Positive bounded solutions for nonlinear polyharmonic problems in the unit ball

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Abstract

In this paper, we study the existence of positive solutions for the following nonlinear polyharmonic equation $(-\Delta)^m u + \lambda f(x, u) = 0$ in B , subject to some boundary conditions, where m is a positive integer, λ is a nonnegative constant and B is the unit ball of $\mathbb{R}^n$ ($n \geq 2$). Under some appropriate assumptions on the nonnegative nonlinearity term $f(x, u)$ and by using the Schäuder fixed point theorem, the existence of positive solutions is obtained. At last, examples are given for illustration.

1 Introduction

The goal of the paper is to study the existence of positive continuous bounded solutions for the following nonlinear elliptic higher order problem :

$$\begin{cases} (-\Delta)^m u + \lambda f(x, u) = 0 \text{ in } B, \\ u > 0 \text{ in } B, \\ \lim_{x \rightarrow \xi \in \partial B} \frac{u(x)}{(1-|x|^2)^{m-1}} = \varphi(\xi), \end{cases} \quad (1.1)$$

where m is a positive integer, $B = \{x \in \mathbb{R}^n : |x| < 1\}$ is the unit ball of $\mathbb{R}^n$ ($n \geq 2$), $\partial B = \{x \in \mathbb{R}^n : |x| = 1\}$ is the boundary of B , λ is a nonnegative constant, φ is a nontrivial nonnegative continuous function on ∂B and $f : B \times [0, \infty) \rightarrow [0, \infty)$ is continuous.

Key Words: Green function, Kato class, Positive solution, Nonlinear polyharmonic equation, Schäuder fixed point theorem.

2010 Mathematics Subject Classification: Primary 35B09, 35J40; Secondary 34B27.

Received: 29.07.2016

Revised: 01.03.2017

Accepted: 06.03.2017

The polyharmonic operator $(-\Delta)^m$, $m \in \mathbb{N}^*$, has been studied several years later. Indeed, in [5], Boggio showed that the Green function $G_{m,n}$ of the operator $(-\Delta)^m$ on B with Dirichlet boundary conditions $u = \frac{\partial}{\partial \nu} u = \dots = \frac{\partial^{m-1}}{\partial \nu^{m-1}} u = 0$ on ∂B , is given by :

$$G_{m,n}(x, y) = k_{m,n} |x - y|^{2m-n} \int_1^{\frac{[x,y]}{|x-y|}} \frac{(\nu^2 - 1)^{m-1}}{\nu^{n-1}} d\nu,$$

where $k_{m,n}$ is a positive constant, $\frac{\partial}{\partial \nu}$ is the outward normal derivative and for x, y in B , $[x, y]^2 = |x - y|^2 + (1 - |x|^2)(1 - |y|^2)$.

In [2], the estimates on the Green function $G_{m,n}$ of $(-\Delta)^m$ on B and particularly the 3G-theorem (see [2], Theorem 2.8), allowed the authors to introduce a large functional class called Kato class denoted by $K_{m,n}$ (see Definition 1 below). This class plays a key role in the study of some nonlinear polyharmonic equations (see [2, 3, 6, 10]). For related results we refer to the recent monograph [12] and the papers [7, 9, 11, 13, 14, 15].

Definition 1. (See [2]) *A Borel measurable function q on B belongs to the Kato class $K_{m,n}$ if q satisfies the following condition :*

$$\lim_{\alpha \rightarrow 0} \left(\sup_{x \in B} \int_{B \cap B(x, \alpha)} \left(\frac{\delta(y)}{\delta(x)} \right)^m G_{m,n}(x, y) |q(y)| dy \right) = 0.$$

Here and always $\delta(x) = 1 - |x|$ is the Euclidian distance between x and ∂B .

As typical example of functions belonging to the class $K_{m,n}$, we quote

Example 1. (See [3]) *The function q defined on B by*

$$q(x) = \frac{1}{(\delta(x))^\lambda (\log \frac{2}{\delta(x)})^\mu},$$

is in $K_{m,n}$ if and only if $\lambda < 2m$ and $\mu \in \mathbb{R}$ or $\lambda = 2m$ and $\mu > 1$.

Before presenting our main result, we lay out a number of potential theory tools and some notations which will be used throughout the paper. Let φ be a nontrivial nonnegative continuous function on ∂B , we denote by $H\varphi$ the bounded continuous solution of the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } B, \\ u|_{\partial B} = \varphi. \end{cases}$$

We set ω the function defined on B by

$$\omega : x \rightarrow (1 - |x|^2)^{m-1} H\varphi(x).$$

We remark that the function ω is a bounded continuous solution of the problem

$$\begin{cases} (-\Delta)^m \omega = 0 & \text{in } B, \\ \lim_{x \rightarrow \xi \in \partial B} \frac{\omega(x)}{(1-|x|^2)^{m-1}} = \varphi(\xi). \end{cases} \quad (1.2)$$

For simplicity, we denote by

$$C_0(B) = \{v \text{ continuous on } B \text{ and } \lim_{x \rightarrow \xi \in \partial B} v(x) = 0\}.$$

We also refer to $V_{m,n}g$ the m -potient of a nonnegative measurable function g on B by

$$V_{m,n}g(x) = \int_B G_{m,n}(x, y) g(y) dy, \quad x \in B.$$

Recall that for each nonnegative measurable function g on B such that g and $V_{m,n}g$ are in $L^1_{loc}(B)$, we have

$$(-\Delta)^m(V_{m,n}g) = g \quad (\text{in the distributional sense}).$$

We assume that the function f satisfies the following assumptions :

(H₁) $f : B \times [0, \infty) \rightarrow [0, \infty)$ is continuous and nondecreasing with respect to the second variable.

(H₂) The function $q = \frac{f(\cdot, \omega)}{\omega}$ belongs to the Kato class $K_{m,n}$.

Theorem 1. *Assume (H₁) – (H₂). Then there exists $\lambda_0 > 0$ such that for each $\lambda \in [0, \lambda_0)$, problem (1.1) has a positive continuous solution u satisfying for each $x \in B$*

$$\left(1 - \frac{\lambda}{\lambda_0}\right) \omega(x) \leq u(x) \leq \omega(x). \quad (1.3)$$

We remark that for $m = 1$, we find again the result of [1] which was our original motivation for deriving our study.

Remark 1. *Note that problem (1.1) is a perturbation of problem (1.2). In view of (1.3), we see that the behavior of the obtained solution is not affected by the perturbation term.*

The outline of this paper is as follows. In Section 2, we state some already known results on the Green function $G_{m,n}$ and the functions in the class $K_{m,n}$ that will be used in our study. Section 3 is devoted to the proof of Theorem 1. The last section is reserved for some examples.

Finally, we mention that the letter c will be a positive generic constant which may vary from line to line.

2 Properties of the Green function $G_{m,n}$ and the class $K_{m,n}$

To make the paper self contained, this section is devoted to recall some results established in [2, 4, 8, 10] that will be useful for our study.

Proposition 1. *If $x, y \in B$ such that $|x - y| \geq r > 0$, then there exists $c > 0$ such that*

$$G_{m,n}(x, y) \leq c \frac{(\delta(x)\delta(y))^m}{r^n}.$$

Proposition 2. *Let q be a function in $K_{m,n}$, then*

- (i) *The constant $\alpha_q = \sup_{x,y \in B} \int_B \frac{G_{m,n}(x,z)G_{m,n}(z,y)}{G_{m,n}(x,y)} |q(z)| dz$ is finite.*
- (ii) *The function $x \mapsto (\delta(x))^{2m-1}q(x)$ is in $L^1(B)$.*
- (iii) *For each nonnegative harmonic function h in B , we have for $x \in B$*

$$\int_B G_{m,n}(x, y)(1 - |y|^2)^{m-1}h(y) |q(y)| dy \leq \alpha_q(1 - |x|^2)^{m-1}h(x).$$

- (iv) *For each $x_0 \in \overline{B}$, we have*

$$\lim_{\alpha \rightarrow 0} \left(\sup_{x \in B} \int_{B \cap B(x_0, \alpha)} \left(\frac{\delta(y)}{\delta(x)} \right)^{m-1} G_{m,n}(x, y) |q(y)| dy \right) = 0.$$

- (v) *The function $x \mapsto \int_B \left(\frac{\delta(y)}{\delta(x)} \right)^{m-1} G_{m,n}(x, y) |q(y)| dy$ is in $C_0(B)$.*

3 Proof of Theorem 1

We begin this section with the following lemma which plays a key role in the proof of Theorem 1.

Lemma 1. *If f satisfies (H_2) , then*

$$\lambda_0 := \inf_{x \in B} \frac{\omega(x)}{V_{m,n}(f(\cdot, \omega))(x)} > 0. \quad (3.1)$$

Proof. Since f satisfies (H_2) , then the function $q = \frac{f(\cdot, \omega)}{\omega}$ belongs to the Kato class $K_{m,n}$. It follows from Proposition 2 (iii) that for each $x \in B$,

$$V_{m,n}(f(\cdot, \omega))(x) = V_{m,n}(q \omega)(x) \leq \alpha_q \omega(x).$$

This gives that for each $x \in B$,

$$\frac{\omega(x)}{V_{m,n}(f(.,\omega))(x)} \geq \frac{1}{\alpha_q}.$$

This implies that

$$\lambda_0 \geq \frac{1}{\alpha_q} > 0.$$

□

Now, we are ready to prove our main result.

Proof of Theorem 1. Let Λ be the non-empty closed convex set given by

$$\Lambda = \{v \in C_0(B) : \left(1 - \frac{\lambda}{\lambda_0}\right) \omega \leq v \leq \omega\}.$$

We define the operator T on Λ by

$$Tv = \omega - \lambda V_{m,n}(f(.,v)).$$

We aim to prove that T has a fixed point in Λ . First, we shall prove that $T\Lambda$ is relatively compact in $C_0(B)$. Since $\omega \in C_0(B)$, it is enough to show that the family

$$\{V_{m,n}(f(.,v)), v \in \Lambda\}$$

is relatively compact in $C_0(B)$.

Let $v \in \Lambda$, then by hypothesis (H_1) we obtain that

$$0 \leq V_{m,n}(f(.,v)) \leq V_{m,n}(f(.,\omega)) = V_{m,n}(q\omega). \quad (3.2)$$

Applying Proposition 2 (iii), we get

$$0 \leq V_{m,n}(f(.,v)) \leq \alpha_q \|\omega\|_\infty.$$

Thus the family $\{V_{m,n}(f(.,v)), v \in \Lambda\}$ is uniformly bounded.

Now, we aim at proving that $\{V_{m,n}(f(.,v)), v \in \Lambda\}$ is equicontinuous on B .

Let $x_0 \in B$ and $\varepsilon > 0$. By Proposition 2 (iv), there exists $\alpha > 0$ such that

$$0 \leq \sup_{z \in B} \int_{B \cap B(x_0, 2\alpha)} \left(\frac{\delta(y)}{\delta(\xi)} \right)^{m-1} G_{m,n}(z, y) q(y) dy \leq \frac{\varepsilon}{2^m \|H\varphi\|_\infty}. \quad (3.3)$$

Let $x, x' \in B \cap B(x_0, \alpha)$, then for each $v \in \Lambda$, we have

$$\begin{aligned}
 & |V_{m,n}(f(\cdot, v))(x) - V_{m,n}(f(\cdot, v))(x')| \\
 & \leq \int_B |G_{m,n}(x, y) - G_{m,n}(x', y)| q(y) \omega(y) dy \\
 & \leq 2^{m-1} \|H\varphi\|_\infty \int_B |G_{m,n}(x, y) - G_{m,n}(x', y)| (\delta(y))^{m-1} q(y) dy \\
 & \leq 2^{m-1} \|H\varphi\|_\infty \int_{B \cap B(x_0, 2\alpha)} |G_{m,n}(x, y) - G_{m,n}(x', y)| (\delta(y))^{m-1} q(y) dy \\
 & \quad + 2^{m-1} \|H\varphi\|_\infty \int_{B \cap B^c(x_0, 2\alpha)} |G_{m,n}(x, y) - G_{m,n}(x', y)| (\delta(y))^{m-1} q(y) dy \\
 & := I_1 + I_2.
 \end{aligned}$$

From (3.3), we get that

$$\begin{aligned}
 I_1 & \leq 2^{m-1} \|H\varphi\|_\infty \int_{B \cap B(x_0, 2\alpha)} \left(\frac{\delta(y)}{\delta(x)} \right)^{m-1} G_{m,n}(x, y) q(y) dy \\
 & \quad + 2^{m-1} \|H\varphi\|_\infty \int_{B \cap B(x_0, 2\alpha)} \left(\frac{\delta(y)}{\delta(x')} \right)^{m-1} G_{m,n}(x', y) q(y) dy \\
 & \leq 2^m \|H\varphi\|_\infty \sup_{z \in B} \int_{B \cap B(x_0, 2\alpha)} \left(\frac{\delta(y)}{\delta(z)} \right)^{m-1} G_{m,n}(z, y) q(y) dy \\
 & \leq \varepsilon
 \end{aligned}$$

On the other hand, if $|y - x_0| \geq 2\alpha$ then $|y - x| \geq \alpha$ and $|y - x'| \geq \alpha$. So by applying Proposition 1, we reach

$$\begin{aligned}
 & |G_{m,n}(x, y) - G_{m,n}(x', y)| (\delta(y))^{m-1} q(y) \\
 & \leq (G_{m,n}(x, y) + G_{m,n}(x', y)) (\delta(y))^{m-1} q(y) \\
 & \leq \frac{c}{\alpha^n} ((\delta(x)\delta(y))^m + (\delta(x')\delta(y))^m) (\delta(y))^{m-1} q(y) \\
 & \leq c(\delta(y))^{2m-1} q(y).
 \end{aligned}$$

Now, since for $y \in B \cap B^c(x_0, 2\alpha)$, $x \mapsto G_{m,n}(x, y)$ is continuous in $B \cap B(x_0, \alpha)$ and from Proposition 2 (ii) the function $y \mapsto (\delta(y))^{2m-1} q(y)$ is in $L^1(B)$ then we deduce by the dominated convergence theorem that

$$I_2 \rightarrow 0 \text{ as } |x - x'| \rightarrow 0.$$

Thus $\{V_{m,n}(f(\cdot, v)), v \in \Lambda\}$ is equicontinuous on B .

Next, we claim that $V_{m,n}(f(., v))(x) \rightarrow 0$ as $x \rightarrow \xi \in \partial B$ uniformly in $v \in \Lambda$. Let $\xi \in \partial B$ and $x \in B \cap B(\xi, \alpha)$. Then for each $v \in \Lambda$, we have from (3.2)

$$\begin{aligned} V_{m,n}(f(., v))(x) &\leq 2^{m-1} \|H\varphi\|_\infty \int_B G_{m,n}(x, y) (\delta(y))^{m-1} q(y) dy \\ &\leq 2^{m-1} \|H\varphi\|_\infty \int_{B \cap B(\xi, 2\alpha)} G_{m,n}(x, y) (\delta(y))^{m-1} q(y) dy \\ &\quad + 2^{m-1} \|H\varphi\|_\infty \int_{B \cap B^c(\xi, 2\alpha)} G_{m,n}(x, y) (\delta(y))^{m-1} q(y) dy \\ &\leq 2^{m-1} \|H\varphi\|_\infty \sup_{z \in B} \int_{B \cap B(\xi, 2\alpha)} \left(\frac{\delta(y)}{\delta(z)} \right)^{m-1} G_{m,n}(z, y) q(y) dy \\ &\quad + 2^{m-1} \|H\varphi\|_\infty \int_{B \cap B^c(\xi, 2\alpha)} G_{m,n}(x, y) (\delta(y))^{m-1} q(y) dy. \end{aligned}$$

By applying Proposition 2 (iv), we obtain that

$$2^{m-1} \|H\varphi\|_\infty \sup_{z \in B} \int_{B \cap B(\xi, 2\alpha)} \left(\frac{\delta(y)}{\delta(z)} \right)^{m-1} G_{m,n}(z, y) q(y) dy \rightarrow 0 \text{ as } \alpha \rightarrow 0.$$

For $y \in B \cap B^c(\xi, 2\alpha)$, we have $|x - y| \geq \alpha$. Hence, it follows from Proposition 1 and Proposition 2 (ii) that

$$\begin{aligned} &2^{m-1} \|H\varphi\|_\infty \int_{B \cap B^c(\xi, 2\alpha)} G_{m,n}(x, y) (\delta(y))^{m-1} q(y) dy \\ &\leq c \left(\int_B (\delta(y))^{2m-1} q(y) dy \right) (\delta(x))^m \rightarrow 0 \text{ as } x \rightarrow \xi. \end{aligned}$$

Therefore by Ascoli's theorem, we conclude that the family $\{V_{m,n}(f(., v)), v \in \Lambda\}$ is relatively compact in $C_0(B)$.

For $v \in \Lambda$, we have from (3.2)

$$\omega - \lambda V_{m,n}(f(., \omega)) \leq Tv \leq \omega.$$

This implies from (3.1) that

$$\left(1 - \frac{\lambda}{\lambda_0}\right) \omega \leq Tv \leq \omega.$$

Combining this with the fact that $Tv \in C_0(B)$, we deduce that $T\Lambda \subset \Lambda$.

Now, we prove the continuity of T in Λ in the supremum norm. Let $(v_k)_k$ be a sequence in Λ which converges uniformly to a function v in Λ . Then for $k \in \mathbb{N}$ and each $x \in B$,

$$|Tv_k(x) - Tv(x)| \leq \int_B G_{m,n}(x, y) |f(y, v_k(y)) - f(y, v(y))| dy.$$

On the other hand, from the monotonicity of the function f , we have for $k \in \mathbb{N}$ and $(x, y) \in B^2$,

$$\begin{aligned} G_{m,n}(x, y) |f(y, v_k(y)) - f(y, v(y))| &\leq 2G_{m,n}(x, y)f(y, \omega(y)) \\ &\leq 2G_{m,n}(x, y)q(y)\omega(y). \end{aligned}$$

Since by Proposition 2 (iii), $\int_B G_{m,n}(x, y)q(y)\omega(y)dy \leq \alpha_q \|\omega\|_\infty < \infty$, we conclude by the continuity of f with respect to the second variable and the dominated convergence theorem that

$$|Tv_k(x) - Tv(x)| \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Consequently, since $T\Lambda$ is relatively compact in $C_0(B)$, we deduce that the pointwise convergence implies the uniform convergence, namely

$$\|Tv_k - Tv\|_\infty \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Thus we have proved that T is a compact mapping from Λ to itself. Hence the Schauder's fixed point theorem implies the existence of $u \in \Lambda$ such that

$$u = Tu,$$

that is

$$u = \omega - \lambda V_{m,n}(f(., u)). \quad (3.4)$$

It is clear that u is continuous and satisfies (1.3) and it remains to verify that u is a solution of (1.1).

Since $0 \leq f(., u) \leq 2^{m-1} \|H\varphi\|_\infty (\delta(.))^{m-1} q$ then by Proposition 2 (ii), we obtain that $f(., u) \in L^1_{loc}(B)$ and from (3.4), we have $V_{m,n}(f(., u)) \in L^1_{loc}(B)$. Hence, we have in the distributional sense

$$(-\Delta)^m V_{m,n}(f(., u)) = f(., u) \text{ in } B.$$

Now, applying $(-\Delta)^m$ on both sides of (3.4), we obtain that

$$(-\Delta)^m u = -\lambda f(., u) \text{ in } B.$$

Finally, we have

$$\lim_{x \rightarrow \xi \in \partial B} \frac{u(x)}{(1 - |x|^2)^{m-1}} = \varphi(\xi) - \lambda \lim_{x \rightarrow \xi \in \partial B} \frac{V_{m,n}(f(., u))(x)}{(1 - |x|^2)^{m-1}}.$$

Since for $x \in B$, we have

$$0 \leq \frac{V_{m,n}(f(., u))(x)}{(1 - |x|^2)^{m-1}} \leq 2^{m-1} \|H\varphi\|_\infty \int_B \left(\frac{\delta(y)}{\delta(x)} \right)^{m-1} G_{m,n}(x, y)q(y)dy,$$

we deduce by Proposition 2 (v) that

$$\lim_{x \rightarrow \xi \in \partial B} \frac{V_{m,n}(f(\cdot, u))(x)}{(1 - |x|^2)^{m-1}} = 0.$$

Hence

$$\lim_{x \rightarrow \xi \in \partial B} \frac{u(x)}{(1 - |x|^2)^{m-1}} = \varphi(\xi).$$

This ends the proof. $\square$

4 Examples

We take up in this section some examples illustrating our main result.

Example 2. Let φ be a positive continuous function on ∂B . Let p be a nonnegative measurable function satisfying for each $x \in B$, $p(x) \leq \frac{c}{(\delta(x))^\lambda (\log(\frac{2}{\delta(x)}))^\mu}$ with $\lambda < 2m$ and $\mu \in \mathbb{R}$ or $\lambda = 2m$ and $\mu > 1$. We consider $g : [0, \infty) \rightarrow [0, \infty)$ nondecreasing and continuous function such that for each $c > 0$, there exists $\eta > 0$ satisfying

$$g(t) \leq \eta t, \quad \forall t \in [0, c].$$

Then by Theorem 1, there exists $\lambda_0 > 0$ such that for each $\lambda \in [0, \lambda_0)$, the problem

$$\begin{cases} (-\Delta)^m u + \lambda p(x)g(u) = 0 & \text{in } B, \\ u > 0 & \text{in } B, \\ \lim_{x \rightarrow \xi \in \partial B} \frac{u(x)}{(1 - |x|^2)^{m-1}} = \varphi(\xi), \end{cases}$$

has a positive continuous solution u satisfying (1.3).

Example 3. Let φ be a positive continuous function on ∂B . Let p be a nonnegative measurable function satisfying for each $x \in B$, $p(x) \leq \frac{1}{(\delta(x))^\lambda}$ with $\lambda < 2m$ and f be the function defined on $B \times [0, \infty)$ by $f(x, u) = p(x)u^\sigma$ with $\sigma \geq 1$. Therefore by Theorem 1, there exists $\lambda_0 > 0$ such that for each $\lambda \in [0, \lambda_0)$, the problem

$$\begin{cases} (-\Delta)^m u + \lambda p(x)u^\sigma = 0 & \text{in } B, \\ u > 0 & \text{in } B, \\ \lim_{x \rightarrow \xi \in \partial B} \frac{u(x)}{(1 - |x|^2)^{m-1}} = \varphi(\xi), \end{cases}$$

has a positive continuous solution u satisfying (1.3).

Acknowledgement 1. The authors would like to thank the anonymous referee for his/her helpful comments which improved the presentation of the original manuscript. Also, the authors express their gratitude to Professor Vicentiu Rădulescu for his interest in this work and useful suggestions.

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