A relaxation theorem for a differential inclusion with "maxima"

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Abstract

We consider a Cauchy problem associated to a nonconvex differential inclusion with "maxima" and we prove a Filippov type existence result. This result allows to obtain a relaxation theorem for the problem considered.

1 Introduction

Differential equations with maximum have proved to be strong tools in the modelling of many physical problems: systems with automatic regulation, problems in control theory that correspond to the maximal deviation of the regulated quantity etc.. As a consequence there was an intensive development of the theory of differential equations with "maxima" [2, 5, 6, 8-14] etc..

A classical example is the one of an electric generator ([2]). In this case the mechanism becomes active when the maximum voltage variation is reached in an interval of time. The equation describing the action of the regulator has the form

$$x'(t) = ax(t) + b \max_{s \in [t-h, t]} x(s) + f(t),$$

where a, b are constants given by the system, $x(\cdot)$ is the voltage and $f(\cdot)$ is a perturbation given by the change of voltage.

In this paper we study the following problem

$$x'(t) \in F(t, x(t), \max_{s \in [0, t]} x(s)) \quad a.e. \ ([0, 1]), \quad x(0) = x_0 \quad (1)$$

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where $F : [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ is a set-valued map and $x_0 \in \mathbb{R}$. Several existing results for problem (1) obtained with fixed point approaches may be found in our previous paper [3].

The aim of this note is to obtain a relaxation theorem for the problem considered. Namely, we prove that the solution set of the problem (1) is dense in the set of the relaxed solutions; i.e. the set of solutions of the differential inclusion whose right hand side is the convex hull of the original set-valued map. In order to prove this result we show, first, that Filippov's ideas ([4]) can be suitably adapted in order to obtain the existence of solutions of problem (1). We recall that for a differential inclusion defined by a lipschitzian set-valued map with nonconvex values Filippov's theorem ([4]) consists in proving the existence of a solution starting from a given "quasi" solution. Moreover, the result provides an estimate between the starting "quasi" solution and the solution of the differential inclusion.

The paper is organized as follows: in Section 2 we briefly recall some preliminary results that we will use in the sequel and in Section 3 we prove the main results of the paper.

2 Preliminaries

In this section we sum up some basic facts that we are going to use later.

Let (X, d) be a metric space. The Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by $d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\}$, $d^*(A, B) = \sup\{d(a, B); a \in A\}$, where $d(x, B) = \inf\{d(x, y); y \in B\}$. Let $I := [0, 1]$ and denote by $\mathcal{L}(I)$ the σ -algebra of all Lebesgue measurable subsets of I . Denote by $\mathcal{P}(\mathbb{R})$ the family of all nonempty subsets of $\mathbb{R}$ and by $\mathcal{B}(\mathbb{R})$ the family of all Borel subsets of $\mathbb{R}$. For any subset $A \subset \mathbb{R}$ we denote by $\text{cl}A$ the closure of A and by $\overline{\text{co}}(A)$ the closed convex hull of A .

As usual, we denote by $C(I, \mathbb{R})$ the Banach space of all continuous functions $x(\cdot) : I \rightarrow \mathbb{R}$ endowed with the norm $|x|_C = \sup_{t \in I} |x(t)|$ and by $L^1(I, \mathbb{R})$ the Banach space of all integrable functions $x(\cdot) : I \rightarrow \mathbb{R}$ endowed with the norm $|x|_1 = \int_0^T |x(t)| dt$. The Banach space of all absolutely continuous functions $x(\cdot) : I \rightarrow \mathbb{R}$ will be denoted by $AC(I, \mathbb{R})$. We recall that for a set-valued map $U : I \rightarrow \mathcal{P}(\mathbb{R})$ the Aumann integral of U , denoted by $\int_I U(t) dt$, is the set

$$\int_I U(t) dt = \left\{ \int_I u(t) dt; u(\cdot) \in L^1(I, \mathbb{R}), u(t) \in U(t) \text{ a.e. } (I) \right\}$$

We recall two results that we are going to use in the next section. The first one is a selection result (e.g., [1]) which is a version of the celebrated Kuratowski and Ryll-Nardzewski selection theorem. The proof of the second one may be found in [7].

Lemma 1. Consider X a separable Banach space, B is the closed unit ball in X , $H : I \rightarrow \mathcal{P}(X)$ is a set-valued map with nonempty closed values and $g : I \rightarrow X, L : I \rightarrow \mathbb{R}_+$ are measurable functions. If

$$H(t) \cap (g(t) + L(t)B) \neq \emptyset \quad \text{a.e.}(I),$$

then the set-valued map $t \rightarrow H(t) \cap (g(t) + L(t)B)$ has a measurable selection.

Lemma 2. Let $U : I \rightarrow \mathcal{P}(\mathbb{R})$ be a measurable set-valued map with closed nonempty images and having at least one integrable selection. Then

$$\text{cl}(\int_0^T \overline{\text{co}}U(t)dt) = \text{cl}(\int_0^T U(t)dt).$$

Let $I(\cdot) : \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ a set-valued map with compact convex values defined by $I(t) = [a(t), b(t)]$, where $a(\cdot), b(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions with $a(t) \leq b(t) \forall t \in \mathbb{R}$. For $x(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$ continuous we define $(\max_I)(t) = \max_{s \in I(t)} x(s)$. Therefore, $\max_I : C(\mathbb{R}, \mathbb{R}) \rightarrow C(\mathbb{R}, \mathbb{R})$ is an operator whose properties are summarized in the next lemma proved in [12].

Lemma 3. If $x(\cdot), y(\cdot) \in C(\mathbb{R}, \mathbb{R})$, then one has

- i) $|\max_{s \in I(t)} x(s) - \max_{s \in I(t)} y(s)| \leq \max_{s \in I(t)} |x(s) - y(s)| \forall t \in \mathbb{R}$.
- ii) $\max_{t \in K} |\max_{s \in I(t)} x(s) - \max_{s \in I(t)} y(s)| \leq \max_{s \in \cup_{t \in K} I(t)} |x(s) - y(s)| \forall t \in \mathbb{R}$.

3 The main results

In what follows we assume the following hypotheses.

Hypothesis. i) $F(\cdot, \cdot, \cdot) : I \times \mathbb{R} \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ has nonempty closed values and is $\mathcal{L}(I) \otimes \mathcal{B}(\mathbb{R} \times \mathbb{R})$ measurable.

ii) There exist $l_1(\cdot), l_2(\cdot) \in L^1(I, \mathbb{R}_+)$ such that, for almost all $t \in I$,

$$d_H(F(t, x_1, y_1), F(t, x_2, y_2)) \leq l_1(t)|x_1 - x_2| + l_2(t)|y_1 - y_2| \forall x_1, x_2, y_1, y_2 \in \mathbb{R}.$$

Theorem 1. Assume Hypothesis satisfied and $|l_1|_1 + |l_2|_1 < 1$. Let $y(\cdot) \in AC(I, \mathbb{R})$ be such that there exists $p(\cdot) \in L^1(I, \mathbb{R}_+)$ verifying $d(y'(t), F(t, y(t), \max_{s \in [0, t]} y(s))) \leq p(t)$ a.e. (I) .

Then there exists $x(\cdot)$ a solution of problem (1) satisfying for all $t \in I$

$$|x - y|_C \leq \frac{1}{1 - (|l_1|_1 + |l_2|_1)} (|x_0 - y(0)| + |p|_1). \quad (2)$$

Proof. We set $x_0(\cdot) = y(\cdot)$, $f_0(\cdot) = y'(\cdot)$.

The set-valued map $t \rightarrow F(t, y(t), \max_{s \in [0, t]} y(s))$ is measurable with closed values and

$$F(t, y(t), \max_{s \in [0, t]} y(s)) \cap \{y'(t) + p(t)[-1, 1]\} \neq \emptyset \quad \text{a.e. } (I).$$

It follows from Lemma 1 that there exists a measurable function $f_1(\cdot)$ such that $f_1(t) \in F(t, x_0(t), \max_{s \in [0, t]} x_0(s))$ a.e. (I) and, for almost all $t \in I$, $|f_1(t) - y'(t)| \leq p(t)$. Define $x_1(t) = x_0 + \int_0^t f_1(s)ds$ and one has

$$|x_1(t) - y(t)| \leq |x_0 - y(0)| + \int_0^t p(s)ds \leq |x_0 - y(0)| + |p|_1.$$

Thus $|x_1 - y|_C \leq |x_0 - y(0)| + |p|_1$.

The set-valued map $t \rightarrow F(t, x_1(t), \max_{s \in [0, t]} x_1(s))$ is measurable. Moreover, the map $t \rightarrow l_1(t)|x_1(t) - x_0(t)| + l_2(t)|\max_{s \in [0, t]} x_1(s) - \max_{s \in [0, t]} x_0(s)|$ is measurable. By the lipschitzianity of $F(t, \cdot, \cdot)$ we have that for almost all $t \in I$

$$d(f_1(t), F(t, x_1(t), \max_{s \in [0, t]} x_1(s))) \leq d_H(F(t, x_0(t), \max_{s \in [0, t]} x_0(s)),$$

$$F(t, x_1(t), \max_{s \in [0, t]} x_1(s))) \leq l_1(t)|x_1(t) - x_0(t)| + l_2(t)|\max_{s \in [0, t]} x_0(s) - \max_{s \in [0, t]} x_1(s)|.$$

Therefore,

$$F(t, x_1(t), \max_{s \in [0, t]} x_1(s)) \cap \{f_1(t) + (l_1(t)|x_1(t) - x_0(t)| + l_2(t)|\max_{s \in [0, t]} x_1(s) - \max_{s \in [0, t]} x_0(s)|)[-1, 1]\} \neq \emptyset.$$

From Lemma 1 we deduce the existence of a measurable function $f_2(\cdot)$ such that $f_2(t) \in F(t, x_1(t), \max_{s \in [0, t]} x_1(s))$ a.e. (I) and for almost all $t \in I$

$$|f_1(t) - f_2(t)| \leq d(f_1(t), F(t, x_1(t), \max_{s \in [0, t]} x_1(s))) \leq d_H(F(t, x_0(t), \max_{s \in [0, t]} x_0(s)),$$

$$F(t, x_1(t), \max_{s \in [0, t]} x_1(s))) \leq l_1(t)|x_1(t) - x_0(t)| + l_2(t)|\max_{s \in [0, t]} x_0(s) - \max_{s \in [0, t]} x_1(s)|.$$

Define $x_2(t) = x_0 + \int_0^t f_2(s)ds$ and one has

$$|x_1(t) - x_2(t)| \leq \int_0^t |f_1(s) - f_2(s)|ds \leq \int_0^t [l_1(s)|x_0(s) - x_1(s)| +$$

$$l_2(s)|\max_{\sigma \in [0, s]} x_0(\sigma) - \max_{\sigma \in [0, s]} x_1(\sigma)|]ds \leq (|l_1|_1 + |l_2|_1)|x_1 - x_0|_C$$

$$\leq (|l_1|_1 + |l_2|_1)(|x_0 - y(0)| + |p|_1).$$

Assume that for some $n \geq 1$ we have constructed $(x_i(\cdot))_{i=1}^n$ with x_n satisfying

$$|x_n - x_{n-1}|_C \leq (|l_1|_1 + |l_2|_1)^{n-1}(|x_0 - y(0)| + |p|_1).$$

The set-valued map $t \rightarrow F(t, x_n(t), \max_{s \in [0, t]} x_n(s))$ is measurable. At the same time, the map $t \rightarrow l_1(t)|x_n(t) - x_{n-1}(t)| + l_2(t)|\max_{s \in [0, t]} x_n(s) - \max_{s \in [0, t]} x_{n-1}(s)|$ is measurable. As before, by the lipschitzianity of $F(t, \cdot, \cdot)$ we have that for almost all $t \in I$

$$F(t, x_n(t), \max_{s \in [0, t]} x_n(s)) \cap \{f_n(t) + (l_1(t)|x_n(t) - x_{n-1}(t)| + l_2(t)|\max_{s \in [0, t]} x_n(s) - \max_{s \in [0, t]} x_{n-1}(s)|)[-1, 1]\} \neq \emptyset.$$

Using again Lemma 1 we deduce the existence of a measurable function $f_{n+1}(\cdot)$ such that $f_{n+1}(t) \in F(t, x_n(t), \max_{s \in [0, t]} x_n(s))$ a.e. (I) and for almost all $t \in I$

$$\begin{aligned} |f_{n+1}(t) - f_n(t)| &\leq d(f_{n+1}(t), F(t, x_{n-1}(t), \max_{s \in [0, t]} x_{n-1}(s))) \leq \\ &d_H(F(t, x_n(t), \max_{s \in [0, t]} x_n(s)), F(t, x_{n-1}(t), \max_{s \in [0, t]} x_{n-1}(s))) \leq \\ &l_1(t)|x_n(t) - x_{n-1}(t)| + l_2(t)|\max_{s \in [0, t]} x_n(s) - \max_{s \in [0, t]} x_{n-1}(s)|. \end{aligned}$$

Define

$$x_{n+1}(t) = x_0 + \int_0^t f_{n+1}(s) ds. \quad (3)$$

We have

$$\begin{aligned} |x_{n+1}(t) - x_n(t)| &\leq \int_0^t |f_{n+1}(s) - f_n(s)| ds \leq \\ &\int_0^t [l_1(s)|x_n(s) - x_{n-1}(s)| + l_2(s)|\max_{\sigma \in [0, s]} x_n(\sigma) - \max_{\sigma \in [0, s]} x_{n-1}(\sigma)|] ds \\ &\leq (|l_1|_1 + |l_2|_1)|x_n - x_{n-1}|_C \leq (|l_1|_1 + |l_2|_1)^n(|x_0 - y(0)| + |p|_1). \end{aligned}$$

Therefore $(x_n(\cdot))_{n \geq 0}$ is a Cauchy sequence in the Banach space $C(I, \mathbb{R})$, so it converges to $x(\cdot) \in C(I, \mathbb{R})$. Since, for almost all $t \in I$, we have

$$\begin{aligned} |f_{n+1}(t) - f_n(t)| &\leq l_1(t)|x_n(t) - x_{n-1}(t)| + l_2(t)|\max_{s \in [0, t]} x_n(s) - \max_{s \in [0, t]} x_{n-1}(s)| \\ &\leq [l_1(t) + l_2(t)]|x_n - x_{n-1}|_C, \end{aligned}$$

$\{f_n(\cdot)\}$ is a Cauchy sequence in the Banach space $L^1(I, \mathbb{R})$ and thus it converges to $f(\cdot) \in L^1(I, \mathbb{R})$.

We note that one may write

$$\begin{aligned} \left| \int_0^t f_n(s) ds - \int_0^t f(s) ds \right| &\leq \int_0^t |f_n(s) - f(s)| ds \leq \int_0^t [l_1(s) + l_2(s)] |x_{n+1} - x|_C ds \\ &\leq (|l_1|_1 + |l_2|_1) |x_{n+1} - x|_C. \end{aligned}$$

Therefore, one may pass to the limit in (3) and we get $x(t) = x_0 + \int_0^t f(s) ds$. Moreover, since the values of $F(., ., .)$ are closed and $f_{n+1}(t) \in F(t, x_n(t), \max_{s \in [0, t]} x_n(s))$ passing to the limit we obtain $f(t) \in F(t, x(t), \max_{s \in [0, t]} x(s))$ a.e. (I).

It remains to prove the estimate (2). One has

$$\begin{aligned} |x_n - x_0|_C &\leq |x_n - x_{n-1}|_C + \dots + |x_2 - x_1|_C + |x_1 - x_0|_C \leq \\ &(|l_1|_1 + |l_2|_1)^{n-1} (|x_0 - y(0)| + |p|_1) + \dots + (|l_1|_1 + |l_2|_1) (|x_0 - y(0)| + |p|_1) + \\ &(|x_0 - y(0)| + |p|_1) \leq \frac{1}{1 - (|l_1|_1 + |l_2|_1)} (|x_0 - y(0)| + |p|_1). \end{aligned}$$

Passage to the limit in the last inequality completes the proof. $\square$

Remark 1. A similar result to the one in Theorem 1 may be found in [3], namely Theorem 3.1. The approach in [3], apart from the requirement that the values of $F(., ., .)$ are compact, does not provides a priori bounds for solutions as in (3.1).

As we already pointed out, Theorem 1 allows to obtain a relaxation theorem for problem (1). In what follows, we are concerned also with the convexified (relaxed) problem

$$x'(t) \in \overline{\text{co}}F(t, x(t), \max_{s \in [0, t]} x(s)), \quad x(0) = x_0. \quad (4)$$

Note that if $F(., ., .)$ satisfies Hypothesis, then so does the set-valued map $(t, x, y) \rightarrow \overline{\text{co}}F(t, x, y)$ (e.g., [1]).

Theorem 2. *We assume that Hypothesis is satisfied and $|l_1|_1 + |l_2|_1 < 1$. Let $\bar{x}(\cdot) : I \rightarrow \mathbb{R}$ be a solution to the relaxed inclusion (4) such that the set-valued map $t \rightarrow F(t, \bar{x}(t), \max_{s \in [0, t]} \bar{x}(s))$ has at least one integrable selection.*

Then for every $\varepsilon > 0$ there exists $x(\cdot)$ a solution of problem (1) such that

$$|x - \bar{x}|_C < \varepsilon.$$

Proof. Since $\bar{x}(\cdot)$ is a solution of the relaxed inclusion (4), there exists $\bar{f}(\cdot) \in L^1(I, \mathbb{R})$, $\bar{f}(t) \in \overline{\text{co}}F(t, \bar{x}(t), \max_{s \in [0, t]} \bar{x}(s))$ a.e. (I) such that $\bar{x}(t) = x_0 + \int_0^t \bar{f}(s) ds$.

From Lemma 2, for $\delta > 0$, there exists $\tilde{f}(t) \in F(t, \bar{x}(t), \max_{s \in [0, t]} \bar{x}(s))$ a.e. (I) such that

$$\sup_{t \in I} \left| \int_0^t (\tilde{f}(s) - \bar{f}(s)) ds \right| \leq \delta.$$

Define $\tilde{x}(t) = x_0 + \int_0^t \tilde{f}(s) ds$. Therefore, $|\tilde{x} - \bar{x}|_C \leq \delta$.

We apply Theorem 1 for the "quasi" solution $\tilde{x}(\cdot)$ of (1). One has

$$p(t) = d(\tilde{f}(t), F(t, \tilde{x}(t), \max_{s \in [0, t]} \tilde{x}(s))) \leq d_H(F(t, \bar{x}(t), \max_{s \in [0, t]} \bar{x}(s)),$$

$$\begin{aligned} F(t, \tilde{x}(t), \max_{s \in [0, t]} \tilde{x}(s))) &\leq l_1(t) |\bar{x}(t) - \tilde{x}(t)| + l_2(t) \left| \max_{s \in [0, t]} \bar{x}(s) - \max_{s \in [0, t]} \tilde{x}(s) \right| \\ &\leq l_1(t) |\tilde{x} - \bar{x}|_C + l_2(t) |\tilde{x} - \bar{x}|_C \leq (l_1(t) + l_2(t)) \delta, \end{aligned}$$

which shows that $p(\cdot) \in L^1(I, \mathbb{R})$.

From Theorem 1 there exists $x(\cdot)$ a solution of (1) such that

$$|x - \tilde{x}|_C \leq \frac{1}{1 - (|l_1|_1 + |l_2|_1)} |p|_1 \leq \frac{|l_1|_1 + |l_2|_1}{1 - (|l_1|_1 + |l_2|_1)} \delta.$$

It remains to take $\delta = [1 - (|l_1|_1 + |l_2|_1)]\varepsilon$ and to deduce that $|x - \bar{x}|_C \leq |x - \tilde{x}|_C + |\tilde{x} - \bar{x}|_C \leq \varepsilon$. $\square$

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