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A note on composition (m, n)-hyperrings

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Abstract

Based on the concepts of composition ring and composition hyperring, in this note we introduce the notion of composition structure for (m, n)-hyperrings and study the connections with composition hyperrings. Moreover we show that particular strong endomorphisms of (m, n)-hyperrings can determine the composition structure of a such (m, n)-hyperrings. Finally, the three isomorphism theorems are presented in the case of composition (m, n)-hyperrings, showing that they are not a pure extension of those for composition hyperrings.

1 Introduction

Today one area of big interest for researchers working on algebraic hyperstructures is represented by the n-ary hyperstructures, since it has been proved they have many applications to computer science, coding theory, topology, combinatorics and quantum physic [9]. They are a generalization of classical algebraic hyperstructures, with a lot of applications in Euclidean and non Euclidean geometries, graphs and hypergraphs, binary relations, lattices, automata, cryptography, coding theory, artificial intelligence, probabilities, chemistry and so on (for more details see [3], [4], [9], [23]).

One type of these n-ary algebraic hyperstructures is represented by (m, n)-hyperrings, based on the notion of n-ary hypergroups, introduced by Davvaz and Vougiouklis [11] as a generalization of the concept of hypergroups, defined

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by Marty [17] in 1934, and a generalization of n-ary groups, defined by Dörnte [12] in 1928. On the other hand, they can be seen as an extension of (m, n)-rings [5], [6] in the framework of hyperstructure theory. Many applications of n-ary hypergroups and (m, n)-hyperrings were established and studied in connection with hyperideals [2], fundamental relations [8], [19], or binary relations [15], [16], [18].

Adler's paper [9] on composition rings after 50 years has opened a new line of research in the hyperrings framework. In the first work on this topic, Cristea and Jančić-Rašović [7] defined and studied the composition hyperrings, emphasizing their interesting properties in relation with endomorphisms of hyperrings. This work can be extended to other two directions: the first one, the topic of this note, deals with composition (m, n)-hyperrings, while the second one (subject investigated in [21]) with n-ary composition hyperrings, i.e. hyperrings endowed with a composition, that is an n-ary hyperoperation. Combining both directions, one can obtained the so called composition (m, n, k)-hyperrings. They are (m, n)-hyperrings with a k-ary hyperoperation called composition. This general case was recently considered and investigated by Davvaz et al. [10], but just from the perspective of isomorphism theorems. Even if the above mentioned article was published after we finished to write our two manuscripts (the current one and the submitted one [21]), for a better understanding of the subject, we prefer to divide our work into two parts: composition (m, n)-hyperrings (studied in the current note), that are $(m, n, 2)$-hyperrings, and n-ary composition hyperrings [21], that are $(2, 2, n)$-hyperrings, using the notation in [10]. In our opinion, the terminology composition (m, n, k)-hyperrings doesn't reflect at the first sight the algebraic structure of the considered hyperrings, in the sense that they are (m, n)-hyperrings endowed with a k-ary composition hyperoperation.

Motivated by these aspects, in Section 2 we recall some basic concepts concerning n-ary hyperstructures necessary for our proposes. In Section 3 we give several examples illustrating our definition and investigate connections between composition (m, n)-hyperrings and composition hyperrings. Besides, we show how composition (m, n)-hyperrings can be determined by endomorphisms of (m, n)-hyperrings. Finally, for the completeness of the study, Section 4 is dedicated to the presentation of the three isomorphism theorems of composition (m, n)-hyperrings, as a particular case of those stated in [10]. Several integrative lemmas are included. We conclude the paper with some remarks connecting the papers already written on this argument and with some proposals of future work.

2 Preliminaries

A mapping $f : \underbrace{H \times \cdots \times H}_{n} \longrightarrow \mathcal{P}^*(H)$ is called an *n-ary hyperoperation*, where $\mathcal{P}^*(H)$ is the set of all the nonempty subsets of H. An algebraic system (H, f), where f is an n-ary hyperoperation defined on H, is called an *n-ary hypergroupoid*.

The sequence $x_i, x_{i+1}, \ldots, x_j$ will be denoted by x_i^j. For $j < i$, x_i^j is the empty set. Using this notation,

$$f(x_1, \ldots, x_i, y_{i+1}, \ldots, y_j, z_{j+1}, \ldots, z_n)$$

will be written as $f(x_1^i, y_{i+1}^j, z_{j+1}^n)$. In the case when $y_{i+1} = \cdots = y_j = y$ the last expression will be written $f(x_1^i, y^{(j-i)}, z_{j+1}^n)$.

If f is an n-ary hyperoperation and $t = l(n-1)+1$, for some $l \geq 0$, then t-ary hyperoperation $f_{(l)}$ is given by

$$f_{(l)}(x_1^{l(n-1)+1}) = \underbrace{f\Big(f\Big(}_{l} \ldots, f\big(f(x_1^n), x_{n+1}^{2n-1}\big), \ldots\Big), x_{(l-1)(n-1)+1}^{l(n-1)+1}\Big).$$

For nonempty subsets $A_1, \ldots, A_n$ of H we define

$$f(A_1^n) = f(A_1, \ldots, A_n) = \bigcup \{f(x_1^n) \mid x_i \in A_i, i = 1, \ldots, n\}.$$

An n-ary hyperoperation f is called *associative* if

$$f\Big(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}\Big) = f\Big(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1}\Big),$$

holds, for every $1 \leq i < j \leq n$ and all $x_1, x_2, \ldots, x_{2n-1} \in H$. An n-ary hypergroupoid with the associative n-ary hyperoperation is called an *n-ary semihypergroup*.

An n-ary hypergroupoid (H, f) in which the equation $b \in f(a_1^{i-1}, x_i, a_{i+1}^n)$ has a solution $x_i \in H$, for every $a_1^{i-1}, a_{i+1}^n, b \in H$ and $1 \leq i \leq n$, is called an *n-ary quasihypergroup*. If (H, f) is an n-ary semihypergroup and an n-ary quasihypergroup, then it is called an *n-ary hypergroup*. An n-ary hypergroupoid (H, f) is *commutative* if, for all $\sigma \in \mathbb{S}_n$ and for every $a_1^n \in H$, we have $f(a_1, \ldots, a_n) = f(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$. If $a_1^n \in H$ then we denote $(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$ by $a_{\sigma(1)}^{\sigma(n)}$.

Definition 2.1. ([18]) Let (H, f) be an n-ary hypergroup and B a nonempty subset of H. B is called an *n-ary subhypergroup* of (H, f), if $f(x_1^n) \subseteq B$, for all $x_1^n \in B$, and the equation $b \in f(b_1^{i-1}, x_i, b_{i+1}^n)$ has a solution $x_i \in B$, for all $b_1^{i-1}, b_{i+1}^n, b \in B$ and $1 \leq i \leq n$.

Definition 2.2. ([18]) Let (H,f) be a commutative n-ary hypergroup. (H,f) is called a *canonical n-ary hypergroup*, if

(1) there exists a unique $e \in H$, such that, for every $x \in H$, $f(x, e^{n-1}) = \{x\}$;

(2) for all $x \in H$, there exists a unique $x^{-1} \in H$, such that $e \in f(x, x^{-1}, e^{n-2})$;

(3) if $x \in f(x_1^n)$, then, for all $1 \leq i \leq n$, we have the following relation $x_i \in f(x, x^{-1}, \dots, x_{i-1}^{-1}, x_{i+1}^{-1}, \dots, x_n^{-1})$.

Definition 2.3. ([18]) An *(m,n)-hyperring* is an algebraic hyperstructure (R,f,g) which satisfies the following axioms:

(1) (R,f) is an m-ary hypergroup.

(2) (R,g) is an n-ary semihypergroup.

(3) The n-ary hyperoperation g is distributive with respect to the m-ary hyperoperation f, i.e., for all $a_1^{i-1}, a_{i+1}^n, x_1^m \in R$, and $1 \leq i \leq n$,

$$g\Big(a_1^{i-1}, f(x_1^m), a_{i+1}^n\Big) = f\Big(g(a_1^{i-1}, x_1, a_{i+1}^n), \dots, g(a_1^{i-1}, x_m, a_{i+1}^n)\Big).$$

A nonempty subset S of R is called an *(m,n)-subhyperring*, if (S,f,g) is an (m,n)-hyperring. Let $i \in \{1,\dots,n\}$. An *i-hyperideal* I of R is an (m,n)-subhyperring of R such that $g(x_1^{i-1}, I, x_{i+1}^n) \subseteq I$, for every $x_1^n \in R$. I is called a *hyperideal*, if I is an i-hyperideal, for all $1 \leq i \leq n$.

Example 2.4. *Consider the set of all integers, $\mathbb{Z}$, with the hyperoperations defined as $x \oplus y = \{x, y, x+y\}$ and $x \otimes y = \{x \cdot y\}$, for all $x, y \in \mathbb{Z}$, where "+" and "·" are ordinary addition and multiplication. Then it is routine to see that $(\mathbb{Z}, \oplus, \otimes)$ is a hyperring. For $x_1^m, y_1^n \in \mathbb{Z}$, set $g(y_1^n) = \bigotimes_{i=1}^{n} y_i = \{\prod_{j=1}^{n} y_j\}$ and*

$$f(x_1^m) = \bigoplus_{i=1}^{m} x_i = \Big\{x_1^m,\ x_{i_1} + x_{i_2},\ \dots\ ,\ x_{i_1} + x_{i_2} + \dots + x_{i_m}\Big\}$$

such that $i_1, i_2, \dots, i_m$ are different natural numbers from 1 to m. Then, $(\mathbb{Z}, f, g)$ is an (m,n)-hyperring.

Let (R,f,g) and (T,f',g') be two (m,n)-hyperrings. A map $\phi : R \longrightarrow T$ is called a *homomorphism* from R to T if, for all $x_1^m, y_1^n \in R$, the following conditions are valid:

1. $\bigcup_{u\in f(x_1^m)} \phi(u) \subseteq f'\big(\phi(x_1),\dots,\phi(x_m)\big)$

2. $\bigcup_{v\in g(y_1^n)} \phi(v) \subseteq g'\big(\phi(y_1),\dots,\phi(y_n)\big)$

If the equalities are valid in the above conditions, then ϕ is called a *strong homomorphism*. A homomorphism from R to R is called *endomorphism* of R. If ϕ_1 and ϕ_2 are endomorphisms on a hyperring R, then their composition $\phi_1\circ\phi_2$, defined by $(\phi_1\circ\phi_2)(x)=\bigcup_{a\in\phi_2(x)}\phi_1(a)$, is also an endomorphism on R.

3 Composition (m,n)-hyperrings

In this section, we introduce the composition (m,n)-hyperrings and give several examples of them. Using the terminology in [10], they are composition $(m,n,2)$-hyperrings. Besides, connections between composition hyperrings and composition (m,n)-hyperrings are established and investigated. In particular, we show how composition (m,n)-hyperrings can be determined by their particular endomorphisms.

Definition 3.1. An algebraic hyperstructure $(R,f,g,\circ)$ is called a *composition (m,n)-hyperring* if the following statements are satisfied.

(1) (R,f,g) is a commutative (m,n)-hyperring.

(2) $(R,\circ)$ is a semihypergroup.

(3) For all $x_1^m, y_1^n, z\in R$, the following properties hold:

$$f(x_1^m)\circ z = f(x_1\circ z,\dots,x_m\circ z) \quad \text{and} \quad g(y_1^n)\circ z = g(y_1\circ z,\dots,y_n\circ z).$$

Remark 3.2. *For $m=n=2$ we get that $(R,f,g,\circ)$ is a composition hyperring, defined in [7]. For this reason throughout this paper, when we talk about a composition (m,n)-hyperring we intend $(m,n)\neq(2,2)$.*

Let $(R,f,g,\circ)$ be a composition (m,n)-hyperring. An element $c\in R$ is called a *constant*, if $c\circ x=c$, for all $x\in R$. If A is an arbitrary subset of R, the set of all constants in A is called a *foundation* of A, denoted by Found(A).

The next theorem presents a method to construct a composition (m,n)-hyperring from a composition hyperring.

Theorem 3.3. (**Construction theorem**) *Every composition hyperring leads to a composition (m,n)-hyperring.*

Proof. Let $(R, +, \cdot, \circ)$ be a composition hyperring. For $x_1^m, y_1^n \in R$, define

$$f(x_1^m) = \sum_{i=1}^{m} x_i \quad \text{and} \quad g(y_1^n) = \prod_{j=1}^{n} y_j.$$

Since $(R, +, \cdot, \circ)$ is a composition hyperring, it is not difficult to see that the three assertions of Definition 3.1 are valid for $(R, f, g, \circ)$. □

An (m, n)-hyperring (R, f, g) is called *Krasner*, if (R, f) is a canonical m-ary hypergroup and (R, g) is an n-ary semigroup with the absorbing element 0, such that $g(x_1^{i-1}, 0, x_{i+1}^n) = 0$, for all $x_1^n \in R$ (see [18]). Under this hypothesis we obtain the following theorem.

Theorem 3.4. *Let $(R, f, g, \circ)$ be a composition (m, n)-hyperring such that (R, f, g) is a Krasner (m, n)-hyperring. Then, $(R, f, g, \circ)$ will derive a composition $(2, n)$-hyperring.*

Proof. Define $x + y = f\big(x, y, 0^{(m-2)}\big)$, for every $x, y \in R$. It is clear that " + " is commutative and associative. Also, 0 is a scalar neutral and a zero element of $(R, +, g, \circ)$. It is easy to see that the n-ary operation g is distributive with respect to the hyperoperation " + ". Therefore $(R, +, g, \circ)$ is a composition $(2, n)$-hyperring. □

We present here several examples, illustrating the given definitions and results.

Example 3.5. *Let $(R, +, \cdot)$ be a commutative hyperring. Consider*

$$R[[x]] = \{(a_0, a_1, \ldots, a_n, \ldots) \mid a_i \in R\},$$

the set of all infinite sequences $(a_0, a_1, \ldots, a_n, \ldots)$ with coefficients in R, with the following hyperoperations:

$$(a_0, a_1, \ldots, a_n, \ldots) \oplus (b_0, b_1, \ldots, b_n, \ldots) = \Big\{(c_0, c_1, \ldots, c_n, \ldots) \mid c_k \in a_k + b_k\Big\}$$

$$(a_0, a_1, \ldots, a_n, \ldots) \odot (b_0, b_1, \ldots, b_n, \ldots) = \Big\{(c_0, c_1, \ldots, c_n, \ldots) \mid c_k \in \sum_{i+j=k} a_i b_j\Big\}.$$

By [7], $(R[[x]], \oplus, \odot)$ is a hyperring, and also if there exists $0 \in R$ such that $0 + 0 = \{0\}$ and $a \cdot 0 = \{0\}$, for all $a \in R$, then $(R[x], \oplus, \odot)$ is a subhyperring of $R[[x]]$, where

$R[x] = \{(a_0, a_1, \ldots, a_n, \ldots) \in R[[x]] \mid a_i = 0$ *except a finite number of indices* $i\}$

is a subhyperring of $R[[x]]$. *Also, consider the hyperoperation "* $\circ$ *" as follows:*

$$h \circ l = a_0 \oplus (a_1 \odot l) \oplus \ldots \oplus (a_n \odot l^n),$$

where $h = (a_0, a_1, \ldots, a_n, \ldots) \in R[x]$ *such that* $a_k = 0$*, for all* $k \geq n+1$*, and* $l \in R[x]$ *and also* $(a_i, 0, \ldots, 0, \ldots)$ *is denoted by short by* a_i. *Then by [7],* $(R[x], \oplus, \odot, \circ)$ *is a composition hyperring. Now, define the following m-ary and n-ary hyperoperations on* $R[[x]]$*:*

$$f\Big((a_{01}, a_{11}, \ldots, a_{n1}, \ldots), \ldots, (a_{0m}, a_{1m}, \ldots, a_{nm}, \ldots)\Big) = \bigoplus_{i=1}^{m} (a_{0i}, a_{1i}, \ldots, a_{ni}, \ldots)$$

$$g\Big((a_{01}, a_{11}, \ldots, a_{n1}, \ldots), \ldots, (a_{0n}, a_{1n}, \ldots, a_{nn}, \ldots)\Big) = \bigodot_{j=1}^{n} (a_{0j}, a_{1j}, \ldots, a_{nj}, \ldots)$$

Hence, by Theorem 3.3, $(R, f, g, \circ)$ *is a composition* (m,n)*-hyperring with* $Found(R[x]) = R$.

Example 3.6. *Let* (R, f, g) *be a commutative* (m,n)*-hyperring. Consider the set* $R[[x]]$ *in Example 3.5, and for* $(a_{0j}, a_{1j}, \ldots, a_{tj}, \ldots) \in R[[x]]$ *such that* $1 \leq j \leq m, n$*, define the following m-ary and n-ary hyperoperations: (Notice that hereafter, for brevity, a sequence of elements of* $R[[x]]$ *such as "*$(a_{01}, a_{11}, \ldots, a_{t1}, \ldots), \ldots, (a_{0m}, a_{1m}, \ldots, a_{tm}, \ldots)$*" is denoted, for all* $m \in \mathbb{N}$*, by* $(a_0, a_1, \ldots, a_t, \ldots)_1^m$.*)*

$$F\Big((a_0, a_1, \ldots, a_t, \ldots)_1^m\Big) = \Big\{(c_0, c_1, \ldots, c_t, \ldots) \mid c_k \in f(a_{k1}, a_{k2}, \ldots, a_{km})\Big\}$$
$$G\Big((a_0, a_1, \ldots, a_t, \ldots)_1^n\Big) = \Big\{(d_0, d_1, \ldots, d_t, \ldots) \mid d_k \in f_{(k)}\big(g(a_{i_1 1}, \ldots, a_{i_n n})^{(z)}\big)\Big\},$$

where $i_1 + \ldots + i_n = k$ *and* $z = k(m-1)+1$. *Then, it is routine to verify that* $(R[[x]], F, G)$ *is an* (m,n)*-hyperring. Also, if there exists* $0 \in R$ *such that*

$$f(a_i, 0^{(m-1)}) = \{a_i\} \quad \text{and} \quad g(a_1^{i-1}, 0, a_{i+1}^n) = \{0\},$$

for all $a_1^n \in R$*, then it can be seen that*

$$R[x] = \Big\{(a_0, a_1, \ldots, a_t, \ldots) \mid \exists t \text{ such that } \forall k \geq t+1,\ a_k = 0\Big\}$$

is an (m,n)*-subhyperring of* $R[[x]]$. *Now, suppose there exists* $1 \in R$ *such that, for every* $a \in R$*, we have* $g(a, 1_R^{(n-1)}) = \{a\}$. *Take* $h = (a_0, a_1, \ldots, a_t, \ldots) \in R[x]$ *such that* $a_k = 0$*, for all* $k \geq t+1$*, and* $l \in R[x]$. *Define the hyperoperation*

" $*$ " as follows: (the sequence $(r, 0, 0, \ldots, 0, \ldots)$ is denoted by "r")

$$h{*}l = \begin{cases} F\Big(a_0, G(a_1, l, 1_R^{(n-2)}), \ldots, G\big(a_t, G(l^{(t)}, 1_R^{(n-t)}), 1_R^{(n-2)}\big), 0^{(m-t)}\Big), \text{if } \ t < m, n \\ F_{(k)}\Big(a_0, G(a_1, l, 1_R^{(n-2)}), \ldots, G\big(a_t, G_{(k')}(l^{(t)}), 1_R^{(n-2)}\big)\Big), \text{if } \ t > m, n \\ F_{(k)}\Big(a_0, G(a_1, l, 1_R^{(n-2)}), \ldots, G\big(a_t, G(l^{(t)}, 1_R^{(n-t)}), 1_R^{(n-2)}\big)\Big), \text{if } \ m < t < n \\ F\Big(a_0, G(a_1, l, 1_R^{(n-2)}), \ldots, G\big(a_t, G_{(k')}(l^{(t)}), 1_R^{(n-2)}\big)\Big), \ \text{if } \ n < t < m \end{cases}$$

such that $t = k(m-1)+1$ and $t = k'(n-1)+1$. Let $t > m, n$. Then

$$\begin{aligned} &\Big\{(a_0, a_1, \ldots, a_t, \ldots)\Big\} = \Big\{(c_0, c_1, \ldots, c_t, \ldots) \mid c_i \in \{a_i\}\Big\} \\ &= \Big\{(c_0, c_1, \ldots, c_t, \ldots) \mid c_i \in f_{(k)}(a_i, 0^{(t-1)})\Big\} \\ &= F_{(k)}\Big((a_0, 0, \ldots, 0, \ldots), (0, a_1, 0, \ldots, 0, \ldots), \ldots, (0, 0, \ldots, 0, a_t, 0, \ldots)\Big), \end{aligned}$$

also, for all $i \in \{0, 1, ..., t\}$, we have

$$\begin{aligned} F\Big((0, \ldots, 0, a_i, 0, \ldots)_1^m\Big) &= \Big\{(0, \ldots, 0, c_i, 0, \ldots) \mid c_i \in f(a_{i1}^{im})\Big\} \\ &= (0, \ldots, 0, f(a_{i1}^{im}), 0, \ldots). \end{aligned}$$

Hence,

$$\begin{aligned} F\big((a_0, \ldots, a_t, \ldots)_1^m\big) &= F\Big(\big\{(a_{01}, \ldots, a_{t1}, \ldots)\big\}, \ldots, \big\{(a_{0m}, \ldots, a_{tm}, \ldots)\big\}\Big) \\ &= F\Big(F_{(k)}\big((a_{01}, 0, \ldots), \ldots, (0, \ldots, 0, a_{t1}, \ldots)\big) \\ &\quad , \ldots, F_{(k)}\big((a_{0m}, 0, \ldots), ..., (0, \ldots, 0, a_{tm}, \ldots)\big)\Big) \\ &= F_{(k)}\Big(F\big((a_0, 0, \ldots)_1^m\big), \ldots, F\big((0, \ldots, 0, a_t, 0, \ldots)_1^m\big)\Big) \\ &= F_{(k)}\Big((f(a_{01}^{0m}), 0, \ldots), \ldots, (0, \ldots, 0, f(a_{t1}^{tm}), \ldots)\Big) \\ &= \Big(f(a_{01}^{0m}), \ldots, f(a_{t1}^{tm}), \ldots\Big). \end{aligned}$$

Therefore, for $t > m, n$ such that in all sequences $(a_0, \ldots, a_t, \ldots)_1^m \in R[x]$, for

$k \geq t+1$ and $1 \leq i \leq m$, $a_{ki}=0$, and $l \in R[x]$ we have

$$F\big((a_0,\ldots,a_t,\ldots)_1^m\big)*l = \big(f(a_{01}^{0m}),\ldots,f(a_{t1}^{tm}),\ldots\big)*l$$
$$= F_{(k)}\Big(f(a_{01}^{0m}), G(f(a_{11}^{1m}),l,1_R^{(n-2)}),\ldots,G\big(f(a_{t1}^{tm}),G_{(k')}(l^{(t)}),1_R^{(n-2)}\big)\Big)$$
$$= F_{(k)}\Big(F\big((a_{01},0,\ldots),\ldots,(a_{0m},0,\ldots)\big),G(F\big((a_{11},0,\ldots),\ldots,(a_{1m},0,\ldots)\big),l,1_R^{(n-2)}),$$
$$\ldots,G\big(F\big((a_{t1},0,\ldots),\ldots,(a_{tm},0,\ldots)\big),G_{(k')}(l^{(t)}),1_R^{(n-2)}\big)\Big)$$
$$= F_{(k)}\Big(F(a_{01}^{0m}),F\big(G\big(a_{11},l,1_R^{(n-2)}\big),\ldots,G\big(a_{1m},l,1_R^{(n-2)}\big)\big),\ldots,$$
$$F\Big(G\big(a_{t1},G_{(k')}(l^{(t)}),1_R^{(n-2)}\big),\ldots,G\big(a_{tm},G_{(k')}(l^{(t)}),1_R^{(n-2)}\big)\Big)\Big)$$
$$= F\Big(F_{(k)}\Big(a_{01},G\big(a_{11},l,1_R^{(n-2)}\big),\ldots,G\big(a_{t1},G_{(k')}(l^{(t)}),1_R^{(n-2)}\big)\Big),\ldots,$$
$$F_{(k)}\Big(a_{0m},G\big(a_{1m},l,1_R^{(n-2)}\big),\ldots,G\big(a_{tm},G_{(k')}(l^{(t)}),1_R^{(n-2)}\big)\Big)\Big)$$
$$= F\Big((a_{01},a_{11},\ldots,a_{t1},\ldots)*l,\ldots,(a_{0m},a_{1m},\ldots,a_{tm},\ldots)*l\Big).$$

Similarly, the related relation is valid also for "G". Moreover, we can show that the assertion (3) *of Definition 3.1 is valid for other conditions of the definition of the hyperoperation "$*$" and the other types of sequences in $R[x]$. Besides, for $h=(a_0,a_1,\ldots,a_t,\ldots) \in R[x]$ such that $t > m,n$, and $a_k = 0$ for $k \geq t+1$, and $l,b \in R[x]$, we have $G(0,b,1_R^{(n-2)}) = \{0\}$, and so*

$$\begin{aligned}1_R * b &= (1_R,0,\ldots,0,\ldots)*b\\ &= F_{(k)}\Big(1_R, G\big(0,b,1_R^{(n-2)}\big),...,G\big(0,G_{(k')}(b^{(t)}),1_R^{(n-2)}\big)\Big)\\ &= F_{(k)}\big((1_R,0,\ldots,0,\ldots),(0,\ldots,0,\ldots)^{(t-1)}\big)\\ &= \Big\{(c_0,c_1,\ldots,c_d,\ldots) \mid c_0 \in f_{(k)}(1_R,0^{(t-1)}), c_1 \in f_{(k)}(0^{(t)}),\ldots,c_d \in f_{(k)}(0^{(t)})\Big\}\\ &= \big\{(1_R,0,\ldots,0,\ldots)\big\} = \{1_R\}.\end{aligned}$$

Hence, for $a \in R$ we have $a*b = (a,0,\ldots,0,\ldots)*b = \{a\}$. Thus, by validity of assertion (3) *of Definition* 3.1*, we conclude that*

$$(h*l)*b = \Big(F_{(k)}\Big(a_0,G\big(a_1,l,1_R^{(n-2)}\big),\ldots,G\big(a_t,G_{(k')}(l^{(t)}),1_R^{(n-2)}\big)\Big)\Big)*b =$$
$$F_{(k)}\Big(a_0*b,G\big(a_1*b,l*b,(1_R*b)^{(n-2)}\big),\ldots,G\big(a_t*b,G_{(k')}(((l*b)^{(t)}),(1_R*b)^{(n-2)}\big)\Big)$$
$$= F_{(k)}\Big(a_0,G\big(a_1,l*b,1_R^{(n-2)}\big),\ldots,G\big(a_t,G_{(k')}((l*b)^{(t)}),1_R^{(n-2)}\big)\Big)$$
$$= h*(l*b).$$

Therefore, $(R[x],*)$ is a semihypergroup. It implies that $(R[x],F,G,*)$ is a composition (m,n)-hyperring.

Example 3.7. *Let (R,f,g) be an arbitrary commutative (m,n)-hyperring and "$\circ$" be defined by $r \circ s = \{r\}$, for all $r,s \in R$. Then it is not difficult to verify that $(R,f,g,\circ)$ is a composition (m,n)-hyperring with Found$(R) = R$.*

In the following, we study the relationship between a composition (m, n)-hyperring and a certain class of its strong endomorphisms. This is another method to define composition (m, n)-hyperrings.

Theorem 3.8. *Let $(R, f, g, \circ)$ be a composition hyperring. For any element $y \in R$, the function $\Phi_y : R \longrightarrow \mathcal{P}^*(R)$ defined by $\Phi_y(x) = x \circ y$, for all $x \in R$, is a strong endomorphism of the hyperring R. Moreover, for a nonempty subset M of R, set $\Phi_M(x) = \bigcup_{m\in M} \Phi_m(x)$, for all $x \in R$. Then we have*

$$\Phi_{\Phi_x(y)}(z) = \bigcup_{t\in\Phi_y(z)} \Phi_x(t), \quad \forall x, y, z \in R. \tag{1}$$

Proof. Let $(R, f, g, \circ)$ be a composition (m, n)-hyperring and let $y \in R$. By the definition of the function Φ_y, for all $a, b \in R$, we have

$$\begin{aligned}\Phi_y(f(x_1^m)) = \bigcup_{u\in f(x_1^m)} \Phi_y(u) &= \bigcup_{u\in f(x_1^m)} u \circ y \\ &= f(x_1^m) \circ y \\ &= f(x_1 \circ y, \ldots, x_m \circ y) \\ &= f(\Phi_y(x_1), \ldots, \Phi_y(x_m)).\end{aligned}$$

Similarly, we have $\Phi_y(g(y_1^n)) = g(\Phi_y(y_1), \ldots, \Phi_y(y_n))$. Thus, Φ_y is a strong endomorphism of the (m, n)-hyperring (R, f, g). Moreover, for all $x, y, z \in R$, we have

$$\Phi_{\Phi_x(y)}(z) = \bigcup_{s\in y\circ x} \Phi_s(z) = \bigcup_{s\in y\circ x} z\circ s = z\circ(y\circ x) = (z\circ y)\circ x = \bigcup_{t\in z\circ y} t\circ x = \bigcup_{t\in\Phi_y(z)} \Phi_x(t).$$

□

Theorem 3.9. *Let (R, f, g) be a commutative (m, n)-hyperring and $(\Phi_y)_{y\in R}$ a family of its strong endomorphisms satisfying the equation*

$$\Phi_{\Phi_x(y)}(z) = \bigcup_{t\in\Phi_y(z)} \Phi_x(t),$$

for all $x, y, z \in R$. Define the hyperoperation "$\circ$" as $x \circ y = \Phi_y(x)$, for every $x, y \in R$. Then $(R, f, g, \circ)$ is a composition (m, n)-hyperring.

Proof. By assumption, for $x_1^m \in R$, we have

$$f(x_1^m) \circ y = \bigcup_{u\in f(x_1^m)} u \circ y = \bigcup_{u\in f(x_1^m)} \Phi_y(u) = \Phi_y(f(x_1^m)) = f(x_1 \circ y, \ldots, x_m \circ y).$$

Similarly, $g(y_1^n) \circ y = f(y_1 \circ y, \ldots, y_n \circ y)$, for $y_1^n \in R$. Also,

$$(x \circ y) \circ z = \bigcup_{s \in \Phi_y(x)} s \circ z = \bigcup_{s \in \Phi_y(x)} \Phi_z(s) = \Phi_{\Phi_z(y)}(x) = \bigcup_{u \in \Phi_z(y)} \Phi_u(x) = \bigcup_{u \in y \circ z} x \circ u = x \circ (y \circ z).$$

Thus, $(R, \circ)$ is a semihypergroup and so $(R, f, g, \circ)$ is a composition (m,n)-hyperring. □

In the following, we determine conditions under which a family of endomorphisms of an (m,n)-hyperring generates the class $(\Phi_y)_{y \in R}$ satisfying the conditions in Theorems 3.8 and 3.9.

Let Ω be a family of endomorphisms of an (m,n)-hyperring (R, f, g). For any $y \in R$, denote $P_y = \bigcup_{\Phi \in \Omega} \Phi(y)$. The set P_y is called the *orbit* of y. An orbit P is said to be *principal* if, for all $x \in P$ and $\Phi_1, \Phi_2 \in \Omega$, it holds:

$$\Phi_1(x) \cap \Phi_2(x) \neq \emptyset \Longrightarrow \Phi_1 = \Phi_2.$$

Let (R, f, g) be a commutative (m,n)-hyperring and $f(a, 0^{(m-1)}) = \{a\}$ for all $a \in R$.

Lemma 3.10. *Let Ω be a family of strong endomorphisms of an (m,n)-hyperring (R, f, g), such that:*

(1) $\Phi_1 \circ \Phi_2 \in \Omega$, *for all* $\Phi_1, \Phi_2 \in \Omega$, *where* $\Phi_1 \circ \Phi_2$ *is defined by:* $(\Phi_1 \circ \Phi_2)(x) = \bigcup_{v \in \Phi_2(x)} \Phi_1(v)$.

(2) $\Phi(0) = 0$, *for all* $\Phi \in \Omega$.

(3) *For all $x, y \in R$ it holds:*

$$\Phi \in \Omega \quad \textit{and} \quad x \in \Phi(y) \Longrightarrow \exists \Phi_1 \in \Omega \quad \textit{such that} \quad y \in \Phi_1(x).$$

Then Ω induces a partition of the set $\Omega(R) = \bigcup_{\Phi \in \Omega, r \in R} \Phi(r)$ into orbits.

Proof. The proof is similar to the proof of Lemma 3.7 in [7] for composition hyperrings. □

Notice that, if the family Ω satisfies the three conditions of the previous lemma and if Ω has at least two elements, then, for any principal orbit P, it holds $0 \notin P$.

Let Ω be a family of strong endomorphisms of an (m,n)-hyperring (R, f, g) satisfying conditions of Lemma 3.10. Also, let $\mathcal{S}$ be a nonempty set of principal orbits with $0 \notin \mathcal{S}$ and for each $P \in \mathcal{S}$, let a_p be an element of P. Under these hypotheses, we get the following result.

Theorem 3.11. *Let for each $y \in R$, the endomorphism $\Phi_y : R \longrightarrow \mathcal{P}^*(R)$ be defined as follows:*

$$\Phi_y(x) = \begin{cases} \Phi(x), & \text{if } \exists P \in \mathcal{S} \text{ such that } y \in P \text{ and } \Phi \in \Omega, \text{ with } y \in \Phi(a_p) \\ 0, & \text{if } \forall P \in \mathcal{S},\ y \notin P \end{cases}$$

Then

(1) *the family $(\Phi_y)_{y\in R}$ satisfies the relation $\Phi_{\Phi_x(y)}(z) = \bigcup_{t\in\Phi_y(z)} \Phi_x(t)$, for all $x, y, z \in R$;*

(2) *this family generates a composition hyperoperation on R.*

Proof. (1) See the proof of Theorem 3.8 in [7].
(2) For all $x, y \in R$, define $x \circ y = \Phi_y(x)$. Then by Theorem 3.9 and assertion (1), we conclude that $(R, f, g, \circ)$ is a composition (m, n)-hyperring. □

Example 3.12. *Let $(\mathbb{R}, +, \cdot)$ be the field of real numbers and $A = \{2^q \mid q \in \mathbb{Q}\}$. Define the hyperoperations $\oplus_A$ and $\odot_A$ on $\mathbb{R}$ as $x \oplus_A y = xA + yA$ and $x \odot_A y = xAy$. By Example 3.10 in [7], $(\mathbb{R}, \oplus_A, \odot_A)$ is a commutative hyperring. Now, we define the m-ary and n-ary hyperoperations "f" and "g" on $\mathbb{R}$ as follows:*

$$f(x_1^m) = x_1 \oplus_A \ldots \oplus_A x_m \quad \text{and} \quad g(y_1^n) = y_1 \odot_A \ldots \odot_A y_n,$$

such that $x_1^m, y_1^n \in \mathbb{R}$. Then, it is easy to see that (R, f, g) is a commutative (m, n)-hyperring. Now, similar to Example 3.10 in [7], define two functions $h : R \longrightarrow \mathcal{P}^*(R)$ and $l : R \longrightarrow \mathcal{P}^*(R)$ by $h(x) = A \cdot x = \{2^q \cdot x \mid q \in \mathbb{Q}\}$ and $l(x) = -A \cdot x = \{-2^q \cdot x \mid q \in \mathbb{Q}\}$. Obviously, h and l are strong endomorphisms of (R, f, g). Also, $h \circ h = l \circ l = h$ and $h \circ l = l \circ h = l$. If $x \in h(y)$, then $x = 2^q y$, for some $q \in \mathbb{Q}$, and so $y = 2^{-q}x \in Ax = h(x)$. Similarly, $x \in l(y)$ implies that $y \in l(x)$. Obviously $h(0) = 0$ and $l(0) = 0$. Let $\Omega = \{h, l\}$. It is easy to verify that Ω satisfies conditions of Lemma 3.10.

Besides, for any $y \in R$, its orbit has the form $P_y = h(y) \cup l(y) = \{\pm 2^q \cdot y \mid q \in \mathbb{Q}\}$.

If $y \neq 0$, then P_y is a principal orbit, since, for any $x \in P_y$, it holds $h(x) \cap l(x) = \emptyset$, because $2^{\mathbb{Q}}x \cap (-2^{\mathbb{Q}}x) = \emptyset$. Thus, by Theorem 3.11, each family $\mathcal{S}$ of principal orbits generates corresponding composition hyperoperation on $\mathbb{R}$. For instance, if $\mathcal{S} = \{P_n \mid n \in \mathbb{N}\}$, then, for $y \in \bigcup_{n\in\mathbb{N}} P_n$ and $y > 0$, we put $\Phi_y = h$ and, for $y < 0$, we put $\Phi_y = l$. If $y \notin \bigcup_{n\in\mathbb{N}} P_n$, then $\Phi_y = 0$. Thus, the corresponding hyperoperation is defined by:

$$x \circ y = \begin{cases} A \cdot x & \text{if } y \in A \cdot \mathbb{N}, \\ -A \cdot x & \text{if } y \in -A \cdot \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

4 Isomorphism theorems of composition (m, n)-hyperrings

One of the main argument regarding the algebraic (hyper)structures concerns the three isomorphism theorems. As we have already mentioned in the first section, they represent the topic of the paper [10], presented in the general case, when the composition hyperoperation defined on the (m, n)-hyperrings is a k-ary hyperoperation. It worth to mention that they are not just a simple generalization of the similar theorems for composition hyperrings, since they need supplementary assumptions stated in the following lemmas, that are not clearly mentioned in [10]. Based on these considerations, in this section we omit the proofs of the isomorphism theorems, except the second one (which is slight different by the second isomorphism theorem in [10]), insisting on the proofs of the lemmas that are fundamental in proving the theorems.

Throughout this section, $(R, f, g, \circ)$ is a composition (m, n)-hyperring, such that (R, f) is a canonical n-ary hypergroup and $g(x_1^{i-1}, 0, x_{i+1}^n) = \{0\}$, for all $x_1^n \in R$.

Like in the classical case, we need to define the concept of hyperideal in a such particular hyperring, called *composition hyperideal.* If $0 \circ x = \{0\}$ for all $x \in R$, then it is a so called *composition* $(m, n, 2)$*-hyperideal*, defined in [10].

Definition 4.1. The nonempty subset I of a composition (m, n)-hyperring $(R, f, g, \circ)$ is called a *composition hyperideal*, if the following conditions are valid.

(1) I is a hyperideal of the (m, n)-hyperring (R, f, g).

(2) $n \circ r \subseteq N$, for all $n \in I$ and $r \in R$.

(3) For $r, s, t \in R$ and $f(r, -s, 0^{(m-2)}) \cap I \neq \emptyset$, it holds that

$$f(t \circ r, -t \circ s, 0^{(m-2)}) \subseteq I.$$

Moreover, if I is an i-hyperideal of (R, f, g), then we say that I is a composition i-hyperideal of R, for $i \in \{1, \dots, n\}$.

Let I be a composition hyperideal of R. Consider the following relation on R:

$$x \rho y \Longleftrightarrow f(x, I, 0^{(m-2)}) = f(y, I, 0^{(m-2)}).$$

It is easy to see that ρ is an equivalence on R and the equivalence class represented by x is $[x]_\rho = f(x, I, 0^{(m-2)})$. Let $R/I = \{f(x, I, 0^{(m-2)}) \mid x \in R\}$ be the set of all equivalence classes of the elements of R with respect to the equivalence relation ρ.

Lemma 4.2. *Let $(R, f, g, \circ)$ be a composition (m, n)-hyperring and I a composition hyperideal of R. Define the hyperoperations $F, G, \circledcirc$ on R/I as follows:*

$$F\Big(f(x_1, I, 0^{(m-2)}), \ldots, f(x_m, I, 0^{(m-2)})\Big) = \{f(z, I, 0^{(m-2)}) \mid z \in f(x_1^m)\}$$
$$G\Big(g(y_1, I, 0^{(m-2)}), \ldots, g(y_n, I, 0^{(m-2)})\Big) = \{f(c, I, 0^{(m-2)}) \mid c \in g(y_1^n)\}$$
$$f(x, I, 0^{(m-2)}) \circledcirc f(y, I, 0^{(m-2)}) = \{f(z, I, 0^{(m-2)}) \mid z \in x \circ y\}.$$

Then $(R/I, F, G, \circledcirc)$ is a composition (m, n)-hyperring, called the quotient composition (m, n)-hyperring related to the equivalence relation ρ.

Proof. It is routine to check the validity of conditions of a composition (m, n)-hyperring for $(R/I, F, G, \circledcirc)$, since $(R, f, g, \circ)$ is a composition (m, n)-hyperring. Hence, it remains to prove only that the hyperoperations F, G and $\circledcirc$ are well-defined on R. Let $f(x_i, I, 0^{(m-2)}) = f(y_i, I, 0^{(m-2)})$, for $x_i, y_i \in R$ and $1 \leq i \leq m, n$. Set $L = F\Big(f(x_1, I, 0^{(m-2)}), \ldots, f(x_m, I, 0^{(m-2)})\Big)$ and $D = F\Big(f(y_1, I, 0^{(m-2)}), \ldots, f(y_m, I, 0^{(m-2)})\Big)$. Let $f(z, I, 0^{(m-2)}) \in L$. Since $z \in f(x_1^m)$, this implies that

$$\begin{aligned} z &\in f(z, 0^{(m-1)}) \\ &\subseteq f(z, I, 0^{(m-2)}) \\ &\subseteq f\big(f(x_1^m), I, 0^{(m-2)}\big) \\ &= f\big(f(x_1^m), f(I^{(m)}), f(0^{(m)})^{(m-2)}\big) \\ &= f\Big(f(x_1, I, 0^{(m-2)}), \ldots, f(x_m, I, 0^{(m-2)})\Big) \\ &= f\Big(f(y_1, I, 0^{(m-2)}), \ldots, f(y_m, I, 0^{(m-2)})\Big) \\ &= f\big(f(y_1^m), f(I^{(m)}), f(0^{(m)})^{(m-2)}\big) \\ &= f\big(f(y_1^m), I, 0^{(m-2)}\big), \end{aligned}$$

then there exist $z' \in f(y_1^m)$ and $n \in I$ such that $z \in f(z', n, 0^{(m-2)})$. Therefore,

$$\begin{aligned} f(z, I, 0^{(m-2)}) &\subseteq f\big(f(z', n, 0^{(m-2)}), I, 0^{(m-2)}\big) \\ &= f\big(z', f(n, I, 0^{(m-2)}), 0^{(m-2)}\big) \\ &\subseteq f(z', I, 0^{(m-2)}). \end{aligned}$$

Also, since (R, f) is canonical, then $z \in f(z', n, 0^{(m-2)})$ implies that $z' \in f(z, -n, 0^{(m-2)})$, and so similarly $f(z', I, 0^{(m-2)}) \subseteq f(z, I, 0^{(m-2)})$. Thus,

$f(z,I,0^{(m-2)}) = f(z',I,0^{(m-2)})$ such that $z' \in f(y_1^m)$. Therefore, $L \subseteq D$. Similarly, it can be proved that $D \subseteq L$.
Now, set

$$L = G\Big(f(x_1,I,0^{(m-2)}),\dots,f(x_n,I,0^{(m-2)})\Big),$$
$$D = G\Big(f(y_1,I,0^{(m-2)}),\dots,f(y_n,I,0^{(m-2)})\Big).$$

As before, $z \in g(x_1^n)$ implies that

$$z \in g\Big(f(y_1,I,0^{(m-2)}),\dots,f(y_n,I,0^{(m-2)})\Big) = f\big(g(y_1^n),I,0^{(m-2)}\big)\big).$$

Hence, for $z' \in g(y_1^n)$ we can conclude that $f(z,I,0^{(m-2)}) = f(z',I,0^{(m-2)}) \in D$. Then $L \subseteq D$ and similarly we have $D \subseteq L$.
Now, suppose that $L = f(x_1,I,0^{(m-2)}) \circledcirc f(x_2,I,0^{(m-2)}) = \{f(z,I,0^{(m-2)}) \mid z \in x_1 \circ x_2\}$ and $D = f(y_1,I,0^{(m-2)}) \circledcirc f(y_2,I,0^{(m-2)}) = \{f(z,I,0^{(m-2)}) \mid z \in y_1 \circ y_2\}$. Since (R,f) is canonical and $x_2 \in f(x_2,I,0^{(m-2)}) = f(y_2,I,0^{(m-2)})$, then for $n \in I$, we have $n \in f(x_2,-y_2,0^{(m-2)})$. Hence, $f(x_2,-y_2,0^{(m-2)}) \cap I \neq \emptyset$. Since I is a composition hyperideal, we have $f(x_1 \circ x_2, -x_1 \circ y_2, 0^{(m-2)}) \subseteq I$, and so

$$\begin{aligned} x_1 \circ x_2 &\subseteq f(x_1 \circ x_2, 0^{(m-1)}) \\ &\subseteq f\big(x_1 \circ x_2, f(x_1 \circ y_2, -x_1 \circ y_2, 0^{(m-2)}), 0^{(m-2)}\big) \\ &= f\big(x_1 \circ y_2, f(x_1 \circ x_2, -x_1 \circ y_2, 0^{(m-2)}), 0^{(m-2)}\big) \\ &\subseteq f(x_1 \circ y_2, I, 0^{(m-2)}). \end{aligned}$$

Since $x_1 \in f(y_1,I,0^{(m-2)})$, there exists $n' \in I$ such that $x_1 \in f(y_1,n',0^{(m-2)})$. Thus

$$\begin{aligned} x_1 \circ x_2 &\subseteq f(x_1 \circ y_2, I, 0^{(m-2)}) \\ &\subseteq f\Big(f(y_1,n',0^{(m-2)}) \circ y_2, I, 0^{(m-2)}\Big) \\ &= f\Big(f\big(y_1 \circ y_2, n' \circ y_2, (0 \circ y_2)^{(m-2)}\big), I, 0^{(m-2)}\Big) \\ &\subseteq f\Big(f\big(y_1 \circ y_2, I, I^{(m-2)}\big), I, 0^{(m-2)}\Big) \\ &= f\big(y_1 \circ y_2, f(I^{(m)}), 0^{(m-2)}\big) \\ &= f\big(y_1 \circ y_2, I, 0^{(m-2)}\big). \end{aligned}$$

Hence, $z \in x_1 \circ x_2$ implies that there exists $z' \in y_1 \circ y_2$ such that $z \in f(z',I,0^{(m-2)})$, that is, $f(z,I,0^{(m-2)}) = f(z',I,0^{(m-2)})$. Then $L \subseteq D$. Similarly we have $D \subseteq L$. □

Definition 4.3. A mapping $h : R_1 \longrightarrow R_2$ is called a *strong homomorphism* of composition (m,n)-hyperrings $(R_1, f_1, g_1, \circ_1)$ and $(R_2, f_2, g_2, \circ_2)$, if, for all $x_1^m, y_1^n, x, y \in R_1$, the following conditions are valid:

(1) $h\big(f_1(x_1^m)\big) = f_2\big(h(x_1), \ldots, h(x_m)\big)$;

(2) $h\big(g_1(y_1^n)\big) = g_2\big(h(y_1), \ldots, h(y_n)\big)$;

(3) $h(x \circ_1 y) = h(x) \circ_2 h(y)$;

(4) $h(0) = 0$.

We say that h is an *isomorphism*, if h is one to one and onto, and write $R_1 \cong R_2$ if R_1 is isomorphic with R_2. Also, if h is a strong homomorphism from R_1 into R_2, then, for all $x \in R_1$, we have $f(-x) = -f(x)$. Moreover, let $\ker h = \{x \in R_1 \mid h(x) = 0\}$, which is a hyperideal of R_1, but generally it is not a composition (m,n)-hyperideal.

Theorem 4.4. *Let $(R_1, f_1, g_1, \circ_1)$ and $(R_2, f_2, g_2, \circ_2)$ be two composition (m,n)-hyperrings. If $h : R_1 \longrightarrow R_2$ is a strong homomorphism such that $\ker h = H$ is a composition hyperideal of R_1, then $R_1/H \cong Imh$.*

Proof. It is the particular case of Theorem 3.8 [10], for $k = 2$. □

Lemma 4.5. *Let A_1^m be composition hyperideals of a composition (m,n)-hyperring$(R, f, g, \circ)$. Then*

(1) *A_i is a composition hyperideal of $f(A_1^m)$, for every $1 \le i \le m$.*

(2) *$(f(A_1^m), f, g, \circ)$ is a composition (m,n)-hyperring.*

(3) *If the hyperideal $\{0\}$ of (R, f, g) is composition, then $f(A_1^{i-1}, 0, A_{i+1}^m) \cap A_i$ is a composition hyperideal of $f(A_1^{i-1}, 0, A_{i+1}^m)$, for $1 \le i \le m$.*

(4) *$\displaystyle\bigcap_{i=1}^{m} A_i$ is a composition hyperideal of R.*

Proof. (1) For every $1 \le i \le m$, we have $A_i = f(A_i, 0^{(m-1)}) \subseteq f(A_1^m)$. Also, for all $a_1^m \in A_i$ we have $f(a_1^m) \subseteq A_i$ (since every A_i is a hyperideal). Moreover, for every $x_1^n \in f(A_1^m)$, there exist $a_{11}^{1m}, \ldots, a_{n1}^{nm} \in A_1^m$ such that $x_i \in f(a_{i1}^{im})$, for all $1 \le i \le n$. Hence,

$$\begin{aligned} g(x_1^{i-1}, A_i, x_{i+1}^n) &\subseteq g\Big(f(a_{11}^{1m}), \ldots, f(a_{(i-1)1}^{(i-1)m}), f(A_1^m), f(a_{(i+1)1}^{(i+1)m}), \ldots, f(a_{n1}^{nm})\Big) \\ &\subseteq g\big(f(A_1^m)^{(n)}\big) \\ &\subseteq f(A_1^m), \end{aligned}$$

since $f(A_1^m)$ is a hyperideal by [18]. Thus, A_i is a hyperideal of $f(A_1^m)$. Moreover, for every $a \in A_i$ and $x \in f(a_1^m) \subseteq f(A_1^m)$, we have

$$x \circ a \subseteq f(a_1^m) \circ a = f(a \circ a_1, \ldots, a \circ a_m) \subseteq A_i,$$

since A_i is a composition hyperideal, for all $1 \leq i \leq m$. In addition, for $r, s, t \in f(A_1^m)$, if $f(r, -s, 0^{(m-2)}) \cap A_i \neq \emptyset$, then we have $f(t \circ r, -t \circ s, 0^{(m-2)}) \subseteq A_i$, since $f(A_1^m) \subseteq R$ and A_i is a composition hyperideal of R. Therefore, A_i is a composition hyperideal of $f(A_1^m)$, for all $1 \leq i \leq m$, by Definition 4.1.
(2) Since A_1^m and $f(A_1^m)$ are hyperideals of R, it is routine to verify that $(f(A_1^m), f, g)$ is an (m,n)-subhyperring of (R, f, g). Also, for every $x, y \in f(A_1^m)$ there exist $a_i, b_i \in A_i$, for $1 \leq i \leq m$, such that $x \in f(a_1^m)$ and $y \in f(b_1^m)$. Hence, we have

$$x \circ y \subseteq f(a_1^m) \circ f(b_1^m) = \bigcup_{s \in f(b_1^m)} f(a_1^m) \circ s = \bigcup_{s \in f(b_1^m)} f(a_1 \circ s, \ldots, a_m \circ s) \subseteq f(A_1^m),$$

since A_1^m are composition hyperideals. Thus, $f(A_1^m) \circ f(A_1^m) \subseteq f(A_1^m)$ and also assertion (3) of Definition 3.1 is clearly valid. Hence, $f(A_1^m)$ is a composition (m,n)-subhyperring of R, meaning that $(f(A_1^m), f, g, \circ)$ is a composition (m,n)-hyperring.
(3) Since $f(A_1^{i-1}, 0, A_{i+1}^m)$ and A_i are hyperideals of R, then $f(A_1^{i-1}, 0, A_{i+1}^m) \cap A_i$ is a hyperideal of R and also is a hyperideal of $f(A_1^{i-1}, 0, A_{i+1}^m)$. Now, let $x \in f(A_1^{i-1}, 0, A_{i+1}^m)$ and $n \in f(A_1^{i-1}, 0, A_{i+1}^m) \cap A_i$. Then $n \in f(A_1^{i-1}, 0, A_{i+1}^m)$, and so by (2) we have

$$n \circ x \subseteq f(A_1^{i-1}, 0, A_{i+1}^m) \circ f(A_1^{i-1}, 0, A_{i+1}^m) \subseteq f(A_1^{i-1}, 0, A_{i+1}^m).$$

For $r, s, t \in f(A_1^{i-1}, 0, A_{i+1}^m)$ such that $f(r, -s, 0^{(m-2)}) \cap (f(A_1^{i-1}, 0, A_{i+1}^m) \cap A_i) \neq \emptyset$, we have

$$\begin{aligned} & f(t \circ r, -t \circ s, 0^{(m-2)}) \subseteq f\Big(f(A_1^{i-1}, 0, A_{i+1}^m) \circ r, -f(A_1^{i-1}, 0, A_{i+1}^m) \circ s, 0^{(m-2)}\Big) \\ & = f\Big(f(A_1^{i-1} \circ r, 0 \circ r, A_{i+1}^m \circ r), -f(A_1^{i-1} \circ s, 0 \circ s, A_{i+1}^m \circ s), 0^{(m-2)}\Big) \\ & \subseteq f\Big(f(A_1^{i-1}, 0, A_{i+1}^m), -f(A_1^{i-1}, 0, A_{i+1}^m), 0^{(m-2)}\Big) \\ & \subseteq f(A_1^{i-1}, 0, A_{i+1}^m), \end{aligned}$$

since $A_1^m, \{0\}$ are composition hyperideals and $f(A_1^{i-1}, 0, A_{i+1}^m)$ is hyperideal. Also, $f(r, -s, 0^{(m-2)}) \cap \big(f(A_1^{i-1}, 0, A_{i+1}^m) \cap A_i\big) \neq \emptyset$ implies that $f(r, -s, 0^{(m-2)}) \cap A_i \neq \emptyset$. Since A_i is composition hyperideal, then $f(t \circ r, -t \circ s, 0^{(m-2)}) \subseteq A_i$. Therefore,

$$f(t \circ r, -t \circ s, 0^{(m-2)}) \subseteq f(A_1^{i-1}, 0, A_{i+1}^m) \cap A_i.$$

Consequently, we obtain that $f(A_1^{i-1}, 0, A_{i+1}^m) \cap A_i$ is a composition hyperideal of $f(A_1^{i-1}, 0, A_{i+1}^m)$.
Since A_1^m are composition hyperideals, the proof of (4) is straightforward. □

We say that a hyperideal A of a composition (m, n)-hyperring $(R, f, g, \circ)$ is *normal*, if $f(r, A, -r, 0^{(m-3)}) \subseteq A$, for all $r \in R$. Also, we recall Lemma 4.6 from [18], for the reader convenience, in the classical case, for composition (m, n)-hyperrings.

Lemma 4.6. *Let A be a normal hyperideal of a composition (m, n)-hyperring $(R, f, g, \circ)$. Then, for all $a \in f(a_1^m)$ and $a_1^m \in R$, we have*

$$f(a, A, 0^{(m-2)}) = f\Big(f(a_1^m), A, 0^{(m-2)}\Big).$$

Proof. The proof is similar to proof of Lemma 4.6 in [18]. □

Theorem 4.7. *If A_1^m and $\{0\}$ are normal composition hyperideals of a composition (m, n)-hyperring $(R, f, g, \circ)$, then*

$$f(A_1^{i-1}, 0, A_{i+1}^m)/f(A_1^{i-1}, 0, A_{i+1}^m) \cap A_i \cong f(A_1^m)/A_i.$$

Proof. Define $h : f(A_1^{i-1}, 0, A_{i+1}^m) \longrightarrow f(A_1^m)/A_i$ by $h(a) = f(a, A_i, 0^{(m-2)})$. It is clear that h is well-defined. For $a_1^m \in f(A_1^{i-1}, 0, A_{i+1}^m)$, we have

$$\begin{aligned} h\big(f(a_1^m)\big) &= \big\{h(x) \mid x \in f(a_1^m)\big\} \\ &= \big\{f(x, A_i, 0^{(m-2)}) \mid x \in f(a_1^m)\big\} \\ &= F\big(f(a_1, A_i, 0^{(m-2)}), \ldots, f(a_m, A_i, 0^{(m-2)})\big) \\ &= F\big(h(a_1), \ldots, h(a_m)\big). \end{aligned}$$

Similarly, $h\big(g(b_1^n)\big) = G\big(h(b_1), \ldots, h(b_n)\big)$ and $h(a \circ b) = f(a, A_i, 0^{(m-2)}) \circledcirc f(b, A_i, 0^{(m-2)})$, for $a, b, a_1^m, b_1^n \in R$. Also, $h(0) = f(0, A_i, 0^{(m-2)}) = A_i = 0_{f(A_1^m)/A_i}$. Hence, by Lemma 4.5, "$h$" is a homomorphism of composition (m, n)-hyperrings. Now, let $f(a, A_i, 0^{(m-2)}) \in f(A_1^m)/A_i$ such that $a \in f(A_1^m)$. Then, for every $1 \leq i \leq m$, there exists $a_i \in A_i$ such that $a \in f(a_1^m)$. Since

A_i is a normal hyperideal, by Lemma 4.6, we have

$$\begin{aligned} f(a, A_i, 0^{(m-2)}) &= f\big(f(a_1^m), A_i, 0^{(m-2)}\big) \\ &= f\big(a_1^{i-1}, f(a_i, A_i, 0^{(m-2)}), a_{i+1}^m\big) \\ &= f(a_1^{i-1}, A_i, a_{i+1}^m) \\ &= f\big(a_1^{i-1}, f(A_i, 0^{(m-1)}), a_{i+1}^m\big) \\ &= f\big(f(a_1^{i-1}, 0, a_{i+1}^m), A_i, 0^{(m-2)}\big) \\ &= f(x, A_i, 0^{(m-2)}) \qquad (\forall x \in f(a_1^{i-1}, 0, a_{i+1}^m)) \\ &= h(x). \end{aligned}$$

This implies that h is onto. Also, for any $x \in f(A_1^{i-1}, 0, A_{i+1}^m)$, by

$$\begin{aligned} x \in \ker h &\Longleftrightarrow h(x) = A_i \\ &\Longleftrightarrow f(x, A_i, 0^{(m-2)}) = A_i \\ &\Longleftrightarrow x \in A_i \end{aligned}$$

we have $\ker h = f(A_1^{i-1}, 0, A_{i+1}^m) \cap A_i$ which by Lemma 4.5 (3), is a composition hyperideal of $f(A_1^{i-1}, 0, A_{i+1}^m)$. Consequently, by Theorem 4.4, the proof is complete. □

Remark 4.8. *We included the proof of the second isomorphism theorem in order to better emphasize the fact that the function "h" is onto.*

Remark 4.9. *According with [20] and [22], if I is a normal hyperideal of the (m,n)-hyperring (R,f,g), then F and G defined in Lemma 4.2 are m-ary and n-ary operations, respectively. Moreover, the composition hyperoperation defined on the quocient R/I is an operation. Therefore, in this case, $(R/I, F, G, \circledcirc)$ is a composition (m,n)-ring, a natural generalization of composition ring. Theorem 4.7 is satisfied in such conditions.*

Lemma 4.10. *If A and B are composition hyperideals of $(R,f,g,\circ)$ such that $A \subseteq B$, then B/A is a composition hyperideal of R/A.*

Proof. Since, A is a composition hyperideal of B, then $(B/A, F, G)$ is a hyperideal of $(R/A, F, G)$ by Lemma 4.2. Let consider $f(x, A, 0^{(m-2)}) \in B/A$ and $f(y, A, 0^{(m-2)}) \in R/A$ such that $x \in B$ and $y \in R$. Since, B is composition hyperideal, we have

$$\begin{aligned} f(x, A, 0^{(m-2)}) \circledcirc f(y, A, 0^{(m-2)}) &= \{f(t, A, 0^{(m-2)}) \mid t \in x \circ y\} \\ &\subseteq \{f(t, A, 0^{(m-2)}) \mid t \in B\} \\ &= B/A. \end{aligned}$$

Now, suppose that $f(x, A, 0^{(m-2)}), f(y, A, 0^{(m-2)}), f(t, A, 0^{(m-2)}) \in R/A$. If

$$F\big(f(x, A, 0^{(m-2)}), -f(y, A, 0^{(m-2)}), 0_{R/A}^{(m-2)}\big) \cap B/A \neq \emptyset,$$

then we have $f(t \circ x, -t \circ y, 0^{(m-2)}) \subseteq B$, since B is composition hyperideal. Hence

$$\begin{aligned}
&F\Big(f(t, A, 0^{(m-2)}) \circledcirc f(x, A, 0^{(m-2)}), -f(t, A, 0^{(m-2)}) \circledcirc f(y, A, 0^{(m-2)}), 0_{R/A}^{(m-2)}\Big)\\
&\quad = F\Big(\{f(z, A, 0^{(m-2)}) \mid z \in t \circ x\}, -\{f(c, A, 0^{(m-2)}) \mid c \in t \circ y\}, 0_{R/A}^{(m-2)}\Big)\\
&\quad = \bigcup_{z \in t \circ x,\ c \in t \circ y} F\Big(f(z, A, 0^{(m-2)}), -f(c, A, 0^{(m-2)}), 0_{R/A}^{(m-2)}\Big)\\
&\quad = \bigcup_{z \in t \circ x,\ c \in t \circ y} \Big\{f(d, A, 0^{(m-2)}) \mid d \in f(z, -c, 0^{(m-2)})\Big\}\\
&\quad \subseteq \Big\{f(d, A, 0^{(m-2)}) \mid d \in f(t \circ x, -t \circ y, 0^{(m-2)})\Big\}\\
&\quad \subseteq \big\{f(d, A, 0^{(m-2)}) \mid d \in B\big\}\\
&\quad = B/A.
\end{aligned}$$

Therefore, B/A is a composition hyperideal of R/A. □

Theorem 4.11. *Let A and B be composition hyperideals of a composition (m, n)-hyperring $(R, f, g, \circ)$ such that $A \subseteq B$. Then $(R/A)/(B/A) \cong R/B$.*

Proof. It is the particular case of Theorem 3.10 [10], for $k = 2$. □

5 Conclusions and future work

Based on the notion of composition rings [1] and taking into account the properties of the hyperrings of polynomials [13], a new type of hyperrings, called composition hyperrings, has been introduced in [7]. Following the same idea, in this note we have investigated the properties of composition (m, n)-hyperrings, emphasizing the relations between them and the composition hyperrings, connection illustrated by several examples. Furthermore, conditions for constructing a composition hyperoperation on (m, n)-hyperrings using particular endomorphisms of such hyperrings have been established.

On the other side, considering on an (m, n)-hyperring a composition k-ary hyperoperation, one obtains the so called composition (m, n, k)-hyperrings, recently studied in [10]. So the composition (m, n)-hyperrings are composition $(m, n, 2)$-hyperrings, using the terminology in [10]. In the same paper, the authors stated and proved the three isomorphism theorems for the composition (m, n, k)-hyperrings, and for this reason, in Section 4, we omit the proofs of the similar theorems for composition (m, n)-hyperrings. We stress the fact that Section 4 in this note is not a repetition of the work in [10], but an integration; we have insisted more on the lemmas that assure the conditions for the isomorphism theorems.

The study can be continued in more directions. One is already considered in the submitted paper [21], where we extend this work to the case of n-ary composition hyperrings. Another one concerns the prime, primary and maximal hyperideals in composition (m, n)-hyperrings. Besides one can define and investigate the fuzzy substructures of composition (m, n)-hyperrings.

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