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Transverse vibrations analysis of a beam with degrading hysteretic behavior by using Euler-Bernoulli beam model

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Abstract

The paper is based on the analytical and experimental results from [14], [15] and reveals, by mathematical methods, the degradation of material stiffness due to the decrease of the first natural frequency, when the driving frequency is slightly lower than the first natural frequency of the undegradated structure. By considering the vibration of the uniform slender cantilever beam as an oscillating system with degrading hysteretic behavior the following equation is considered

$$\frac{\partial^2 y(x,t)}{\partial t^2} + 2\zeta(t)\omega(t)\frac{\partial y(x,t)}{\partial t} + l^4\omega^2(t)\frac{\partial^4 y(x,t)}{\partial x^4} = 0$$

subjected to the boundary conditions

$$y(0,t) = y_0 \sin \omega_{input} t,\ \frac{\partial y(0,t)}{\partial x} = 0,\ \frac{\partial^2 y(l,t)}{\partial x^2} = 0,\ \frac{\partial^3 y(l,t)}{\partial x^3} = 0.$$

To approximate the solution of the this problem, we use the method of Newton interpolating series (see [6]) and the Taylor series method (see [8]).

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1 Introduction

The transverse vibration of a uniform slender cantilever beam excited by an imposed harmonic motion is described by Euler-Bernoulli beam model [9]

$$\frac{\partial^2 y(x,t)}{\partial t^2} + c^2 \frac{\partial^4 y(x,t)}{\partial x^4} = 0,\ c = \sqrt{\frac{EI}{\rho A}}, \tag{1}$$

where ρ is the density, A is the cross section area, l is the length and EI is the bending stiffness of the beam, subjected to the boundary conditions

$$y(0,t) = y_0 \sin \omega_{input} t,\ \frac{\partial y(0,t)}{\partial x} = 0,\ \frac{\partial^2 y(l,t)}{\partial x^2} = 0,\ \frac{\partial^3 y(l,t)}{\partial x^3} = 0. \tag{2}$$

The structural stiffness degradation of an oscillating system could be beneficial if the system natural vibration frequencies are lower than those of the main components of the input (see [15]). The generalized Bouc-Wen differential model (see, for example, [2]-[4], [12]) is one of the most useful empirical model for the degrading hysteresis behavior. For the vibration of a hysteretic cantilever beam with a concentrated mass on top, in [14] and [15], it was considered an empiric model similar to an SDOF system with time dependent visco-elastic characteristic, to approximate the effect of beam structural degradation in the range of its lowest mode. For some theoretical and experimental results, we refer the reader to [1], [5], [10] and [11].

The equation of motion was taken

$$\ddot{x}(t) + 2\zeta(t)\omega(t)\dot{x}(t) + \omega(t)^2 x(t) = -\ddot{x}_0 \tag{3}$$

where

$$\omega(t) = \omega_{\text{input}} \nu(t), \tag{4}$$

$$\nu(t) = \begin{cases} 1.09 - 2.1 \cdot 10^{-4} t + 6.22 \cdot 10^{-7} t^2 - 1.43 \cdot 10^{-8} t^3, & 0 \le t < 112 \\ 0.988 + 9.64 e^{-\frac{t}{22.5}}, & 112 \le t < 250, \end{cases} \tag{5}$$

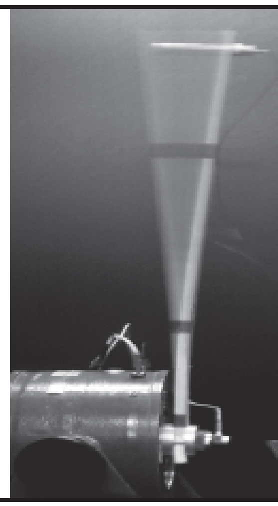
and

$$\zeta(t) = 0.17 - 0.135 e^{-\frac{t}{745}} \tag{6}$$

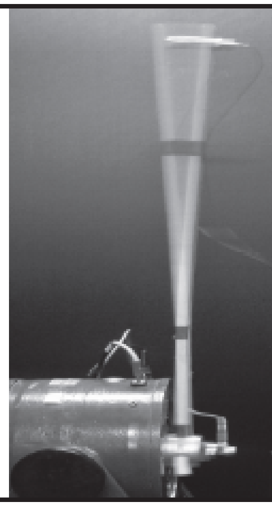
are obtained by interpolating experimental data.

Figure 1 (see [15])) presents three moments of the evolution in time of beam forced vibrations from below to above resonance. This shifting of vibration regime is a direct consequence of structural degradation, which results in a gradual decrease of beam bending stiffness and, therefore, of its modal frequencies.

a. $f_{\text{syst}} > f_{\text{input}}$



b. $f_{\text{syst}} \cong f_{\text{input}}$



c. $f_{syst} < f_{input}$

Fig. 1 Three moments of the evolution in time of beam forced vibrations

To study the structural stiffness degradation of an oscillating system, in Section 2, we use a two-dimensional model which generalizes the equations (1) and (3). Numerical results are presented in Section 3.

2 Description of the method

By introducing the dimensionless parameters (see [13])

$$\tau = \omega_b t,\ \xi = \frac{x}{l}, \overline{\nu} = \frac{\omega_{input}}{\omega_b},\ \eta(\xi,\tau) = \frac{y(l\xi, \tau/\omega_b)}{l},\ \eta_0 = \frac{y_0}{l}, \tag{7}$$

where $\omega_b = \frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$, from (1) and (2), we get the equation

$$\frac{\partial^4 \eta(\xi,\tau)}{\partial \xi^4} + \frac{\partial^2 \eta(\xi,\tau)}{\partial \tau^2} = 0 \tag{8}$$

and the boundary conditions

$$\eta(0,\tau) = \eta_0 \sin \overline{\nu}\tau,\ \frac{\partial \eta(0,\tau)}{\partial \xi} = 0,\ \frac{\partial^2 \eta(1,\tau)}{\partial \xi^2} = 0,\ \frac{\partial^3 \eta(1,\tau)}{\partial \xi^3} = 0. \tag{9}$$

By following [13] we seek a solution of the form

$$\eta(\xi,\tau) = \eta_0 \phi(\xi) \sin \overline{\nu}\tau. \tag{10}$$

Then from (8) and (9) it follows that

$$\phi^{(iv)}(\xi) - \overline{\nu}^2\phi(\xi) = 0, \tag{11}$$

$$\phi(0) = 1,\ \phi'(0) = 0,\ \phi''(1) = 0,\ \phi'''(1) = 0. \tag{12}$$

To approximate the solution of the problem (11), (12) we use the method of Newton interpolating series (see [6] or [7]).

Let $\{\beta_i\}_{i\geq 1}$ be the sequence of elements from $[0,1]$ defined by $\beta_1 = 0,\ \beta_2 = 1,\ \beta_3 = \frac{1}{2}$, and for $k \geq 4$,

$$\beta_k = \frac{2s+1}{2^{m+1}},\ \text{when } k \in (2^m+1, 2^{m+1}+1],\quad s = k - 2^m - 2. \tag{13}$$

An infinite series of the form

$$\sum_{k=0}^{\infty} a_k u_k(\xi), \tag{14}$$

where a_k are real number and

$$u_k(\xi) = \prod_{j=1}^{k} (\xi - \beta_j) \tag{15}$$

is called *Newton interpolating series* with real coefficients at $\{\beta_j\}_{j\geq 1}$. If $f : [0,1] \to \mathbb{R}$ is a continuous function, we say that the function f is *representable into a Newton interpolating series at* $\{\beta_j\}_{j\geq 1}$ if there exists a series of the form (14), which converges absolutely and uniformly to f on $[0,1]$.

The solution $\phi(\xi)$ of the equation (11) is representable into a Newton interpolating series at $\{\beta_j\}_{j\geq 1}$ (see [6] or [7]) and the coefficients a_k are the divided differences of $\phi(\xi)$ with respect to β_j, $j = 1, 2, ...$ The coefficients are determined by replacing the series (14) into the equation (11) and by using the boundary conditions (12).

As a generalization of the equations (1) and (3) for the study of the vibration of the uniform slender cantilever beam, as an oscillating system with degrading hysteretic behavior, the following partial differential equation is considered

$$\frac{\partial^2 y(x,t)}{\partial t^2} + 2\zeta(t)\omega(t)\frac{\partial y(x,t)}{\partial t} + l^4\omega^2(t)\frac{\partial^4 y(x,t)}{\partial x^4} = 0. \tag{16}$$

By (4) and (5) we get $\omega_{\text{input}} = \omega(0)/\nu(0) = \frac{\omega(0)}{1.09}$. Thus

$$\omega(t) = \frac{\omega(0)\nu(t)}{1.09}. \tag{17}$$

The boundary conditions are given by (2).

Since we'll seek an analytic solution of the equation (16) we'll approximate $\nu(t)$ by a polynomial. By least-square method we found the following polynomial, denoted also by $\nu(t)$.

$$\nu(t) = -.36955446\cdot 10^{-11}t^5 + .22954260\cdot 10^{-8}t^4 - .47389354\cdot 10^{-6}t^3 + .3534202510^{-4}t^2 - .10994788 \cdot 10^{-2}t + 1.09413538. \tag{18}$$

To approximate the solution we use the Taylor series method (see, for example, [8]). Thus the set $\mathbb{N}^2$ of pairs of non-negative integers is ordered by

$$\mathbf{i} = (i_1, i_2) < \mathbf{i}' = (i'_1, i'_2) \text{ if either } i_1 + i_2 < i'_1 + i'_2 \text{ or } i_1 + i_2 = i'_1 + i'_2 \text{ and } i_2 < i'_2. \tag{19}$$

Let P be a polynomial in two variables x_1, x_2 with real coefficients. The terms of the polynomials can be ordered by (19) with respect to exponents of the unknowns. Hence we can write

$$P(\mathbf{x}) = \sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{m}} c_{\mathbf{i}} x_1^{i_1} x_2^{i_2}, \tag{20}$$

where $\mathbf{0} = (0, 0)$, $c_{\mathbf{m}} = c_{(m_1,m_2)} \neq 0$, and $\mathbf{m}$ is the greatest index $\mathbf{i}$ with respect to the order defined by (19) such that $c_{\mathbf{i}} \neq 0$. Then $\mathbf{m} = \mathbf{d}_P$ is called the degree of P. The ordinary degree of the polynomial P is $\deg(P) = m_1 + m_2$.

Let P be a polynomial in two variables written in the form of (20) with respect to the order defined by (19). Thus

$$\begin{aligned} P(x_1, x_2) = {} & c_{(0,0)} + c_{(1,0)}x_1 + c_{(0,1)}x_2 + c_{(2,0)}x_1^2 + c_{(1,1)}x_1x_2 + c_{(0,2)}x_2^2 \\ & + c_{(3,0)}x_1^3 + c_{(2,1)}x_1^2x_2 + c_{(1,2)}x_1x_2^2 + c_{(0,3)}x_2^3 + \dots \ . \end{aligned} \tag{21}$$

To find the position of a term in (21), first we write $P(x_1, x_2)$ as

$$P(x_1, x_2) = a_1 + a_2x_1 + a_3x_2 + a_4x_1^2 + a_5x_1x_2 + a_6x_2^2 + a_7x_1^3 + a_8x_1^2x_2 + a_9x_1x_2^2 + \dots \ . \tag{22}$$

Then γ from $\mathbb{N}^2$ to $\mathbb{N}^*$ defined by

$$\gamma(i_1, i_2) = \frac{(i_1 + i_2)(i_1 + i_2 + 1)}{2} + i_2 + 1. \tag{23}$$

is a one-to-one mapping which keeps the order (see [8]). By using (21) and (23), we obtain the position $\gamma(i_1, i_2)$ of the term $c_{\mathbf{i}}\mathbf{x}^{\mathbf{i}}$. For example, $\gamma(1, 2) = 9$, which implies that the term $c_{(2,1)}x_1x_2^2$ is located in the ninth term in (22).

By using Taylor series, the solution of the partial differential equation from (16) with the boundary conditions (2) is considered of the form

$$y(x, \tau) = \sum_{(i,j)\geq(0,0)} c_{(i,j)} x^i t^j, \tag{24}$$

where the terms of the Taylor series are ordered by following (19) with respect to indexes $(i,j) \in \mathbb{N}^2$. Then by using (16) and (2) the solution is approximated by a partial sum

$$P_{\mathbf{m}}(x,t) = \sum_{(i,j)\geq(0,0)}^{\mathbf{m}} c_{(i,j)} x^i t^j. \tag{25}$$

By denoting

$$2\zeta(t)\omega(t) = \sum_{j=0}^{n_1} d_j t^j,$$
$$l^4\omega^2(t) = \sum_{j=0}^{n_2} e_j t^j,$$

by (16) we get

$$\sum_{(i,j)\geq(0,0)} (j+1)(j+2)c_{(i,j+2)}x^i t^j + \sum_{j=0}^{n_1} d_j t^j \sum_{(i,j)\geq(0,0)} (j+1)c_{(i,j+1)}x^i t^j$$
$$+ \sum_{j=0}^{n_2} e_j t^j \sum_{(i,j)\geq(0,0)} (i+1)(i+2)(i+3)(i+4)c_{(i+4,j)}x^i t^j = 0.$$

Hence, for $(i+4,j) \leq \mathbf{m}$, it follows that

$$c_{(i+4,j)} = \frac{1}{e_0(i+1)(i+2)(i+3)(i+4)} \Big(-(j+1)(j+2)c_{(i,j+2)}$$
$$- \sum_{k=0}^{j} d_k (j+1-k)c_{(i,j+1-k)} - \sum_{k=0}^{j} e_k (i+1)(i+2)(i+3)(i+4)c_{(i+4,j-k)} \Big). \tag{26}$$

Then by (2) and (25) we get

$$c_{(0,j)} = \begin{cases} 0 & \text{if } j = 2k \\ \frac{y_0(-1)^k \omega_{input}^{2k+1}}{(2k+1)!} & \text{if } j = 2k+1, \end{cases} \qquad c_{(1,j)} = 0, \tag{27}$$

$$\sum_{i\geq 0}^{(i+2,j)\leq\mathbf{m}} (i+1)(i+2)l^i c_{(i+2,j)} = 0, \ j = 0,1,..., \tag{28}$$

$$\sum_{i\geq 0}^{(i+3,j)\leq\mathbf{m}} (i+1)(i+2)(i+3)l^i c_{(i+3,j)} = 0, \ j = 0,1,\ldots\,. \tag{29}$$

By (27)-(29) we obtain $c_{(i,j)}$, for every $(i,j) \leq \mathbf{m}$ and by (25) it follows the approximate solution $P_{\mathbf{m}}(x,t)$.

We note that equation (1) is obtained from (16), for $\zeta \equiv 0$ and $\nu \equiv 1$.

3 Numerical results

Consider $l = 0.30$ m, $y_0 = 0.0025$ m and $\omega_{input} = 5$ Hz. By choosing $\mathbf{m} = (0, 60)$ results for forced vibrations are represented in Figures 2-8. The continuous lines represent the values for the free end and the points those which correspond to the beam base. Numerical results for the acceleration are given in Table 1.

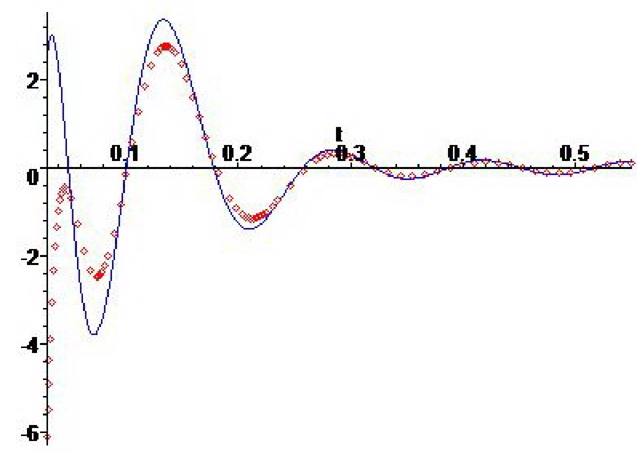


Fig. 2 Displacements at $x = 1.00$ and $t_0 = 0$.

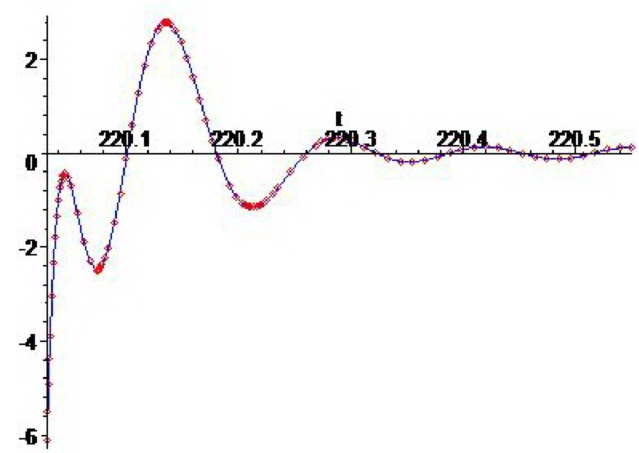


Fig. 3 Acceleration at $x = 1.00$ and $t_0 = 0$

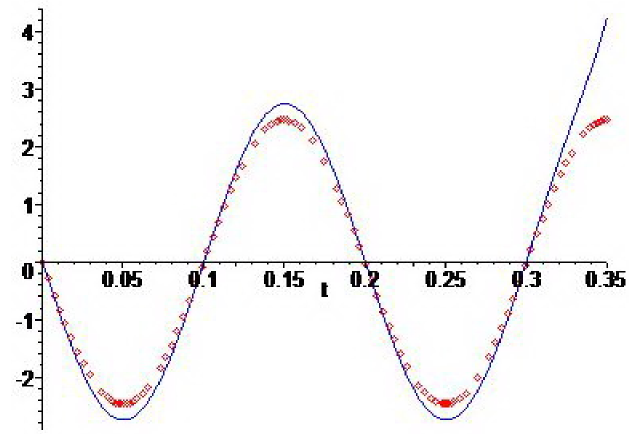


Fig. 4 Displacements $y(x,t)$ at $x \in [0, 0.30]$ and $t_0 = 0$

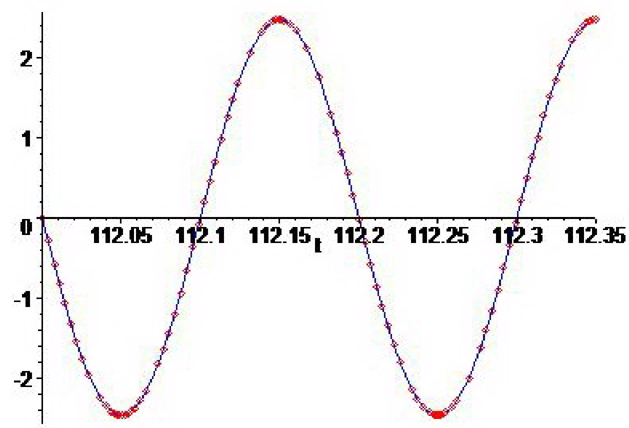


Fig. 5 Displacements at $x = 1.00$ and $t_0 = 112$

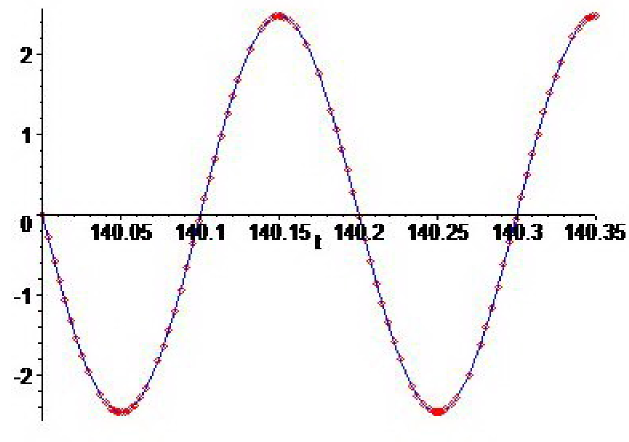


Fig. 6 Acceleration at $x = 1.00$ and $t_0 = 112$

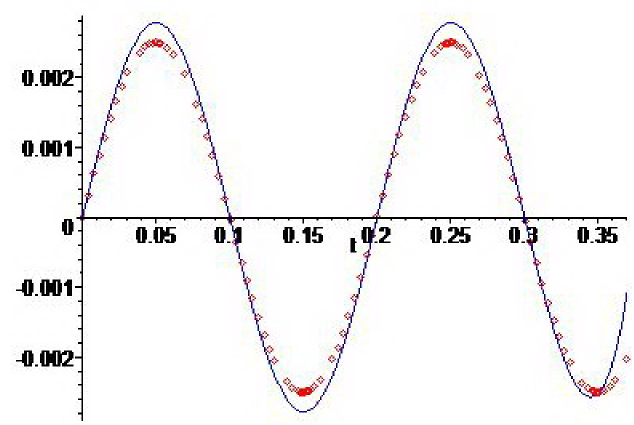


Fig. 7 Displacements at $x = 1.00$ and $t_0 = 140$

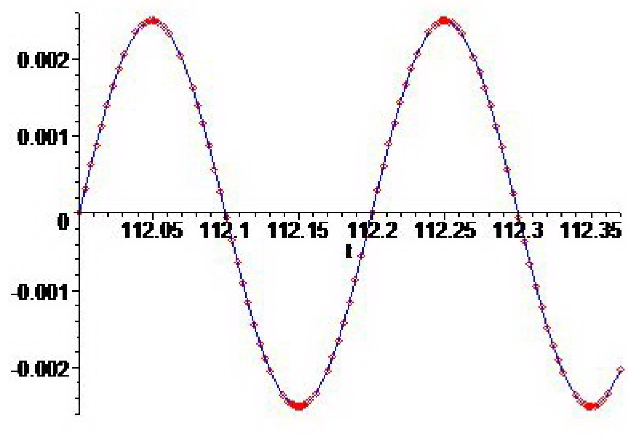


Fig. 8 Acceleration at $x = 1.00$ and $t_0 = 140$

If we define an amplification factor $A(t_0)$ as $A(t_0) := \frac{\max\limits_{t\in[t_0,t_0+\delta]} |\ddot{y}(l,t)|}{\max\limits_{t\in[t_0,t_0+\delta]} |\ddot{y}(0,t)|}$, where $[t_0, t_0 + \delta]$ is a suitable interval where $\ddot{y}(l,t)$ and $\ddot{y}(0,t)$ are bounded, we get $A(0) = 2.714433$, $A(50) = 2.144169$, $A(112) = 1.040357$, $A(140) = 1.007035$ and $A(204) = 1.000000$. Thus, for forced vibrations the amplification factor decreases when ζ increases, that it is smaller when the degradation increases.

To study the free transverse vibrations we consider the same partial differential equation but the first boundary condition is replaced by

$$y(0,t) = \begin{cases} y_0 \sin \omega_{input} t, & \text{if } t \in [t_0, t_0 + 0.2] \\ 0, & \text{if } t > t_0 + 0.2. \end{cases}$$

By studying the free vibrations (see Figures 9 and 10) it follows that, when the degradation is greater (the case $t_0 = 220$), after $t > t_0 + 0.2$ the values of the acceleration are smaller than in the initial case, $t_0 = 0$.

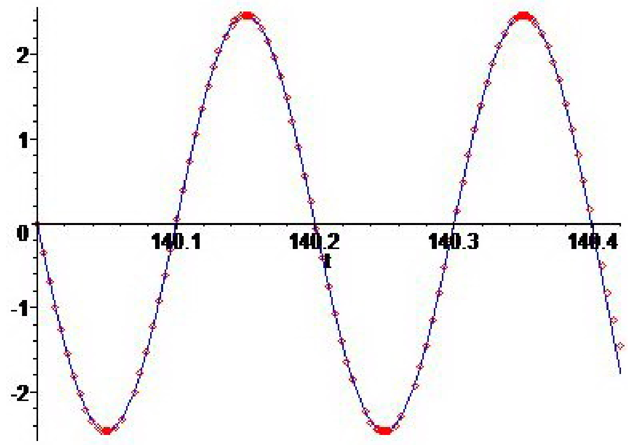


Fig. 9 Acceleration at $x = 1.00$ and $t_0 = 0$

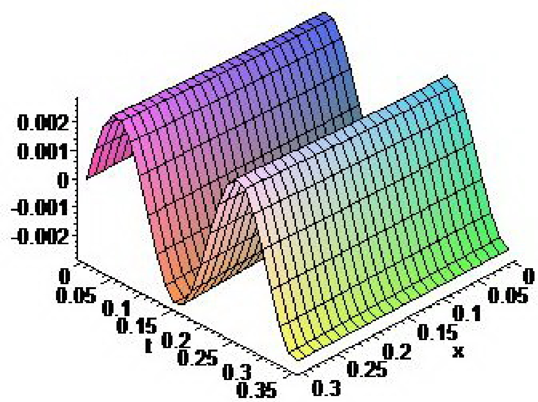


Fig. 10 Acceleration at $x = 1.00$ and $t_0 = 220$

Table 1

	$t_0 = 0\ s$	$t_0 = 50\ s$	$t_0 = 112\ s$	$t_0 = 140\ s$	$t_0 = 204\ s$
t_0	0.371724	0.381938	0.015509	0.001708	0.000089
$t_0+0.02$	−3.632027	−2.728699	−1.440674	−1.448955	−1.450230
$t_0+0.05$	−6.690919	−5.184976	−2.472290	−2.467457	−2.467401
$t_0+0.07$	−5.631675	−4.429885	−2.009155	−1.997214	−1.996222
$t_0+0.10$	−0.371928	−0.387456	−0.015344	−0.001697	−0.000088
$t_0+0.12$	3.631970	2.754801	1.440744	1.448963	1.450231
$t_0+0.15$	6.691102	5.237521	2.472184	2.467457	2.467401
$t_0+0.17$	5.631942	4.475981	2.008974	1.997208	1.996221
$t_0+0.20$	0.372105	0.393154	0.015183	0.001687	0.000088
$t_0+0.22$	−3.632186	−2.781689	−1.440806	−1.448970	−1.450231
$t_0+0.25$	−6.697594	−5.290524	−2.471928	−2.467437	−2.467399
$t_0+0.27$	−5.674031	−4.517869	−2.007969	−1.997094	−1.996206
$t_0+0.30$	−0.928329	−0.401417	−0.007500	−0.000661	0.000108
$t_0+0.32$	0.928040	2.487189	1.466843	1.452690	1.451151

4 Conclusions

To study the structural stiffness degradation of a uniform slender cantilever beam as a consequence of transverse vibrations due to an imposed harmonic motion, we use a two-dimensional model which generalizes those based on an SDOF oscillating system. The numerical results obtained describes closed enough the real behavior of this oscillating system found by experimental results in [14].

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