DOI: 10.1515/auom-2017-0018
An. Şt. Univ. Ovidius Constanţa

VERSITA
Vol. 25(2),2017, 25–38

Numerical Analysis of Transmission Lines Equation by new β-method Schemes

Fatima Allali , Nour Eddine Alaa, Abdelilah Ghammaz and Hicham Rouijaa

Abstract

In this paper we develop a new β-method applied to the resolution of homogeneous transmission lines. A comparison with conventional methods used for this type of problems like FDTD method or classical β-method is also given. Furthermore, various numerical experiments are presented to confirm the accuracy, efficiency and stability of our proposed method. In particular, these simulations show that our new scheme is unconditionally stable and fourth-order accurate in space and time.

1 Introduction

One of the basic technologies of the twentieth century is without doubt electromagnetism which is the study of electric and magnetic fields and their interaction [1],[2]. In this study we are concerned with the transmission lines which serve to guide the propagation of electromagnetic energy from a source terminal to a load terminal. These lines take many physical forms, including a twisted pair line used for telephone or internet connections, coaxial cable, or any number of systems wave guiding multi-conductor. Transmission lines is a multidisciplinary field. Its domain of applications range from the theory of electromagnetism, methods of numerical simulations to geometric modelling

Key Words: transmission lines, finite-difference time-domain (FDTD), electromagnetic propagation, fourth order, beta method, numerical stability, truncation error

2010 Mathematics Subject Classification: Primary 65L06, 65L12, 65L20; Secondary 78M20, 78M25.

Received: May, 2016.

Revised: May, 2016.

Accepted: , 2016.

and visualization. Numerical algorithms have important roles to play in this study. The goal of this paper is the proposal of new numerical simulation methods of the transmission line equation. Typically, modelling of the transmission line is one-dimensional and is represented by voltages and currents according to the line axis of the transmission line.
A transmission line is a specialized cable or other structure designed to carry alternating current. It can be modelled by a resistance R, an inductance L (which represents the series impedance along the cable), the capacitance C and the conductance G that form the shunt admittance across the line. The time-dependent transmission line governing the line voltages and currents are expressed as (see [1], [7]):

$$\begin{cases} \dfrac{\partial V(z,t)}{\partial z} + L\dfrac{\partial I(z,t)}{\partial t} + R\ I(z,t) = 0 \\ \dfrac{\partial I(z,t)}{\partial z} + C\dfrac{\partial V(z,t)}{\partial t} + G\ V(z,t) = 0 \\ V(0,t) = V_s(t) \\ I(0,t) = I_s(t) \end{cases} \tag{1}$$

Where the axis Oz corresponds to the direction of the line $V(z,t)$ is the line voltage at position z along the transmission line axis at time t, $I(z,t)$ is the line current, and L and C are the per unit length inductance (H/m) and capacitance (F/m). Ideal transmission lines model [12], [13] ideal guiding structures, that is, interconnections without losses, uniform in space and with parameters independent of frequency. We set for an ideal transmission line conductors $R = G = 0$.
For the simulation of this type of equations, the full-wave techniques are used which are finite difference time domain method (FDTD); the method of moments (MoM); and the finite element method (FEM). There is another class of numerical methods for solving these equations and are called asymptotic techniques. These methods require advanced approximations and manipulations. They are applied for instance in physical optics (PO),geometrical optics (GO) and uniform theory of diffraction (UTD). These methods are very powerful, yet in the case of our problem, the underlying approximations fail to reproduce the physical phenomena. Generally, the accuracy of the numerical methods is related to the discretization (i.e. mesh size). The finer is the mesh, the better is the accuracy of the methods. The method of moments (MoM) is preferred for the radiation in the frequency domain and distribution problems involving perfectly or highly son and / or conductive surfaces. However MoM is not suitable for problems involving non-homogeneous dielectric materials. The finite element method (FEM) has been widely used in structural mechanics and thermodynamics. It is the most practical method for device simulation microwave and eigenproblem analysis. Its association with MoM can be accurate and efficient for scattering problems involving penetrable electromagnetic

media and specialized antenna problems.
The finite difference time domain (FDTD) method is of a similar vintage to the MoM and FEM in electromagnetic. Unlike the FEM, the FDTD method does not use variational formulations or weighted residuals. Differential operators in (1) are directly approximated on a staggered grid in time and space. The FDTD method is an "explicit" finite difference approach, i.e. no matrix equation is set up and solved. There are several very good references on the FDTD method. Kunz and Luebbers's book [6] presented a comprehensive analysis of FDTD methods in 1993, but Taflove and hagness's book [8] is considered a standard reference in the subject.
For wideband systems, the FDTD method is preferred, it is also effective for any radiation diffusion electromagnetic problem. But it does not give great accuracy. In 1989, Fang [5] proposed two higher order FDTD schemes. The first one is fourth order accurate in space and second order accurate in time, while the second is fourth order in both space and time. These schemes were developed by using fourth order central differences approximations. In 2004, S. Zhao and G.W. Wei [4] proposed the hierarchical DM method seeking high order accurate solutions of time-domain Maxwell's equation with material interfaces in the framework of FDTD. Here our approach is different, we will adopt the β-schemes that were introduced in 1987 by Desideri et al [11]. Several studies show that we can control the degrees of dispersion and diffusion of the scheme by the choice of the parameter β. Despite their high order and precise control over dispersion and diffusion, these methods produce oscillations near such discontinuities because they are linear greater than 1. These oscillations negatively influence the quality of our simulations, especially since our system is hyperpolic. We will see this flaw in our numerical simulations of the following paragraphs. The original idea of this work is to couple the β-methods of high order with Rung Kutta method in order to get high order schemes without oscillations. This part will be presented in section 4.
Our paper is organized as follows. In the next section we give the discretization of classical FDTD method for our system (1). Section three is devoted to the classical beta method. After presenting the algorithm adapted to equation (1), we will give some results about its stability and accuracy. In the fourth section, we present the principle of our new method adapted for transmissions lines. Finally, section five is dedicated to the analysis of our new algorithm. It is fourth order in time and space and eliminates the oscillations at the discontinuities. In fact, numerical experiments are carried out to demonstrate the accuracy, stability, and cost-efficiency of the proposed method, in comparing with the classical versions of the FDTD and β-method.

2 Fourth order proposed model for transmission line system

The classical finite difference time domain (FDTD) method was introduced by Yee ([14]) in 1966. This is a non-dissipative regime throughout space and time. It is a well-known technique for the analysis of electromagnetic problems and for solving telegrapher's equation system. The FDTD method employs finite differences as approximations to both the spatial and temporal derivatives that appear in transmission line equation system and requires that the spatial grid size and time step should satisfy a very restricted condition in order to prevent the numerical solution from diverging.

Now explain the principle: The first step in obtaining a $FDTD$-method solution is to set up a regular grid in space and time. The points on this grid can be designated as (z_k, t_n), where $z_k = (k-1) * dz$ and $t_n = (n-1) * dt$ for $k = 1, 2, .., NDZ + 1$ and $n = 1, 2, ..., NDT$. As it has already been noted in previous work, additional points are put in place at half time and half space. These additional points can be designated as $(z_{k+\frac{1}{2}}, t_{n+\frac{1}{2}})$. We shall compute $V(z,t)$ at the points (z_k, t_n), and $I(z,t)$ at the points $(z_{k+\frac{1}{2}}, t_{n+\frac{1}{2}})$, i.e. the voltage and currents are computed at offset locations in space and also in time. We denote

$$V_k^n \equiv V(z_k, t_n) \tag{2}$$

$$I_k^n \equiv I(z_{k+\frac{1}{2}}, t_{n+\frac{1}{2}}) \tag{3}$$

The finite difference approximation for the equation (1) give:

• for $k = 1$ or $k = NDZ + 1$:

The essential problem in incorporating the terminal conditions is that the voltages and currents at each end of the line, V_1, I_1, and $V_{NDZ+1}, I_N DZ$, are not collocated in space or time, whereas the terminal conditions relate the voltage and current at the same position and at the same time. We will denote the voltage at the source (z = 0) as V_s and the current at the load (z = L) as I_L. The second transmission line equation given in 1 is discretized at the source by averaging the source voltage V_s in order to obtain a value that is located in time at the same time point as I_1^{n+1}

$$V_1^{n+1} = (\frac{R_s C}{F} + 1)^{-1} \left[(\frac{R_s C}{F} - 1) V_1^n - 2 R_s I_1^n + (V_s^{n+1} + V_s^n) \right] \tag{4}$$

where $F = \frac{dt}{dz}$. Similarly, the second transmission-line equation, is discretized at the load by averaging the load currents I_L in order to obtain a value that is located in time at the same point as I_{NDZ}^{n+1} as

$$V_{NDZ+1}^{n+1} = (\frac{R_L C}{F} + 1)^{-1} ((\frac{R_s C}{F} - 1) V_{NDZ+1}^n + 2 R_L I_{NDZ}^n + (V_L^{n+1} + V_L^n)) \tag{5}$$

As for the current, a simple Taylor's development takes place and so:

$$I_1^{n+1} = I_1^n - \frac{F}{L}(V_2^{n+1} - V_1^{n+1}) \tag{6}$$

• for $k = 2$:

$$\begin{aligned} V_2^{n+1} &= V_2^n - \frac{F}{C}\left(I_2^n - (V_s^{n+1} + V_s^n)\right) \\ I_2^{n+1} &= I_2^n - \frac{F}{L}\left(V_3^{n+1} - V_2^{n+1}\right) \end{aligned} \tag{7}$$

• for $k = 3 : NDZ$

$$\begin{aligned} V_k^{n+1} &= V_k^n - \frac{F}{C}\left(I_k^n - I_{k-2}^n\right) \\ I_k^{n+1} &= I_k^n - \frac{F}{L}\left(V_{k+1}^{n+1} - V_k^{n+1}\right) \end{aligned} \tag{8}$$

However, it is a conditionally stable technique since the maximum time-step size is limited by Courant-Friedrichs-Lewy (CFL) condition $\frac{dt}{dz} \leq \frac{1}{\nu}$ where ν is a propagation speed (in the case of a many conductors, this corresponds to the higher speed) see for example Paul's book [7].

2.1 Principle of classical beta method

However, the FDTD method is conditionally stable as technical as the maximum size of time is regulated by the Courant condition. For this reason, problems occur when the size of geometric features are much smaller than the wavelength. The Courant condition enforces small time-step sizes that oversample the signal, making the FDTD method computationally inefficient.
To analyze the transmission line as a whole and to better predict its output, a finite difference method for fourth order compact [9] may be applied. This method has a satisfactory accuracy but it also presents oscillations as will be seen in the numerical simulations. Now we use the β -schemes that were introduced in 1987 by Desideri et al. Their adaptation to the system (1) can be written as follows:
• for $k = 1$ and $k = N + 1$, the terms V_k^{n+1} and I_k^{n+1} remain unchanged, they are given by the formulas (2) and (3).
• for $k = 3 : NDZ - 1$:

$$V_k^{n+1} = V_k^n - \frac{F}{C}\left[-\frac{\beta}{4}I_{k+2}^n + \frac{1+\beta}{2}I_{k+1}^n - \frac{1+\beta}{2}I_{k-1}^n + \frac{\beta}{4}I_{k-2}^n\right] \tag{9}$$

$$I_k^{n+1} = I_k^n - \frac{F}{L}\left[-\frac{\beta}{4}V_{k+2}^n + \frac{1+\beta}{2}V_{k+1}^n - \frac{1+\beta}{2}V_{k-1}^n + \frac{\beta}{4}V_{k-2}^n\right] \tag{10}$$

• for $k = NDZ$, the terms V_k^{n+1} and I_k^{n+1} remain unchanged, they are given by the formulas (8).

3 Stability and order:

The use of the β-method may introduce certain instability problems; therefore the stability issue of the β-method demands careful studies. For simplicity, the stability issues are dealt with in this section primarily for the scheme of order 4. We first consider the stability of the scheme under the Courant condition. We, then examine the accuracy of the proposed high-order method.

Stability:

It's well known (see for example [10]), that the standard stability condition, by considering the Von Neumann condition for a uniform cartesian grid, can be given as:

$$\nu \frac{dt}{dz} \leq \min_{\theta \in [0,2\pi]} \frac{4}{|2(1+\beta)\sin(\theta) - \beta \sin(2\theta)|}$$

As the stability of the β-method or scheme was one of concerns. To this end we substitute values in scheme and after Taylor development to order 3, we rewrite the system (V, I) into the form:

$$\begin{cases} L\dfrac{\partial I(z,t)}{\partial t} = -\dfrac{\partial V(z,t)}{\partial z} - dz^2 \dfrac{\partial^3 V(z,t)}{\partial z^3}(\dfrac{1-3\beta}{6} - \dfrac{dt^2}{24LCdz^2}) + O(dz^4, dt^4) \\ C\dfrac{\partial V(z,t)}{\partial t} = -\dfrac{\partial I(z,t)}{\partial z} - dz^2 \dfrac{\partial^3 I(z,t)}{\partial z^3}(\dfrac{1-3\beta}{6} - \dfrac{dt^2}{24LCdz^2}) + O(dz^4, dt^4) \end{cases}$$

The β-scheme (9), (10) is second order in space and time on a uniform mesh, moreover, numerically it has been found that when $\beta < \frac{1}{3}$ and $\dfrac{dt}{dz} = \sqrt{4LC(1-3\beta)}$, the scheme is of order 4 in space and time.

Remark that the order of this scheme depend on $\frac{dt}{dz}$ which it self depends on β. A method where the dependence between these two falls will be introduced in what follows.

It would be unfortunate to use such a scheme with an integration time of lower order. We'll choose a scheme Runge - Kutta of order 2.

3.1 Proposed method

In order to eliminate the oscillations that are result of the hyperbolic characteristic of our system and to provide an accurate solution, we propose a

solved system by coupling the β-method scheme with the optimized time advancement Runge-Kutta algorithm (RK-2). Using the RK-2 formulation the mathematical system can be written as:
Compute of voltage:
- for $k = 1$:

$$V_1^{n+1} = (\frac{R_s C}{F} + 1)^{-1} \left[(\frac{R_s C}{F} - 1) V_1^n - 2 R_s I_1^{n+1} + (V_s^{n+1} + V_s^n) \right] \tag{11}$$

- for $k = 2$:

$$V_2^{n+1} = V_2^n - \frac{F}{2C}(I_3^n - I_1^n) \tag{12}$$

- for $k = 3 : NDZ$:

$$V_k^{n+1} = V_k^n - \frac{F}{4C}\left[-\frac{\beta}{2} I_{k+2}^n + (1+\beta) I_{k+1}^n - (1+\beta) I_{k-1}^n + \frac{\beta}{2} I_{k-2}^n \right] \tag{13}$$

- for $k = NDZ + 1$:

$$V_{NDZ+1}^{n+1} = (\frac{R_L C}{F} + 1)^{-1} \left[(\frac{R_s C}{F} - 1) V_{NDZ+1}^n + 2 R_L I_{NDZ}^{n+1} + (V_L^{n+1} + V_L^n) \right] \tag{14}$$

Compute of current:
- for $k = 1$:

$$I_1^{n+1} = I_1^n - \frac{F}{L}(V_2^n - I_1^n) \tag{15}$$

- for $k = 2$:

$$I_2^{n+1} = I_2^n - \frac{F}{2C}(V_3^n - V_1^n) \tag{16}$$

- for $k = 3 : NDZ$:

$$I_k^{n+1} = I_k^n - \frac{F}{4L}\left[-\frac{\beta}{2} V_{k+2}^n + (1+\beta) V_{k+1}^n - (1+\beta) V_{k-1}^n + \frac{\beta}{2} V_{k-2}^n \right] \tag{17}$$

Then we update the new value of current and voltage as follows:
- for $k = 3 : NDZ$:

$$V_k^{n+1} = V_k^n - \frac{F}{2C}(I_{k+1}^{n+1} - I_{k-1}^{n+1}) \tag{18}$$

$$I_k^{n+1} = I_k^n - \frac{F}{2L}(V_{k+1}^{n+1} - V_{k-1}^{n+1}) \tag{19}$$

3.2 Stability of the proposed model

The scheme is stable under the (CFL) condition:

$$\nu \frac{dt}{dz} \leq \min_{\theta\in[0,2\pi]} \frac{4}{|2(1+\beta)sin(\theta)-\beta sin(2\theta)|}$$

In fact, Von Neumann's condition will be applied.
We note : $Q_k^n = \begin{pmatrix} V_k^n \\ I_k^n \end{pmatrix}$, and we define its Fourier's transformation as $\hat{Q_k^n} = exp(ik\theta dz)g^n(\theta)$.
Basing on our scheme, one gets :

$$\hat{Q}_k^{n+1} = G_\theta \hat{Q}_k^n$$

Where

$$G_\theta = I_d + \begin{pmatrix} \frac{-\alpha^2F^2}{8LC} & \frac{-i\alpha F}{32L_C^2}(16LC-\alpha^2F^2) \\ \frac{-i\alpha F}{16L^2C}(8LC-\alpha^2F^2) & \frac{-\alpha^2F^2}{(8LC)^2}(16LC-\alpha^2F^2) \end{pmatrix}$$

Von Neumann's necessary and sufficient condition for stability, is that the amplification factor have to be bounded by 1. In our case, the condition is written:

$$\forall\theta\in[0,2\pi],\ \max_{i=1,2} \mid \lambda_\theta^i \mid\leq 1$$

where λ_θ^i represents none but G_θ's eigenvalues. Simple calculations leed to:

$$P_{G_\theta - I_d}(\lambda) = \lambda^2 + \frac{\alpha^2F^2}{(8LC)^2}(24LC-\alpha^2F^2)\lambda + \frac{\alpha^2F^2}{(8LC)^2}(16LC-\alpha^2F^2)$$

The problem indeed in Runge-Kutta methods presents its self in the difficulty of demonstrating, if exists, the stability of the scheme. Remark that when calculating the determinant of this equation, ones gets a six-order equation, which is hard to determine its sign, and so it is hard to study the (CFL) condition.

We propose a numerical method to avoid this hitch.

- Extracting the roots:
  Depending on Matlab's formel calculations, we're going to extract the roots of the equation using the function **solve**.

```
>> syms x aF L C
solve(x^2+ aF^2*(24*L*C-aF^2)*x/((8*L*C)^2) +...
    aF^2*(16*L*C-aF^2)/((8*L*C)^2),x)
```

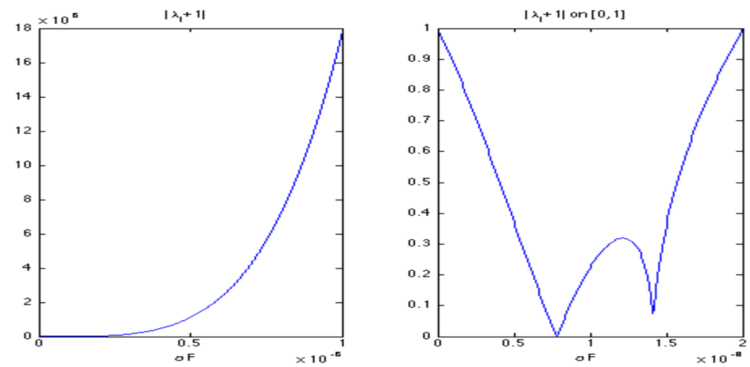

Figure 1: Stability region in the case of the β-scheme.

We get two roots that are conjugated: $\lambda^{1,2}$
We write a simple script, making the founded roots extensible on matrix calculations.

```
function y=l1(aF,L,C)
y=(aF.^4 - 24*C*L.*aF.^2+1i*64*aF.*(((aF.^2 - 8*C*L).*(512*C^2*L^2+...
    - 40*C*L*aF.^2 + aF.^4))./4096).^(1/2))./(128*C^2*L^2);
end
```

Remark: α and F have the same power, we can consider them as one variable. This will help us find the CFL condition on F in function of α.

- Ploting the roots:
  We consider a descritization of αF on $[0, 10^{-8}]$. $L = 0.25 * 10^{-6}$ and $C = 10^{-10}$ are the same constants in the real problem. We plot $\mid 1+\lambda^i \mid$:

- The CFL condition:
  In order to achieve the CFL condition, we have to bound αF so that :

$$\forall \theta \in [0, 2\pi], \max_{i=1,2} \mid \lambda^i + 1 \mid \leq 1 \tag{20}$$

hence

$$0 \leq \alpha F \leq 2.10^{-8} \tag{21}$$

therefore

$$F \leq \frac{2.10^{-8}}{\mid \alpha \mid} \tag{22}$$

and so

$$\frac{dt}{dz} \leq \min_{\theta\in[0,2\pi]} \frac{2.10^{-8}}{\mid 2(1+\beta)sin(\theta) - \beta sin(2\theta) \mid} \tag{23}$$

we get

$$\nu\frac{dt}{dz} \leq \min_{\theta\in[0,2\pi]} \frac{4}{\mid 2(1+\beta)sin(\theta) - \beta sin(2\theta) \mid} \tag{24}$$

where $\nu = 2.10^8$. This achieve the demonstration of the proposition.

4 Validation of results

4.1 Numerical studies for FDTD method

In this section we numerically investigate the performance of the proposed β -scheme for transmission line application Comparisons with the other considered schemes are shown. Let us consider a lossless transmission line with two rectangular conductors ($l = 400m$ length), placed with electrical reference for conducting ground plane perfectly. These conductors are excited by an echelon characterized by a rise time $Tr = 0.1e-6s$ and amplitude $30V$. The characteristic internal impedance of line is (50 Ω) and velocity $\nu = 2.10^8 m/s$. The linear parameters L, C are respectively $L = 0.25\mu H/m; C = 0.1\mu F/m$.
We first study this transmission line by considering the basic FDTD scheme, with the computational parameter NDZ and NDT. Each NDZ and NDT parameter is investigated in order to understand how it influences the accuracy and stability. The final simulation time Tf was $20\mu s$, resulting in $2000, 4000$ times divisions and 200 space divisions respectively.
We initially consider the magic time step occurs $dz = \nu dt$, therefore for $NDZ = 200$ or $dz = 2m$ we tack $NDT = 2000$ or $dt = 10ns$. This gives the exact solution to the system with no approximation error plotted in figure 2 (a), that good results are produced, exactly same those for branin method [7]. Thus for $NDZ = 200$ and $NDT = 4000$ or $dt = 5ns$, the(CFL) condition is satisfied but we can observe less ringing on the leading edge of each transition. These parameters present the worst accuracy and stability plotted in figure 2 (b) .

We next study the same transmission line by considering the β-scheme, In the present approach it's interesting to compare the plots for different computational parameter In order to examine the CFL condition. By using the computational parameter NDZ=200, NDT=2000,4000, the numerical results of this case are given in figure 3 (a) and figure 4 (a). We note that the β-scheme produces ringing on the leading edge of each transition.

Finally, figures 3(b) and 4(b) show that the combination of high-order schemes with Runge-Kutta schemes achieve a very high accuracy and stability.

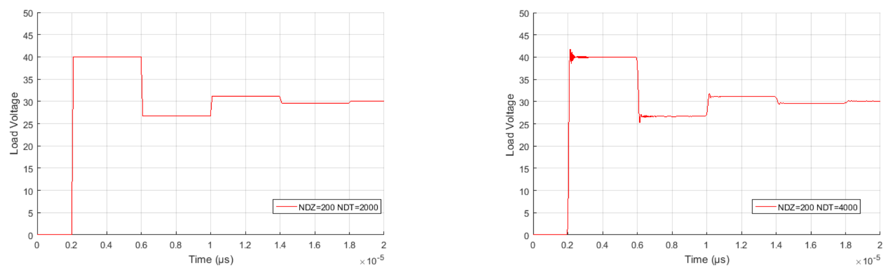

Figure 2: Illustration of the load voltage by using different values of computational parameters : (a) time magic condition $NDZ = 200$ and $NDT = 2000$ (b) $NDZ = 200$ and $NDT = 4000$.

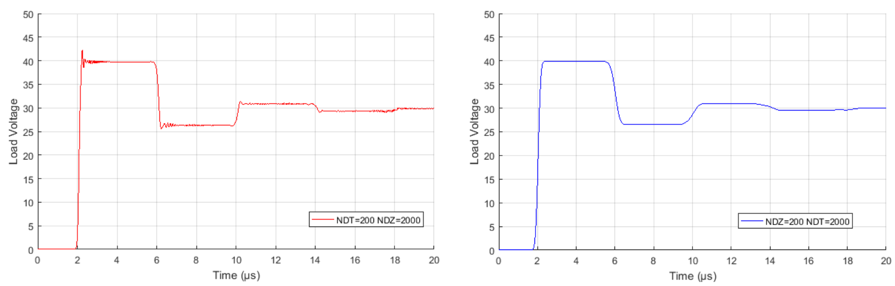

Figure 3: Comparison between the load voltage obtained by the classical β-method (left figure (a)) and the proposed method (figure(b))

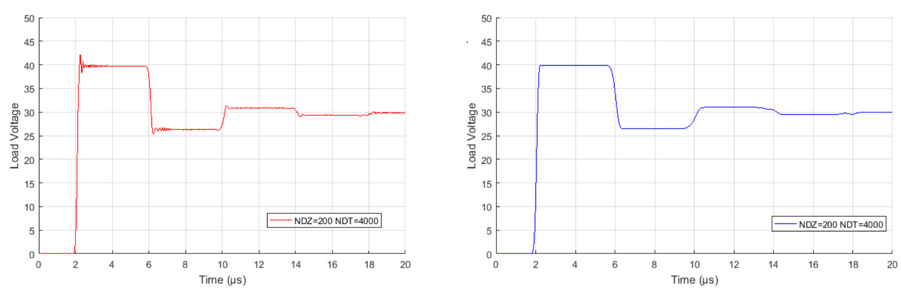

Figure 4: Comparison between the load voltage obtained by the classical β-method (left figure(a)) and the proposed method (figure(b))

It is to note that the proposed method with different NDT and NDZ lead to better result independently of the choice of the time step. In fact the choice of the parameters need to only satisfy the standard CFL condition.

5 Conclusion and Perspective

In this work we are interested in the simulation of the transmission line model with two conductors. The conventional method and the most popular one used for the treatment of this type of system is the FDTD method. This method allows to model variable losses (skin effect) in the line, but its main drawback is its low order approximation besides the step size of the spatial grid and the time step must meet a very tight condition (time magic) to avoid the risk of obtaining a diverged numerical solution. As a remedy, we have developed a new numerical scheme which is actually a generalization of the conventional beta method. We demonstrated that this new algorithm is stable under a relaxed CFL condition. The simulation results are compared to those of the FDTD and beta conventional method. They confirm the accuracy, robustness and stability of the proposed method. In particular, these simulations show that our new model is high order in space and time. In the near future we will develop an algorithm for the simulation of transmission lines with losses ie when R and G are non-zero in equation (1). Inspired by the technique developed in [7] for the FDTD method and based on Proni's factors and the algorithm that we have developed here.

Acknowledgement

The authors are grateful to LAMAI and LSET laboratories of Cadi Ayyad University for their financial support.
The authors are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

References

[1] David B. Davidson, Computational Electromagnetics for RF for RF and Microwave Engineering Second Cambridge university Press, 2011.

[2] J. Clerk Maxwell, A Treatise on Electricity and Magnetism, 3rd ed., vol. 2. Oxford: Clarendon, 1892, pp.68-73.

[3] Erturk E, Corke TC, Gokcol C.numerical solutions of 2- D steady incompressible driven cavity flow at high Reynolds numbers. International journal for Numerical Methods in fluids 2005; 48:747-774.

[4] S. Zhao and G.W. Wei, High-order FDTD methods via derivative matching for Maxwell's equations with material interfaces, Journal of Computational Physics 2004, 200, pp. 60-103.

[5] J. Fang, Time domain finite difference computation for Maxwell's equations, Ph.D. dissertation, Univ. California, Berkeley, CA, 1989.

[6] Kunz, Karl S and Luebbers, Raymond J. The finite difference time domain method for electromagnetics, CRC press 1993.

[7] Clayton R. Paul, Introduction to ElectroMagnetic Compatibility, (second Edition), Wiley Series in Microwave and Optical Engineering, Kai Chang, Series Editor 1992.

[8] Taflove, Allen and Hagness, Susan C, Computational electrodynamics, Artech house publishers 2000.

[9] N. Canouet, L. Fézoui, S. Piperno, Méthode volumes finis pour la résolution du système de Maxwell 1D sur des grilles raffinées localement, INRIA report N°4301, October 2001.

[10] S. Piperno, Schémas TVD d'ordre élevé pour la résolution de l'équation de Burgers, , INRIA report N°4301, December 1996.

[11] J. A. Desideri, A. Goudjo, V. Selmin, Third order schems for hyperbolic problems, INRIA report N°607, 1987.

[12] Heaviside, O. (1951). Electromagnetic Theory, Volume I (first edition 1893; second edition 1951), London: E. and F.N. SPON LTD.

[13] Franceschetti, G. (1997). Electromagnetics, New York: Plenum Press.

[14] Yee K.S., Numerical solution of initial boundary value problems involving Maxwell's equation in isotropic media, IEEE Trans. Antennas Propagat. 14 (1966) 302.

Fatima Allali
University Cadi Ayyad, Laboratory LSET Morocco, Marrakech
fatimallali@gmail.com
NourEddine Alaa
University Cadi Ayyad, Laboratory LAMAI Morocco, Marrakech
n.alaa@uca.ac.ma
Abdelilah GHAMMAZ
University Cadi Ayyad, Laboratory LSET Morocco, Marrakech
aghammaz@yahoo.fr
Hicham ROUIJAA
University Cadi Ayyad, Laboratory LSET Morocco, Marrakech
hrouijaa@hotmail.com