted by P_n^k , is a graph such that for all $1 \leq i < j \leq n$, $\{i, j\} \in E(P_n^k)$ if and only if $0 < j - i \leq k$. If $n \leq k + 1$, then P_n^k is a complete graph on n vertices. If $n \geq k + 2$, then

$$E(P_n^k) = \cup_{i=1}^{n-k} \{\{i, i+1\}, \{i, i+2\}, \dots, \{i, i+k\}\} \cup \cup_{j=n-k+1}^{n-1} \{\{j, j+1\}, \{j, j+2\}, \dots, \{j, n\}\}.$$

For $n \geq 3$, the k^{th} power of a cycle, denoted by C_n^k , is a graph such that for all vertices $1 \leq i, j \leq n$, $\{i, j\} \in E(C_n^k)$ if and only if $|j - i| \leq k$ or $|j - i| \geq n - k$. If $n \leq 2k + 1$, then C_n^k is a complete graph on n vertices. If $n \geq 2k + 2$, then

$$E(C_n^k) = E(P_n^k) \cup \cup_{l=1}^k \{\{l, l+n-k\}, \{l, l+n-k+1\}, \{l, l+n-k+2\}, \dots, \{l, n\}\}.$$

For examples of powers of paths and cycles see Figures 1 and 2.

If $n \leq k+1$, then $I(P_n^k)$ is a squarefree Veronese ideal of degree 2. If $n \geq k+2$, then

$$\mathcal{G}(I(P_n^k)) = \cup_{i=1}^{n-k} \{x_i x_{i+1}, x_i x_{i+2}, \dots, x_i x_{i+k}\} \cup \cup_{j=n-k+1}^{n-1} \{x_j x_{j+1}, x_j x_{j+2}, \dots, x_j x_n\}.$$

If $n \leq 2k+1$, then $I(C_n^k)$ is a squarefree Veronese ideal of degree 2. If $n \geq 2k+2$, then

$$\mathcal{G}(I(C_n^k)) = \mathcal{G}(I(P_n^k)) \cup \cup_{l=1}^k \{x_l x_{l+n-k}, x_l x_{l+n-k+1}, \dots, x_l x_n\}.$$

Figure 1: From left to right, P_{12}^3 and P_{12}^4 respectively.

Figure 2: From left to right, C_{10}^3 and C_{10}^4 respectively.

Lemma 2.1 ([17, Lemma 3]). *If $n \geq k+1$, then $|\mathcal{G}(I(P_n^k))| = nk - \frac{k(k+1)}{2}$.*

Remark 2.2. If $n \geq 2k+1$, then $|\mathcal{G}(I(C_n^k))| = nk$.

Let G be a graph and $i \in [n]$, then $N_G(x_i) := \{x_j : x_i x_j \in \mathcal{G}(I(G))\}$, where $j \in [n] \setminus \{i\}$. For $k \geq 2$, $0 \leq i \leq k-1$ and $n \geq 2k+2$, let A_{n-k-1} ,

A_{n-k+i} , B_{n-k+i} and D_{n-k+i} denote the monomial prime ideals of S , such that $A_{n-k-1} = (0)$, $A_{n-k+i} := (x_{n-k}, x_{n-k+1}, \dots, x_{n-k+i})$,

$$\begin{aligned} B_{n-k+i} &:= (x_j : x_j \in N_{P_n^k}(x_{n-k+i})) \\ &= (x_{n-2k+i}, x_{n-2k+i+1}, \dots, x_{n-k+i-1}, x_{n-k+i+1}, \dots, x_n), \end{aligned}$$

and $D_{n-k+i} := (x_j : x_j \in N_{C_n^k}(x_{n-k+i}))$. Thus if $i = 0$, then

$$D_{n-k+i} = (x_{n-2k}, x_{n-2k+1}, \dots, x_{n-k-1}, x_{n-k+1}, \dots, x_n)$$

and if $1 \leq i \leq k-1$, then

$$D_{n-k+i} = (x_{n-2k+i}, x_{n-2k+i+1}, \dots, x_{n-k+i-1}, x_{n-k+i+1}, \dots, x_n, x_1, \dots, x_i).$$

These monomial prime ideals and the following function play important role in the proof of our main theorems. For $k \geq 2$ and $2k+2 \leq n \leq 3k+1$, we define a function

$$\begin{aligned} f : \{n-k, n-k+1, \dots, n-k+i, \dots, n-1\} &\longrightarrow \mathbb{Z}^+ \cup \{0\}, \text{ by} \\ f(n-k+i) &= \begin{cases} k, & \text{if } n-2k-1+i \geq k+1; \\ n-2k-2+i, & \text{if } 2 \leq n-2k-1+i < k+1. \end{cases} \end{aligned}$$

In the following we recall some known results that we refer several times in this paper.

Lemma 2.3 (Depth Lemma). *If $0 \longrightarrow U \longrightarrow M \longrightarrow N \longrightarrow 0$ is a short exact sequence of modules over a local ring S , or a Noetherian graded ring with S_0 local, then*

1. $\text{depth } M \geq \min\{\text{depth } N, \text{depth } U\}$.
2. $\text{depth } U \geq \min\{\text{depth } M, \text{depth } N + 1\}$.
3. $\text{depth } N \geq \min\{\text{depth } U - 1, \text{depth } M\}$.

Lemma 2.4 ([23, Lemma 2.2]). *Let $0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$ be a short exact sequence of $\mathbb{Z}^n$ -graded S -modules. Then*

$$\text{sdepth}(V) \geq \min\{\text{sdepth}(U), \text{sdepth}(W)\}.$$

Lemma 2.5 ([13, Lemma 3.3]). *Let $I \subset S$ be a squarefree monomial ideal with $\text{supp}(I) = [n]$ and $v := x_{i_1}x_{i_2} \cdots x_{i_q} \in S/I$ such that $x_jv \in I$ for all $j \in [n] \setminus \text{supp}(v)$. Then $\text{sdepth}(S/I) \leq q$.*

The above Lemma can also be seen as an immediate consequence of the result of J. Apel [1, Sec.3].

3 Depth and Stanley of cyclic modules associated to the edge ideals of the powers of a path

We start this section with some results. These results are essential for computations of depth and Stanley depth of $S/I(P_n^k)$.

Lemma 3.1. *Let $a \geq 2$ be an integer, $\{E_i : 1 \leq i \leq a\}$ and $\{G_i : 0 \leq i \leq a\}$ be some families of $\mathbb{Z}^n$ -graded S -modules such that we have the following short exact sequences:*

$$0 \longrightarrow E_1 \longrightarrow G_0 \longrightarrow G_1 \longrightarrow 0 \quad (1)$$

$$0 \longrightarrow E_2 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow 0 \quad (2)$$

$$\vdots$$

$$0 \longrightarrow E_{a-1} \longrightarrow G_{a-2} \longrightarrow G_{a-1} \longrightarrow 0 \quad (a-1)$$

$$0 \longrightarrow E_a \longrightarrow G_{a-1} \longrightarrow G_a \longrightarrow 0 \quad (a)$$

and $\text{depth}(G_a) \geq \text{depth}(E_a)$, $\text{depth}(E_i) \geq \text{depth}(E_{i-1})$ for all $2 \leq i \leq a$. Then $\text{depth}(G_0) = \text{depth}(E_1)$.

Proof. By assumption, we have $\text{depth}(G_a) \geq \text{depth}(E_a)$, applying Depth Lemma on the exact sequence (a) we get $\text{depth}(G_{a-1}) = \text{depth}(E_a)$. We also have by assumption

$$\text{depth}(G_{a-1}) = \text{depth}(E_a) \geq \text{depth}(E_{a-1}).$$

By applying Depth Lemma on the exact sequence (a-1) we have $\text{depth}(G_{a-2}) = \text{depth}(E_{a-1})$. We repeat the same steps on all exact sequences one by one from bottom to top and we get $\text{depth}(G_{i-1}) = \text{depth}(E_i)$ for all i . Thus if $i = 1$ then we have $\text{depth}(G_0) = \text{depth}(E_1)$. $\square$

Lemma 3.2. *Let $k \geq 2$ and $n \geq 2k + 2$. Then*

$$S/(I(P_n^k), A_{n-1}) \cong S_{n-k-1}/I(P_{n-k-1}^k)[x_n].$$

Proof. Since

$$\begin{aligned} \mathcal{G}(I(P_n^k)) = \bigcup_{i=1}^{n-k} \{x_i x_{i+1}, x_i x_{i+2}, \dots, x_i x_{i+k}\} \cup \\ \bigcup_{i=n-k+1}^{n-1} \{x_i x_{i+1}, x_i x_{i+2}, \dots, x_i x_n\}, \end{aligned}$$

so we have

$$\begin{aligned}
 I(P_n^k) + A_{n-1} &= A_{n-1} + \\
 &\left[\sum_{i=1}^{n-2k-1} (x_i x_{i+1}, x_i x_{i+2}, \dots, x_i x_{i+k}) + \sum_{i=n-2k}^{n-k} (x_i x_{i+1}, x_i x_{i+2}, \dots, x_i x_{i+k}) + \right. \\
 &\left. \sum_{i=n-k+1}^{n-1} (x_i x_{i+1}, x_i x_{i+2}, \dots, x_i x_n) \right] = \sum_{i=1}^{n-2k-1} (x_i x_{i+1}, x_i x_{i+2}, \dots, x_i x_{i+k}) + \\
 &\sum_{i=n-2k}^{n-k-2} (x_i x_{i+1}, x_i x_{i+2}, \dots, x_i x_{n-k-1}) + A_{n-1} = I(P_{n-k-1}^k) + A_{n-1}.
 \end{aligned}$$

Thus the required result follows. $\square$

Lemma 3.3. *Let $k \geq 2$, $0 \leq i \leq k-1$ and $n \geq 3k+2$. Then*

$$S/(I(P_n^k) : x_{n-k+i}) \cong S_{n-2k-1+i}/I(P_{n-2k-1+i}^k)[x_{n-k+i}].$$

Proof. It is enough to prove that $(I(P_n^k) : x_{n-k+i}) = (I(P_{n-2k-1+i}^k), B_{n-k+i})$. Clearly

$$I(P_{n-2k-1+i}^k) \subset I(P_n^k) \subset (I(P_n^k) : x_{n-k+i}).$$

Let $u \in B_{n-k+i}$, then by definition of $I(P_n^k)$, $ux_{n-k+i} \in I(P_n^k)$ that is $u \in (I(P_n^k) : x_{n-k+i})$. Thus $B_{n-k+i} \subset (I(P_n^k) : x_{n-k+i})$ and we have $(I(P_{n-2k-1+i}^k), B_{n-k+i}) \subset (I(P_n^k) : x_{n-k+i})$. Now for the other inclusion, let w be a monomial generator of $(I(P_n^k) : x_{n-k+i})$, then $w = \frac{v}{\gcd(v, x_{n-k+i})}$, where $v \in \mathcal{G}(I(P_n^k))$. If $\text{supp}(v) \cap \mathcal{G}(B_{n-k+i}) \neq \emptyset$, then we have $w \in \mathcal{G}(B_{n-k+i})$ and if $\text{supp}(v) \cap \mathcal{G}(B_{n-k+i}) = \emptyset$, then $w \in \mathcal{G}(I(P_n^k)) \cap K[x_1, x_2, \dots, x_{n-2k-1+i}] = \mathcal{G}(I(P_{n-2k-1+i}^k))$. $\square$

Lemma 3.4. *Let $n \geq 3k+2$ and $0 \leq i \leq k-1$, then we have*

$$S/((I(P_n^k), A_{n-k+(i-1)}) : x_{n-k+i}) \cong S_{n-2k-1+i}/I(P_{n-2k-1+i}^k)[x_{n-k+i}].$$

Proof. As $((I(P_n^k), A_{n-k+(i-1)}) : x_{n-k+i}) = ((I(P_n^k) : x_{n-k+i}), A_{n-k+(i-1)})$. Now using the proof of Lemma 3.3 we obtain

$$\begin{aligned}
 ((I(P_n^k) : x_{n-k+i}), A_{n-k+(i-1)}) &= \\
 (I(P_{n-2k-1+i}^k), B_{n-k+i}, A_{n-k+(i-1)}) &= (I(P_{n-2k-1+i}^k), B_{n-k+i}),
 \end{aligned}$$

as $A_{n-k+(i-1)} \subset B_{n-k+i}$. Thus the required result follows by Lemma 3.3. $\square$

Remark 3.5. Let $m \geq 2$ and $I(P_m^{m-1}) \subset S_m = K[x_1, x_2, \dots, x_m]$ be the edge ideal of the $(m-1)^{th}$ power of path P_m . Then $I(P_m^{m-1})$ is a squarefree Veronese ideal of degree 2 in variables $x_1, x_2, \dots, x_m$. Thus by [10, Corollary 10.3.7] and Theorem 3.9

$$\text{depth}(S_m/I(P_m^{m-1})) = \text{sdepth}(S_m/I(P_m^{m-1})) = 1.$$

Remark 3.6. Let $k \geq 2$ and $2k+2 \leq n \leq 3k+1$, then it is easy to see that

(1) If $n = 2k+2$, then

$$S/(I(P_n^k) : x_{n-k}) = S/(x_2, \dots, x_{n-k-1}, x_{n-k+1}, \dots, x_n) \cong K[x_1, x_{n-k}].$$

(2) If $0 \leq i \leq k-1$ and $n > 2k+2$, then

$$\begin{aligned} S/(I(P_n^k) : x_{n-k+i}) &= S/((I(P_n^k), A_{n-k+(i-1)}) : x_{n-k+i}) \\ &\cong S_{n-2k-1+i}/I(P_{n-2k-1+i}^{f(n-k+i)})[x_{n-k+i}] \\ &= \begin{cases} S_{n-2k-1+i}/I(P_{n-2k-1+i}^k)[x_{n-k+i}], & \text{if } n-2k-1+i \geq k+1; \\ S_{n-2k-1+i}/I(P_{n-2k-1+i}^{n-2k-2+i})[x_{n-k+i}], & \text{otherwise.} \end{cases} \end{aligned}$$

We recall a lemma from [11] which is heavily used in this paper.

Lemma 3.7 ([11, Lemma 3.6]). *Let $J \subset I$ be monomial ideals of S , and let $T = S[x_{n+1}]$ be the polynomial ring over S in the variable x_{n+1} . Then $\text{depth}(IT/JT) = \text{depth}(I/J) + 1$ and $\text{sdepth}(IT/JT) = \text{sdepth}(I/J) + 1$.*

Theorem 3.8. *Let $n \geq 2$. Then $\text{depth}(S/I(P_n^k)) = \lceil \frac{n}{2k+1} \rceil$.*

Proof. (a) If $n \leq k+1$, then $I(P_n^k)$ is a squarefree Veronese ideal thus by Remark 3.5, $\text{depth}(S/I(P_n^k)) = 1 = \lceil \frac{n}{2k+1} \rceil$.

(b) For $n \geq k+2$, we consider the following cases:

(1) If $k = 1$, then by [18, Lemma 2.8] we have $\text{depth}(S/I(P_n^1)) = \lceil \frac{n}{3} \rceil = \lceil \frac{n}{2k+1} \rceil$.

(2) If $k \geq 2$ and $k+2 \leq n \leq 2k+1$, then we get $\text{depth}(S/I(P_n^k)) \geq 1$ as $\mathfrak{m} \notin \text{Ass}(S/I(P_n^k))$. Since $x_{k+1} \notin I(P_n^k)$ and $x_s x_{k+1} \in \mathfrak{g}(I(P_n^k))$ for all $s \in \{1, \dots, k, k+2, \dots, n\}$, therefore we have $(I(P_n^k) : x_{k+1}) = (x_1, \dots, x_k, x_{k+2}, \dots, x_n)$. By [23, Corollary 1.3], we have

$$\begin{aligned} \text{depth}(S/I(P_n^k)) &\leq \text{depth}(S/(I(P_n^k) : x_{k+1})) \\ &= \text{depth}(S/(x_1, \dots, x_k, x_{k+2}, \dots, x_n)) = 1. \end{aligned}$$

$$\text{Thus } \text{depth}(S/I(P_n^k)) = 1 = \lceil \frac{n}{2k+1} \rceil.$$

- (3) For $k \geq 2$, $2k + 2 \leq n \leq 3k + 1$ and $0 \leq i \leq k - 1$, consider the family of short exact sequences

$$0 \longrightarrow S/((I(P_n^k), A_{n-k+(i-1)}) : x_{n-k+i}) \xrightarrow{\cdot x_{n-k+i}} S/(I(P_n^k), A_{n-k+(i-1)}) \longrightarrow S/(I(P_n^k), A_{n-k+i}) \longrightarrow 0$$

By Lemma 3.2, $S/(I(P_n^k), A_{n-1}) \cong S_{n-k-1}/I(P_{n-k-1}^k)[x_n]$. Since we are considering the case $2k + 2 \leq n \leq 3k + 1$ which implies that $k + 1 \leq n - k - 1 \leq 2k$. If $n - k - 1 = k + 1$ then $S_{n-k-1}/I(P_{n-k-1}^k) = S_{k+1}/I(P_{k+1}^k)$, by Remark 3.5 and Lemma 3.7 we have $\text{depth}(S/(I(P_n^k), A_{n-1})) = 2$. If $k + 1 < n - k - 1 \leq 2k$, then by case(b)(2) $\text{depth}(S_{n-k-1}/I(P_{n-k-1}^k)) = 1$. Thus by Lemma 3.7 we have $\text{depth}(S/(I(P_n^k), A_{n-1})) = 2$. Now we show that $\text{depth}(S/(I(P_n^k) : x_{n-k})) = 2$. For this we consider two cases: If $n = 2k + 2$, then by Remark 3.6

$$S/(I(P_n^k) : x_{n-k}) = S/(x_2, x_3, \dots, x_{n-k-1}, x_{n-k+1}, \dots, x_n) \cong K[x_1, x_{n-k}],$$

and thus $\text{depth}(S/(I(P_n^k) : x_{n-k})) = 2$. If $n > 2k + 2$, by Remark 3.6 we have

$$S/(I(P_n^k) : x_{n-k}) \cong S_{n-2k-1}/I(P_{n-2k-1}^{n-2k-2})[x_{n-k}],$$

where $2 \leq n - 2k - 1 \leq k$. Thus by Remark 3.5 and Lemma 3.7 we get $\text{depth}(S/(I(P_n^k) : x_{n-k})) = 2$. Now for $1 \leq i \leq k - 1$, by Remark 3.6 we obtain

$$S/((I(P_n^k), A_{n-k+(i-1)}) : x_{n-k+i}) = S/(I(P_n^k) : x_{n-k+i}) \cong S_{n-2k-1+i}/I(P_{n-2k-1+i}^{f(n-k+i)})[x_{n-k+i}].$$

Let $T := S_{n-2k-1+i}/I(P_{n-2k-1+i}^{f(n-k+i)})[x_{n-k+i}]$. We consider the following cases:

- (i) If $k + 1 = n - 2k - 1 + i$, then $T = S_{k+1}/I(P_{k+1}^k)[x_{n-k+i}]$, thus by case(a) and Lemma 3.7 we have $\text{depth}(T) = 2$.
- (ii) For $k + 1 < n - 2k - 1 + i$, $T = S_{n-2k-1+i}/I(P_{n-2k-1+i}^k)[x_{n-k+i}]$. Since $k + 2 \leq n - 2k - 1 + i \leq 2k - 1$, thus by case(b)(2) and Lemma 3.7 we have $\text{depth}(T) = 2$.

- (iii) If $2 \leq n - 2k - 1 + i < k + 1$, then
 $T = S_{n-2k-1+i}/I(P_{n-2k-1+i}^{n-2k-2+i})[x_{n-k+i}]$, by Remark 3.5 and Lemma 3.7 we have $\text{depth}(T) = 2$.

Thus by Lemma 3.1 we have $\text{depth}(S/I(P_n^k)) = 2$.

- (4) For $k \geq 2$, $n \geq 3k + 2$ and $0 \leq i \leq k - 1$, consider the family of short exact sequences

$$0 \longrightarrow S/((I(P_n^k), A_{n-k+(i-1)}) : x_{n-k+i}) \xrightarrow{\cdot x_{n-k+i}} S/((I(P_n^k), A_{n-k+(i-1)}) : x_{n-k+i}) \longrightarrow S/((I(P_n^k), A_{n-k+i}) : x_{n-k+i}) \longrightarrow 0$$

By Lemma 3.2, $S/((I(P_n^k), A_{n-1}) : x_n) \cong S_{n-k-1}/I(P_{n-k-1}^k)[x_n]$. Thus by induction on n and Lemma 3.7 we have $\text{depth}(S/((I(P_n^k), A_{n-1}) : x_n)) = \lceil \frac{n-k-1}{2k+1} \rceil + 1$. By Lemma 3.4 we have

$$S/((I(P_n^k), A_{n-k+(i-1)}) : x_{n-k+i}) \cong S_{n-2k-1+i}/I(P_{n-2k-1+i}^k)[x_{n-k+i}].$$

Thus by induction on n and Lemma 3.7 we have

$$\text{depth}(S/((I(P_n^k), A_{n-k+(i-1)}) : x_{n-k+i})) = \lceil \frac{n-2k-1+i}{2k+1} \rceil + 1.$$

Here we can see that

$$\begin{aligned} \text{depth}(S/((I(P_n^k), A_{n-1}) : x_n)) &= \lceil \frac{n-k-1}{2k+1} \rceil + 1 \geq \\ &\lceil \frac{n-k-2}{2k+1} \rceil + 1 = \text{depth}(S/((I(P_n^k), A_{n-2}) : x_{n-1})), \end{aligned}$$

and for all $1 \leq i \leq k - 1$,

$$\begin{aligned} \text{depth}(S/((I(P_n^k), A_{n-k+(i-1)}) : x_{n-k+i})) &= \lceil \frac{n-2k-1+i}{2k+1} \rceil + 1 \geq \\ \lceil \frac{n-2k-2+i}{2k+1} \rceil + 1 &= \text{depth}(S/((I(P_n^k), A_{n-k+(i-2)}) : x_{n-k+(i-1)})). \end{aligned}$$

Thus by Lemma 3.1 we have $\text{depth}(S/I(P_n^k)) = \lceil \frac{n-2k-1}{2k+1} \rceil + 1 = \lceil \frac{n}{2k+1} \rceil$.

□

Let $d \in [n]$ and $I_{n,d} := (u \in S \text{ square free monomial} : \deg(u) = d)$. Then $I_{n,d}$ is called squarefree Veronese ideal of degree d in the variables $x_1, x_2, \dots, x_n$. Cimpoeas proved the following theorems:

Theorem 3.9 ([5, Theorem 1.1]). (1) $\text{sdepth}(S/I_{n,d}) = d - 1$.

(2) $d \leq \text{sdepth}(I_{n,d}) \leq \frac{n-d}{d+1} + d$.

Theorem 3.10 ([7, Theorem 1.4]). Let M be a $\mathbb{Z}^n$ -graded S -module. If $\text{sdepth}(M) = 0$, then $\text{depth}(M) = 0$. Conversely, if $\text{depth}(M) = 0$ and $\dim_K(M_a) = 1$ for any $a \in \mathbb{Z}^n$, then $\text{sdepth}(M) = 0$.

Lemma 3.11 ([25, Lemma 4]). Let $n \geq 2$, then $\text{sdepth}(S/I(P_n^1)) = \lceil \frac{n}{3} \rceil$.

Example 3.12. Let $n \geq 2$, and $n \leq 2k + 1$, then $\text{sdepth}(S/I(P_n^k)) = 1$.

Proof. If $n \leq k + 1$, then by Theorem 3.9 $\text{sdepth}(S/I(P_n^k)) = 1$. Now if $k + 2 \leq n \leq 2k + 1$, then $\text{depth}(S/I(P_n^k)) \geq 1$ as $\mathfrak{m} \notin \text{Ass}(S/I(P_n^k))$, thus by Theorem 3.10 $\text{sdepth}(S/I(P_n^k)) \geq 1$. Since $x_{k+1} \notin I(P_n^k)$ and $x_i x_{k+1} \in \mathcal{G}(I(P_n^k))$ for all $i \in \{1, \dots, k, k+2, \dots, n\}$, therefore $(I(P_n^k) : x_{k+1}) = (x_1, \dots, x_k, x_{k+2}, \dots, x_n)$. Thus by [4, Proposition 2.7] $\text{sdepth}(S/I(P_n^k)) \leq \text{sdepth}(S/(I(P_n^k) : x_{k+1})) = \text{sdepth}(S/(x_1, \dots, x_k, x_{k+2}, \dots, x_n)) = 1$. $\square$

Proposition 3.13. Let $k \geq 2$ and $n \geq 2k + 2$. Then

$$\text{sdepth}(S/I(P_n^k)) \geq \lceil \frac{n}{2k+1} \rceil.$$

Proof. (1) If $2k + 2 \leq n \leq 3k + 1$, then by applying Lemma 2.4 on the exact sequences in case(b)(3) of Theorem 3.8 we get $\text{sdepth}(S/I(P_n^k)) \geq 2 = \lceil \frac{n}{2k+1} \rceil$.

(2) If $n \geq 3k + 2$, then the proof is similar to Theorem 3.8. We apply Lemma 2.4 on the exact sequences in case(b)(4) of Theorem 3.8 and obtain

$$\begin{aligned} \text{sdepth}(S/I(P_n^k)) &\geq \min \{ \text{sdepth}(S/(I(P_n^k), A_{n-1})), \\ &\min_{i=0}^{k-1} \{ \text{sdepth}(S/((I(P_n^k), A_{n-k+(i-1)}) : x_{n-k+i})) \} \} \geq \lceil \frac{n}{2k+1} \rceil. \end{aligned}$$

$\square$

Theorem 3.14. Let $n \geq 2$, then $\text{sdepth}(S/I(P_n^k)) = \lceil \frac{n}{2k+1} \rceil$.

Proof. If $k = 1$, then the result follows by Lemma 3.11. Let $k \geq 2$. If $n \leq 2k + 1$, then by Example 3.12 we have the required result. If $n \geq 2k + 2$, then by Proposition 3.13 we have

$$\text{sdepth}(S/I(P_n^k)) \geq \lceil \frac{n}{2k+1} \rceil.$$

We need to prove that $\text{sdepth}(S/I(P_n^k)) \leq \lceil \frac{n}{2k+1} \rceil$, for this we consider the following three cases:

- (1) If $n = (2k + 1)l$, where $l \geq 1$. We see that

$$v = x_{k+1}x_{3k+2}x_{5k+3} \cdots x_{(2k+1)l-k} \in S \setminus I(P_n^k),$$

but $x_{t_1}v \in I(P_n^k)$ for all $t_1 \in [n] \setminus \text{supp}(v)$, thus by Lemma 2.5,

$$\text{sdepth}(S/I(P_n^k)) \leq l = \lceil \frac{n}{2k+1} \rceil.$$

- (2) If $n = (2k + 1)l + r$, where $r \in \{1, 2, 3, \dots, k + 1\}$ and $l \geq 1$, then we have

$$v = x_{k+1}x_{3k+2}x_{5k+3} \cdots x_{(2k+1)l-k}x_{(2k+1)l+r} \in S \setminus I(P_n^k),$$

and $x_{t_2}v \in I(P_n^k)$ for all $t_2 \in [n] \setminus \text{supp}(v)$, so by Lemma 2.5,

$$\text{sdepth}(S/I(P_n^k)) \leq l + 1 = \lceil \frac{n}{2k+1} \rceil.$$

- (3) If $n = (2k + 1)l + s$, where $s \in \{k + 2, k + 3, \dots, 2k\}$ and $l \geq 1$, since

$$v = x_{k+1}x_{3k+2}x_{5k+3} \cdots x_{(2k+1)l+k+1} \in S \setminus I(P_n^k),$$

but $x_{t_3}v \in I(P_n^k)$ for all $t_3 \in [n] \setminus \text{supp}(v)$, by Lemma 2.5, we get

$$\text{sdepth}(S/I(P_n^k)) \leq l + 1 = \lceil \frac{n}{2k+1} \rceil.$$

□

4 Depth and Stanley depth of cyclic modules associated to the edge ideals of the powers of a cycle

In this section, we compute bounds for depth and Stanley depth of cyclic modules associated to the edge ideals of powers of a cycle. In order to complete the main task of this section we prove the following three lemmas.

Lemma 4.1. *Let $k \geq 2$ and $n \geq 3k+2$, then $S/(I(C_n^k), A_{n-1}) \cong S_{n-k}/I(P_{n-k}^k)$.*

Proof. Since $\mathcal{G}(I(C_n^k)) = \mathcal{G}(I(P_n^k)) \cup \bigcup_{l=1}^{k-1} \{x_l x_{l+n-k}, x_l x_{l+n-k+1}, \dots, x_l x_{n-1}\} \cup \{x_1 x_n, x_2 x_n, \dots, x_k x_n\}$, we have

$$\begin{aligned} I(C_n^k) + A_{n-1} &= \\ I(P_n^k) + \sum_{l=1}^{k-1} (x_l x_{l+n-k}, x_l x_{l+n-k+1}, \dots, x_l x_{n-1}) &+ (x_1 x_n, x_2 x_n, \dots, x_k x_n) + A_{n-1}. \end{aligned}$$

Thus by the proof of Lemma 3.2, we obtain $I(P_n^k) + A_{n-1} = I(P_{n-k-1}^k) + A_{n-1}$.
 As

$$\sum_{l=1}^{k-1} (x_l x_{l+n-k}, x_l x_{l+n-k+1}, \dots, x_l x_{n-1}) + A_{n-1} = A_{n-1}.$$

$$\begin{aligned} \text{Therefore } S/(I(C_n^k), A_{n-1}) &= S/(I(P_{n-k-1}^k), A_{n-1}, (x_1 x_n, x_2 x_n, \dots, x_k x_n)) \\ &\cong K[x_1, x_2, \dots, x_{n-k-1}, x_n] / (I(P_{n-k-1}^k), (x_1 x_n, x_2 x_n, \dots, x_k x_n)). \end{aligned}$$

After renumbering the variables, we have

$$K[x_1, \dots, x_{n-k-1}, x_n] / (I(P_{n-k-1}^k), (x_1 x_n, x_2 x_n, \dots, x_k x_n)) \cong S_{n-k} / I(P_{n-k}^k).$$

□

Lemma 4.2. *Let $k \geq 2$ and $n \geq 3k + 2$ and $0 \leq i \leq k - 1$, then*

$$S/(I(C_n^k) : x_{n-k+i}) \cong S_{n-2k-1} / I(P_{n-2k-1}^k)[x_{n-k+i}].$$

Proof. Let w be a monomial generator of $(I(C_n^k) : x_{n-k+i})$. Then $w = \frac{v}{\gcd(v, x_{n-k+i})}$, where $v \in \mathcal{G}(I(C_n^k))$. If $\text{supp}(v) \cap \mathcal{G}(D_{n-k+i}) \neq \emptyset$, then we have $w \in \mathcal{G}(D_{n-k+i})$ and if $\text{supp}(v) \cap \mathcal{G}(D_{n-k+i}) = \emptyset$ then $w \in E := \mathcal{G}(I(C_n^k)) \cap K[x_{i+1}, x_{i+2}, \dots, x_{n-2k-1+i}]$. So we obtain $(I(C_n^k) : x_{n-k+i}) \subset E + D_{n-k+i}$. The other inclusion being trivial we get $(I(C_n^k) : x_{n-k+i}) = E + D_{n-k+i}$, which further implies that $S/(I(C_n^k) : x_{n-k+i}) = S/(E + D_{n-k+i})$. After renumbering the variables, we have

$$S/(I(C_n^k) : x_{n-k+i}) = S/(E, D_{n-k+i}) \cong S_{n-2k-1} / I(P_{n-2k-1}^k)[x_{n-k+i}].$$

□

Lemma 4.3. *Let $k \geq 2$, $n \geq 3k + 2$ and $0 \leq i \leq k - 1$. Then*

$$S/((I(C_n^k), A_{n-k+(i-1)}) : x_{n-k+i}) \cong S_{n-2k-1} / I(P_{n-2k-1}^k)[x_{n-k+i}].$$

Proof. As $((I(C_n^k), A_{n-k+(i-1)}) : x_{n-k+i}) = ((I(C_n^k) : x_{n-k+i}), A_{n-k+(i-1)})$. By using the same arguments as in the proof of Lemma 4.2 we have

$$((I(C_n^k) : x_{n-k+i}), A_{n-k+(i-1)}) = (E, D_{n-k+i}, A_{n-k+(i-1)}) = (E, D_{n-k+i})$$

as $A_{n-k+(i-1)} \subset D_{n-k+i}$. Thus the required result follows by Lemma 4.2. □

Corollary 4.4 ([10, Corollary 10.3.7]). *Let $2 \leq d < n$. Then*

$$\text{depth}(S/I_{n,d}^t) = \max\{0, n - t(n - d) - 1\}.$$

Theorem 4.5. *Let $n \geq 3$, then*

$$\begin{aligned} \text{depth}(S/I(C_n^k)) &= 1, \quad \text{if } n \leq 2k+1; \\ \text{depth}(S/I(C_n^k)) &\geq \lceil \frac{n-k}{2k+1} \rceil, \quad \text{if } n \geq 2k+2. \end{aligned}$$

Proof. (a) If $n \leq 2k+1$, then $I(C_n^k)$ is a squarefree Veronese ideal of degree 2. Thus by Corollary 4.4, $\text{depth}(S/I(C_n^k)) = 1$.

(b) For $n \geq 2k+2$, we consider the following cases:

- (1) If $k = 1$, then by [6, Proposition 1.3] $\text{depth}(S/I(C_n^1)) = \lceil \frac{n-1}{3} \rceil$.
- (2) If $k \geq 2$ and $2k+2 \leq n \leq 3k+1$, then we have $\text{depth}(S/I(C_n^k)) \geq 1 = \lceil \frac{n-k}{2k+1} \rceil$ as $\mathfrak{m} \notin \text{Ass}(S/I(C_n^k))$.
- (3) For $k \geq 2$, $n \geq 3k+2$ and $0 \leq i \leq k-1$, consider the family of short exact sequences

$$\begin{aligned} 0 \longrightarrow S/((I(C_n^k), A_{n-k+(i-1)}) : x_{n-k+i}) &\xrightarrow{\cdot x_{n-k+i}} \\ S/(I(C_n^k), A_{n-k+(i-1)}) &\longrightarrow S/(I(C_n^k), A_{n-k+i}) \longrightarrow 0 \end{aligned}$$

By Lemma 4.1 we have $S/(I(C_n^k), A_{n-1}) \cong S_{n-k}/I(P_{n-k}^k)$. Now by Lemma 4.3, we get

$$S/((I(C_n^k), A_{n-k+(i-1)}) : x_{n-k+i}) \cong S_{n-2k-1}/I(P_{n-2k-1}^k)[x_{n-k+i}].$$

By Theorem 3.8 and Lemma 3.7, we obtain

$$\text{depth}(S/((I(C_n^k), A_{n-k+(i-1)}) : x_{n-k+i})) = \lceil \frac{n-2k-1}{2k+1} \rceil + 1 = \lceil \frac{n}{2k+1} \rceil.$$

Again by Theorem 3.8, we have $\text{depth}(S/(I(C_n^k), A_{n-1})) = \lceil \frac{n-k}{2k+1} \rceil$. Thus by applying Lemma 2.3(1) on the family of short exact sequences we get $\text{depth}(S/I(C_n^k)) \geq \lceil \frac{n-k}{2k+1} \rceil$. □

Corollary 4.6. *Let $n \geq 3$, If $n \geq 2k+2$, then*

$$\begin{aligned} \text{depth}(S/I(C_n^k)) &= \lceil \frac{n}{2k+1} \rceil, \quad \text{if } n \equiv 0, k+1, \dots, 2k \pmod{2k+1}; \\ \lceil \frac{n}{2k+1} \rceil - 1 &\leq \text{depth}(S/I(C_n^k)) \leq \lceil \frac{n}{2k+1} \rceil, \quad \text{if } n \equiv 1, \dots, k \pmod{2k+1}. \end{aligned}$$

Proof. By Theorem 4.5, it is enough to prove that $\text{depth}(S/I(C_n^k)) \leq \lceil \frac{n}{2k+1} \rceil$, for $k \geq 2$ and $n \geq 2k+2$. Since $x_{n-k} \notin I(C_n^k)$, thus by [23, Corollary 1.3] we have $\text{depth}(S/I(C_n^k)) \leq \text{depth}(S/(I(C_n^k) : x_{n-k}))$. Now we consider two cases:

- (1) Let $2k + 2 \leq n \leq 3k + 1$, then $S/(I(C_n^k) : x_{n-k}) = S/(I(P_n^k) : x_{n-k})$ so by the proof of Theorem 3.8 we have $\text{depth}(S/(I(P_n^k) : x_{n-k})) = 2 = \lceil \frac{n}{2k+1} \rceil$. Therefore

$$\text{depth}(S/I(C_n^k)) \leq \text{depth}(S/(I(C_n^k) : x_{n-k})) = 2 = \lceil \frac{n}{2k+1} \rceil.$$

- (2) Let $n \geq 3k + 2$, then by Lemma 4.2,

$$S/(I(C_n^k) : x_{n-k}) \cong S_{n-2k-1}/I(P_{n-2k-1}^k)[x_{n-k}].$$

By Lemma 3.7 and Theorem 3.8, $\text{depth}(S_{n-2k-1}/I(P_{n-2k-1}^k)[x_{n-k}]) = \lceil \frac{n}{2k+1} \rceil$. Thus $\text{depth}(S/I(C_n^k)) \leq \text{depth}(S/(I(C_n^k) : x_{n-k})) = \lceil \frac{n}{2k+1} \rceil$. $\square$

Theorem 4.7. *Let $n \geq 3$, then*

$$\begin{aligned} \text{sdepth}(S/I(C_n^k)) &= 1, & \text{if } n \leq 2k + 1; \\ \text{sdepth}(S/I(C_n^k)) &\geq \lceil \frac{n-k}{2k+1} \rceil, & \text{if } n \geq 2k + 2. \end{aligned}$$

Proof. (a) If $n \leq 2k + 1$, then $\text{sdepth}(S/I(C_n^k)) = 1$ by Theorem 3.9.

- (b) For $n \geq 2k + 2$, consider the following cases:

- (1) If $k = 1$, then by [6, Proposition 1.8] $\text{sdepth}(S/I(C_n^1)) \geq \lceil \frac{n-1}{3} \rceil$.
- (2) If $k \geq 2$ and $2k + 2 \leq n \leq 3k + 1$, then $\text{depth}(S/I(C_n^k)) \geq 1$ as $\mathfrak{m} \notin \text{Ass}(S/I(C_n^k))$, thus by Theorem 3.10, $\text{sdepth}(S/I(C_n^k)) \geq 1 = \lceil \frac{n-k}{2k+1} \rceil$.
- (3) For $k \geq 2$, $n \geq 3k + 2$ and $0 \leq i \leq k - 1$, consider the family of short exact sequences

$$\begin{aligned} 0 \longrightarrow S/((I(C_n^k), A_{n-k+(i-1)}) : x_{n-k+i}) &\xrightarrow{\cdot x_{n-k+i}} \\ S/(I(C_n^k), A_{n-k+(i-1)}) &\longrightarrow S/(I(C_n^k), A_{n-k+i}) \longrightarrow 0. \end{aligned}$$

By Lemma 4.1 we have $S/(I(C_n^k), A_{n-1}) \cong S_{n-k}/I(P_{n-k}^k)$. Now by Lemma 4.3, we get

$$S/((I(C_n^k), A_{n-k+(i-1)}) : x_{n-k+i}) \cong S_{n-2k-1}/I(P_{n-2k-1}^k)[x_{n-k+i}].$$

By Theorem 3.14 and Lemma 3.7, we obtain

$$\begin{aligned} \text{sdepth}(S/((I(C_n^k), A_{n-k+(i-1)}) : x_{n-k+i})) &= \\ \lceil \frac{n-2k-1}{2k+1} \rceil + 1 &= \lceil \frac{n}{2k+1} \rceil. \end{aligned}$$

Again by Theorem 3.14, we have $\text{sdepth}(S/(I(C_n^k), A_{n-1})) = \lceil \frac{n-k}{2k+1} \rceil$.
 By applying Lemma 2.4 on the above family of short exact sequences we get $\text{sdepth}(S/I(C_n^k)) \geq \lceil \frac{n-k}{2k+1} \rceil$.

□

Corollary 4.8. *Let $n \geq 3$, if $n \geq 2k + 2$, then*

$$\begin{aligned} \text{sdepth}(S/I(C_n^k)) &= \lceil \frac{n}{2k+1} \rceil, & \text{if } n \equiv 0, k+1, \dots, 2k \pmod{2k+1}; \\ \lceil \frac{n}{2k+1} \rceil - 1 &\leq \text{sdepth}(S/I(C_n^k)) \leq \lceil \frac{n}{2k+1} \rceil, & \text{if } n \equiv 1, \dots, k \pmod{2k+1}. \end{aligned}$$

Proof. When $k = 1$, then by [6, Theorem 1.9], $\text{sdepth}(S/I(C_n^k)) \leq \lceil \frac{n}{3} \rceil$. By Theorem 4.7 it is enough to prove that $\text{sdepth}(S/I(C_n^k)) \leq \lceil \frac{n}{2k+1} \rceil$ for $k \geq 2$ and $n \geq 2k + 2$. Since $x_{n-k} \notin I(C_n^k)$, thus by [4, Proposition 2.7] we have

$$\text{sdepth}(S/I(C_n^k)) \leq \text{sdepth}(S/(I(C_n^k) : x_{n-k})).$$

Now we consider two cases:

- (1) Let $2k + 2 \leq n \leq 3k + 1$, then $S/(I(C_n^k) : x_{n-k}) = S/(I(P_n^k) : x_{n-k})$ so by the proof of Theorem 3.14 we have $\text{sdepth}(S/(I(P_n^k) : x_{n-k})) = 2 = \lceil \frac{n}{2k+1} \rceil$. Therefore

$$\text{sdepth}(S/I(C_n^k)) \leq \text{sdepth}(S/(I(C_n^k) : x_{n-k})) = 2 = \lceil \frac{n}{2k+1} \rceil.$$

- (2) Let $n \geq 3k + 2$, then by Lemma 4.2

$$S/(I(C_n^k) : x_{n-k}) \cong S_{n-2k-1}/I(P_{n-2k-1}^k)[x_{n-k}].$$

By Lemma 3.7 and Theorem 3.14, $\text{sdepth}(S_{n-2k-1}/I(P_{n-2k-1}^k)[x_{n-k}]) = \lceil \frac{n}{2k+1} \rceil$. Thus $\text{sdepth}(S/I(C_n^k)) \leq \text{sdepth}(S/(I(C_n^k) : x_{n-k})) = \lceil \frac{n}{2k+1} \rceil$.

□

5 Lower bounds for Stanley depth of edge ideals of k^{th} powers of paths and cycles and a conjecture of Herzog

In this section we compute some lower bounds for Stanley depth of $I(P_n^k)$ and $I(C_n^k)$. These bounds are good enough to prove that Conjecture 1.1 is true for $I(P_n^k)$ and $I(C_n^k)$. Let $0 \leq i \leq k - 1$, define

$$R_{n-k+i} := K[\{x_1, x_2, \dots, x_n\} \setminus \{x_{n-k}, x_{n-k+1}, \dots, x_{n-k+i}\}]$$

and

$$B'_{n-k+i} := (x_j : x_j \in N_{P_n^k}(x_{n-k+i}) \setminus \{x_{n-k}, x_{n-k+1}, \dots, x_{n-k+(i-1)}\}).$$

Thus R_{n-k+i} is a subring of S and B'_{n-k+i} is a monomial prime ideal of S . Let $I \subset Z = K[x_{i_1}, x_{i_1}, \dots, x_{i_r}]$ be a monomial ideal and $Z' := Z[x_{i_r+1}]$. Then we write $IZ' = I[x_{i_r+1}]$. Now we recall a useful remark of Cimpoeas.

Remark 5.1. [4, Remark 1.7] Let I be a monomial ideal of S , and $I' = (I, x_{n+1}, x_{n+2}, \dots, x_{n+m})$ be a monomial ideal of $S' = S[x_{n+1}, x_{n+2}, \dots, x_{n+m}]$. Then

$$\text{sdepth}_{S'}(I') \geq \min\{\text{sdepth}_S(I) + m, \text{sdepth}_S(S/I) + \lceil \frac{m}{2} \rceil\}.$$

Theorem 5.2. Let $n \geq 2$, then $\text{sdepth}(I(P_n^k)) \geq \lceil \frac{n}{2k+1} \rceil + 1$.

Proof. (a) If $n \leq 2k+1$, then as the minimal generators of $I(P_n^k)$ have degree 2, by [15, Lemma 2.1] we have $\text{sdepth}(I(P_n^k)) \geq 2 = \lceil \frac{n}{2k+1} \rceil + 1$.

(b) For $n \geq 2k+2$, if $k=1$, then by [19, Theorem 2.3], $\text{sdepth}(I(P_n^1)) \geq n - \lfloor \frac{n-1}{2} \rfloor = \lceil \frac{n-1}{2} \rceil + 1 \geq \lceil \frac{n}{3} \rceil + 1$. Now for $k \geq 2$, we prove this result by induction on n . We consider the following decomposition of $I(P_n^k)$ as a vector space:

$$I(P_n^k) = I(P_n^k) \cap R_{n-k} \oplus x_{n-k}(I(P_n^k) : x_{n-k})S.$$

Similarly, we can decompose $I(P_n^k) \cap R_{n-k}$ as follows:

$$I(P_n^k) \cap R_{n-k} = I(P_n^k) \cap R_{n-k+1} \oplus x_{n-k+1}(I(P_n^k) \cap R_{n-k} : x_{n-k+1})R_{n-k}.$$

Continuing in the same way for $1 \leq i \leq k-1$ we have

$$\begin{aligned} I(P_n^k) \cap R_{n-k+i} &= I(P_n^k) \cap R_{n-k+(i+1)} \oplus \\ &\quad x_{n-k+(i+1)}(I(P_n^k) \cap R_{n-k+i} : x_{n-k+(i+1)})R_{n-k+i}. \end{aligned}$$

Finally we get the following decomposition of $I(P_n^k)$:

$$\begin{aligned} I(P_n^k) &= I(P_n^k) \cap R_{n-1} \oplus \\ &\quad \oplus_{i=1}^{k-1} x_{n-k+i}(I(P_n^k) \cap R_{n-k+(i-1)} : x_{n-k+i})R_{n-k+i} \oplus x_{n-k}(I(P_n^k) : x_{n-k})S. \end{aligned}$$

Therefore

$$\begin{aligned} \text{sdepth}(I(P_n^k)) &\geq \min \{ \text{sdepth}(I(P_n^k) \cap R_{n-1}), \text{sdepth}((I(P_n^k) : x_{n-k})S), \\ &\quad \min_{i=1}^{k-1} \{ \text{sdepth}((I(P_n^k) \cap R_{n-k+(i-1)} : x_{n-k+i})R_{n-k+i}) \} \}. \end{aligned}$$

As $I(P_n^k) \cap R_{n-1} = \mathcal{G}(I(P_{n-k-1}^k))[x_n]$, thus by induction on n and Lemma 3.7 we have $\text{sdepth}(I(P_n^k) \cap R_{n-1}) \geq \lceil \frac{n-k-1}{2k+1} \rceil + 1 + 1 \geq \lceil \frac{n}{2k+1} \rceil + 1$. Now we need to show that $\text{sdepth}((I(P_n^k) : x_{n-k})S) \geq \lceil \frac{n}{2k+1} \rceil + 1$ and

$$\text{sdepth}((I(P_n^k) \cap R_{n-k+(i-1)} : x_{n-k+i})R_{n-k+i}) \geq \lceil \frac{n}{2k+1} \rceil + 1.$$

For this we consider the following cases:

- (1) Let $2k+2 \leq n \leq 3k+1$. If $n = 2k+2$, then $(I(P_n^k) : x_{n-k})S = (x_2, \dots, x_{n-k-1}, x_{n-k+1}, \dots, x_n)S$, thus by [2, Theorem 2.2] and Lemma 3.7 we have

$$\text{sdepth}((I(P_n^k) : x_{n-k})S) = \lceil \frac{n-2}{2} \rceil + 2 \geq \lceil \frac{n}{2k+1} \rceil + 1.$$

If $2k+3 \leq n \leq 3k+1$, then by Remark 3.6, we get

$$(I(P_n^k) : x_{n-k})S = (\mathcal{G}(I(P_{n-2k-1}^{f(n-k)})), B_{n-k})[x_{n-k}].$$

Since $\text{sdepth}(I(P_{n-2k-1}^{f(n-k)})) + |\mathcal{G}(B_{n-k})| \geq 2$, by Remark 3.5 we have

$$\text{sdepth}(S_{n-2k-1}/I(P_{n-2k-1}^{f(n-k)})) + \lceil \frac{|\mathcal{G}(B_{n-k})|}{2} \rceil \geq 2,$$

then by Remark 5.1, $\text{sdepth}(\mathcal{G}(I(P_{n-2k-1}^{f(n-k)})), B_{n-k}) \geq 2$, and by Lemma 3.7 we have $\text{sdepth}((I(P_n^k) : x_{n-k})S) \geq 3 = \lceil \frac{n}{2k+1} \rceil + 1$. Now since

$$\begin{aligned} (I(P_n^k) \cap R_{n-k+(i-1)} : x_{n-k+i})R_{n-k+i} = \\ (\mathcal{G}(I(P_{n-2k-1+i}^{f(n-k+i)})), B'_{n-k+i})[x_{n-k+i}]. \end{aligned}$$

So by the same arguments we have

$$\text{sdepth}((I(P_n^k) \cap R_{n-k+(i-1)} : x_{n-k+i})R_{n-k+i}) \geq 3 = \lceil \frac{n}{2k+1} \rceil + 1.$$

- (2) If $n \geq 3k+2$, then by the proof of Lemma 3.3 $(I(P_n^k) : x_{n-k})S = (\mathcal{G}(I(P_{n-2k-1}^k)), B_{n-k})[x_{n-k}]$ and

$$\begin{aligned} (I(P_n^k) \cap R_{n-k+(i-1)} : x_{n-k+i})R_{n-k+i} = \\ (\mathcal{G}(I(P_{n-2k-1+i}^k)), B'_{n-k+i})[x_{n-k+i}]. \end{aligned}$$

By Remark 5.1 we have

$$\begin{aligned} \text{sdepth}(\mathcal{G}(I(P_{n-2k-1}^k)), B_{n-k}) &\geq \min \left\{ \text{sdepth}(\mathcal{G}(I(P_{n-2k-1}^k))) + \right. \\ &\quad \left. |\mathcal{G}(B_{n-k})|, \text{sdepth}(S_{n-2k-1}/I(P_{n-2k-1}^k)) + \lceil \frac{|\mathcal{G}(B_{n-k})|}{2} \rceil \right\}. \end{aligned}$$

By induction on n we have $\text{sdepth}(\mathcal{G}(I(P_{n-2k-1}^k))) \geq \lceil \frac{n-2k-1}{2k+1} \rceil + 1 = \lceil \frac{n}{2k+1} \rceil$, and by Theorem 3.14, $\text{sdepth}(S_{n-2k-1}/I(P_{n-2k-1}^k)) = \lceil \frac{n}{2k+1} \rceil - 1$. Therefore $\text{sdepth}(\mathcal{G}(I(P_{n-2k-1}^k)), B_{n-k}) \geq \lceil \frac{n}{2k+1} \rceil + 1$. Thus by Lemma 3.7 we have $\text{sdepth}((I(P_n^k) : x_{n-k})S) > \lceil \frac{n}{2k+1} \rceil + 1$. Now using Remark 5.1 again, we get

$$\begin{aligned} \text{sdepth}(\mathcal{G}(I(P_{n-2k-1+i}^k)), B'_{n-k+i}) &\geq \\ \min \left\{ \text{sdepth}(\mathcal{G}(I(P_{n-2k-1+i}^k))) + |\mathcal{G}(B'_{n-k+i})|, \right. \\ &\quad \left. \text{sdepth}(S_{n-2k-1+i}/I(P_{n-2k-1+i}^k)) + \lceil \frac{|\mathcal{G}(B'_{n-k+i})|}{2} \rceil \right\}. \end{aligned}$$

By induction on n we have $\text{sdepth}(\mathcal{G}(I(P_{n-2k-1+i}^k))) \geq \lceil \frac{n-2k-1+i}{2k+1} \rceil + 1$, and by Theorem 3.14 we have $\text{sdepth}(S_{n-2k-1+i}/I(P_{n-2k-1+i}^k)) = \lceil \frac{n-2k-1+i}{2k+1} \rceil$. Therefore

$$\text{sdepth}(\mathcal{G}(I(P_{n-2k-1+i}^k)), B'_{n-k+i}) \geq \lceil \frac{n-2k-1+i}{2k+1} \rceil + 1.$$

Thus by Lemma 3.7

$$\text{sdepth}((I(P_n^k) \cap R_{n-k+(i-1)} : x_{n-k+i})R_{n-k+i}) \geq \lceil \frac{n}{2k+1} \rceil + 1.$$

This completes the proof. $\square$

Proposition 5.3. *Let $n \geq 2k + 1$, then $\text{sdepth}(I(C_n^k)/I(P_n^k)) \geq \lceil \frac{n+k+1}{2k+1} \rceil$.*

Proof. When $k = 1$, then by [6, Proposition 1.10] we have the required result. Now assume that $k \geq 2$ and consider the following cases:

(1). If $2k + 1 \leq n \leq 3k + 1$, then as $I(C_n^k)$ is a monomial ideal generated by degree 2 so by [11, Theorem 2.1] $\text{sdepth}(I(C_n^k)/I(P_n^k)) \geq 2 = \lceil \frac{n+k+1}{2k+1} \rceil$.

(2). If $3k + 2 \leq n \leq 4k + 1$, then we use [11] to show that there exist Stanley decompositions of desired Stanley depth. Let $s \in \{1, 2, \dots, k\}$, $j_s \in \{1, 2, \dots, k + 1 - s\}$ and

$$L := \bigoplus_{s=1}^k \left(\bigoplus_{j_s=1}^{k+1-s} x_{j_s} x_{n+1-s} K[x_{j_s}, x_{j_s+k+1}, x_{n+1-s}] \right).$$

It is easy to see that $L \subset I(C_n^k) \setminus I(P_n^k)$. Now let $u_i \in I(C_n^k) \setminus I(P_n^k)$ be a squarefree monomial such that $u_i \notin L$ then clearly $\deg(u_i) \geq 3$. Since

$$I(C_n^k)/I(P_n^k) \cong L \oplus_{u_i} u_i K[\text{supp}(u_i)]$$

Thus $\text{sdepth}(I(C_n^k)/I(P_n^k)) \geq 3 = \lceil \frac{n+k+1}{2k+1} \rceil$ as required.

(3). If $n \geq 4k + 2$, then we have the following K -vector space isomorphism:

$$\begin{aligned} I(C_n^k)/I(P_n^k) &\cong \\ &\bigoplus_{j_1=1}^k x_{j_1} x_n \frac{K[x_{j_1+k+1}, x_{j_1+k+2}, \dots, x_{n-k-1}]}{(x_{j_1+k+1}x_{j_1+k+2}, x_{j_1+k+1}x_{j_1+k+3}, \dots, x_{n-k-2}x_{n-k-1})} [x_{j_1}, x_n] \oplus \\ &\bigoplus_{j_2=1}^{k-1} x_{j_2} x_{n-1} \frac{K[x_{j_2+k+1}, x_{j_2+k+2}, \dots, x_{n-k-2}]}{(x_{j_2+k+1}x_{j_2+k+2}, x_{j_2+k+1}x_{j_1+k+3}, \dots, x_{n-k-3}x_{n-k-2})} [x_{j_2}, x_{n-1}] \oplus \\ &\quad \cdot \\ &\quad \cdot \\ &\quad \cdot \\ &\bigoplus_{j_{k-1}=1}^2 x_{j_{k-1}} x_{n-k+2} \frac{K[x_{j_{k-1}+k+1}, x_{j_{k-1}+k+2}, \dots, x_{n-2k+1}]}{(x_{j_{k-1}+k+1}x_{j_{k-1}+k+2}, \dots, x_{n-2k}x_{n-2k+1})} [x_{j_{k-1}}, x_{n-k+2}] \oplus \\ &\quad x_1 x_{n-(k-1)} \frac{K[x_{k+2}, x_{k+3}, \dots, x_{n-2k}]}{(x_{k+2}x_{k+3}, x_{k+2}x_{k+4}, \dots, x_{n-2k-1}x_{n-2k})} [x_1, x_{n-(k-1)}]. \end{aligned}$$

Thus

$$I(C_n^k)/I(P_n^k) \cong \bigoplus_{s=1}^k \left(\bigoplus_{j_s=1}^{k+1-s} x_{j_s} x_{n+1-s} (S_{j_s+k+1, n-s-k} / (\mathcal{G}(I(P_n^k)) \cap S_{j_s+k+1, n-s-k}) [x_{j_s}, x_{n+1-s}] \right),$$

where $S_{j_s+k+1, n-s-k} = K[x_{j_s+k+1}, x_{j_s+k+2}, \dots, x_{n-s-k}]$. Indeed, if $u \in I(C_n^k)$ such that $u \notin I(P_n^k)$ then $(x_{j_s} x_{n+1-s})|u$ for only one pair of s and j_s . If $(x_{j_s} x_{n+1-s})|u$ then $u = x_{j_s}^\delta x_{n+1-s}^\delta v$ and $v \in S_{j_s+k+1, n-s-k}$. Since $v \notin I(P_n^k)$, it follows that $v \notin \mathcal{G}(I(P_n^k)) \cap S_{j_s+k+1, n-s-k}$. Clearly

$$S_{j_s+k+1, n-s-k} / \mathcal{G}(I(P_n^k)) \cap S_{j_s+k+1, n-s-k} \cong S_{n-(j_s+2k+s)} / I(P_{n-(j_s+2k+s)}^k).$$

Thus by Theorem 3.14 and Lemma 3.7, we have

$$\text{sdepth}(I(C_n^k)/I(P_n^k)) \geq \min_{s=1}^k \left\{ \lceil \frac{n - (j_s + s + 2k)}{2k + 1} \rceil + 2 \right\}.$$

It is easy to see that $\max\{j_s + s\} = k + 1$. Therefore

$$\text{sdepth}(I(C_n^k)/I(P_n^k)) \geq \lceil \frac{n - (3k + 1)}{2k + 1} \rceil + 2 = \lceil \frac{n + k + 1}{2k + 1} \rceil.$$

□

Theorem 5.4. *Let $n \geq 3$, then*

$$\begin{aligned} \text{sdepth}(I(C_n^k)) &\geq 2, & \text{if } n \leq 2k + 1; \\ \text{sdepth}(I(C_n^k)) &\geq \lceil \frac{n - k}{2k + 1} \rceil + 1, & \text{if } n \geq 2k + 2. \end{aligned}$$

Proof. (a) If $n \leq 2k + 1$, then as the minimal generators of $I(C_n^k)$ have degree 2, so by [15, Lemma 2.1] $\text{sdepth}(I(C_n^k)) \geq 2$.

(b) If $n \geq 2k + 2$, then consider the short exact sequence

$$0 \longrightarrow I(P_n^k) \longrightarrow I(C_n^k) \longrightarrow I(C_n^k)/I(P_n^k) \longrightarrow 0,$$

by Lemma 2.4 we have

$$\text{sdepth}(I(C_n^k)) \geq \min\{\text{sdepth}(I(P_n^k)), \text{sdepth}(I(C_n^k)/I(P_n^k))\}.$$

By Theorem 5.2, $\text{sdepth}(I(P_n^k)) \geq \lceil \frac{n}{2k+1} \rceil + 1$, and by Proposition 5.3, we obtain $\text{sdepth}(I(C_n^k)/I(P_n^k)) \geq \lceil \frac{n+k+1}{2k+1} \rceil = \lceil \frac{n-k}{2k+1} \rceil + 1$.

□

Corollary 5.5. *Let $n \geq 3$, if $n \leq 2k + 1$, then $\text{sdepth}(I(C_n^k)) \geq 2 = \text{sdepth}(S/I(C_n^k)) + 1$. If $n \geq 2k + 2$, then*

$$\begin{aligned} \text{sdepth}(I(C_n^k)) &\geq \text{sdepth}(S/I(C_n^k)), & \text{if } n \equiv 1, \dots, k \pmod{2k + 1}; \\ \text{sdepth}(I(C_n^k)) &\geq \text{sdepth}(S/I(C_n^k)) + 1, & \text{if } n \equiv 0, k + 1, \dots, 2k \pmod{2k + 1}. \end{aligned}$$

Proof. Proof follows by Corollary 4.8, Theorem 4.7 and Theorem 5.4. □

Acknowledgement

The authors would like to thank the referee for a careful reading of the paper and for valuable comments. This research is partially supported by HEC Pakistan.

References

- [1] J. Apel, On a conjecture of R. P. Stanley; Part II Quotients modulo monomial ideals, *J. Algebraic Combin.*, **17**, (2003) 57-74.
- [2] C. Biro, D. M. Howard, M. T. Keller, W. T. Trotter, S. J. Young, *Interval partitions and Stanley depth*, *J. Combin. Theory, Ser. A*, **117**, (2010) 475-482.
- [3] J. A. Bondy, U. S. R. Murty, *Graph Theory*, Springer, 2008.
- [4] M. Cimpoeas, *Several inequalities regarding Stanley depth*, *Romanian Journal of Math. and Computer Science*, **2**, (2012) 28-40.
- [5] M. Cimpoeas, *Stanley depth of squarefree Veronese ideals*, *An. St. Univ. Ovidius Constanta*, **21**(3), (2013) 67-71.
- [6] M. Cimpoeas, *On the Stanley depth of edge ideals of line and cyclic graphs*, *Romanian Journal of Math. and Computer Science*, **5**(1), (2015) 70-75.
- [7] M. Cimpoeas, *Some remarks on the Stanley depth for multigraded modules*, *Le Matematiche* **LXIII**, (2008) 165-171.
- [8] A. M. Duval, B. Goeckneker, C. J. Klivans, J. L. Martine, *A non-partitionable Cohen-Macaulay simplicial complex*, *Adv. Math.*, **299**, (2016) 381-395.
- [9] J. Herzog, *A survey on Stanley depth*, In *Monomial ideals, computations and applications*, *Lecture Notes in Math.* Springer, Heidelberg, **2083**, (2013) 3-45.
- [10] J. Herzog, T. Hibi, *Monomial Ideals*, Springer-Verlag London Limited, (2011).
- [11] J. Herzog, M. Vladoiu, X. Zheng, *How to compute the Stanley depth of a monomial ideal*, *J. Algebra*, **322**(9), (2009) 3151-3169.
- [12] J. Herzog, D. Popescu, M. Vladoiu, *Stanley depth and size of a monomial ideal*, *Proc. Amer. Math. Soc.*, **140**(2), (2012) 493-504.
- [13] Z. Iqbal, M. Ishaq, M. Aamir, *Depth and Stanley depth of edge ideals of square paths and square cycles*, *Comm. Algebra*, **46**(3), (2018), 1188-1198.
- [14] M. Ishaq, M. I. Qureshi, *Upper and lower bounds for the Stanley depth of certain classes of monomial ideals and their residue class rings*, *Comm. Algebra*, **41**(3), (2013) 1107-1116.

- [15] M. T. Keller, Y. Shen, N. Streib, S. J. Young, *On the Stanley Depth of Squarefree Veronese Ideals*, J. Algebraic Combin., **33**(2), (2011) 313-324.
- [16] M. T. Keller, S. J. Young, *Combinatorial reductions for the Stanley depth of I and S/I* , Electron. J. Combin. **24**(3), (2017) #P3.48.
- [17] M. C. Lin, D. Rautenbach, F. J. Soullignac, J. L. Szwarcfiter, *Powers of cycles, powers of paths, and distance graphs*, Discrete Appl. Math., **159**, (2011) 621-627.
- [18] S. Morey, *Depths of powers of the edge ideal of a tree*, Comm. Algebra, **38**(11), (2010) 4042-4055.
- [19] R. Okazaki, *A lower bound of Stanley depth of monomial ideals*, J. Commut. Algebra, **3**(1), (2011) 83-88.
- [20] D. Popescu, M. I. Qureshi, *Computing the Stanley depth*, J. Algebra, **323**(10), (2010) 2943-2959.
- [21] D. Popescu, *Stanley conjecture on intersection of four monomial prime ideals*, Comm. Algebra, **41**(11), (2013) 4351-4362.
- [22] M. R. Pournaki, S. A. Seyed Fakhari, S. Yassemi, *Stanley depth of powers of the edge ideals of a forest*, Proc. Amer. Math. Soc., **141**(10), (2013) 3327-3336.
- [23] A. Rauf, *Depth and Stanley depth of multigraded modules*, Comm. Algebra, **38**(2), (2010) 773-784.
- [24] R. P. Stanley, *Linear Diophantine equations and local cohomology*, Invent. Math., **68**(2), (1982) 175-193.
- [25] A. Stefan, *Stanley depth of powers of path ideal*, <http://arxiv.org/pdf/1409.6072.pdf>.
- [26] Z. Tang, *Stanley depths of certain StanleyReisner rings*, J. Algebra, **409**(1), (2014) 430-443.

Zahid Iqbal,
School of Natural Sciences,
National University of Sciences and Technology Islamabad,
Sector H-12, Islamabad Pakistan.
Email: 786zahidwarraich@gmail.com

Muhammad Ishaq,
School of Natural Sciences,
National University of Sciences and Technology Islamabad,
Sector H-12, Islamabad Pakistan.
Email: ishaq_maths@yahoo.com