DOI: 10.1515/auom-2017-0012
An. Şt. Univ. Ovidius Constanţa **Vol. 25(1),2017, 145–161**

Isomorphism theorems of fuzzy hypermodules

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Abstract

In this paper we define and study a new class of subfuzzy hypermodules of a fuzzy hypermodule that we call normal subfuzzy hypermodules. The connection between hypermodules and fuzzy hypermodules can be used as a tool for proving results in fuzzy hypermodules. In this manner we analyse three isomorphism theorems for fuzzy hypermodules.

1 Introduction

Fuzzy hyperstructures is an application of fuzzy set theory ([29]) to algebra which was initiated by Rosenfeld, who defined fuzzy groups ([25]). A large number of publications on this topic, as well as the recent book Fuzzy algebraic hyperstructures ([17]), prove that this represents a significant area of research.

There are three directions of research in the study of fuzzy hyperstructures. The first one studies crisp hyperoperations, defined through fuzzy sets and was initiated by Corsini [7, 8]. Some interesting papers in this direction are [10] and [11]. The second approach consists in defining a fuzzy subset on crisp hyperstructures and was introduced by Zahedi et al.[30]. Some interesting papers in this direction are [9], [15], [16], [31], and [32]. Finally, the third group of papers on fuzzy hyperstructures associates a fuzzy set with each pair of elements of a set. This idea was introduced by Corsini and Tofan in [9] and then, Sen, Ameri and Chowdhury introduced and analyzed fuzzy semihypergroups in [26]. This idea was extended to fuzzy hyperrings and fuzzy hypermodules by Davvaz and Leoreanu-Fotea in [20] and [21]. The fuzzy

Key Words: Fuzzy hyperring, fuzzy hypermodule, normal subfuzzy hypermodule.
2010 Mathematics Subject Classification: Primary 20N20; Secondary 20N25.
Received: 11.02.2016
Accepted: 28.04.2016

transposition hypergroups and fuzzy topological hypergroupoids were studied by Chowdhury in [5], and Cristea and Hoskova in [13] and fuzzy hyperalgebras in [1] by Ameri and Nozari.

In [21], Leoreanu-Fotea studied a connection between hypermodules and fuzzy hypermodules by p-cuts. This connection represents a tool for proving some results in fuzzy hypermodules in our paper. In order to do this, we define the notion of normal subfuzzy hypermodules of fuzzy hypermodules and investigate three isomorphism theorems for fuzzy hypermodules.

2 Preliminaries

2.1 Hypermodules

Let H be a nonempty set and let $\mathcal{P}^*(H)$ be the set of all nonempty subsets of H. A *hyperoperation* on H is a map "$\circ$" $: H \times H \longrightarrow \mathcal{P}^*(H)$, and the couple $(H, \circ)$ is called a *hypergroupoid*. If $x \in H$ and $A, B \in \mathcal{P}^*(H)$, then we denote $A \circ B = \bigcup_{a\in A, b\in B} a \circ b$ and $A \circ x = A \circ \{x\}$. A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for all x, y, z of H, we have $(x \circ y) \circ z = x \circ (y \circ z)$, which means that $\bigcup_{u\in x\circ y} u \circ z = \bigcup_{v\in y\circ z} x \circ v$. We say that a semihypergroup $(H, \circ)$ is a *hypergroup* if for all $x \in H$, $x \circ H = H \circ x = H$. A subhypergroup $(K, \circ)$ of $(H, \circ)$ is a nonempty set K, such that for all $k \in K$, we have $k \circ K = K \circ k = K$. A commutative hypergroup $(H, \circ)$ is called a *canonical hypergroup*, if

(1) there exists a unique $e \in H$, such that for all $x \in H$, $x \circ e = \{x\}$;

(2) for all $x \in H$, there exists a unique $x^{-1} \in H$, such that $e \in x \circ x^{-1}$;

(3) if $x \in y \circ z$, then $y \in x \circ z^{-1}$ and $z \in y^{-1} \circ x$, for all $x, y, z \in H$.

Definition 2.1. The triple $(R, \uplus, \circ)$ is a *hyperring*, if

(1) $(R, \uplus)$ is a commutative hypergroup;

(2) $(R, \circ)$ is a semihypergroup;

(3) "$\circ$" is distributive over "$\uplus$".

Definition 2.2. Let $(R, \uplus, \circ)$ be a hyperring. A nonempty set M, endowed with a hyperoperation "$+$", and an external hyperoperation "$\cdot$" is called a left *hypermodule* over $(R, \uplus, \circ)$ if the following conditions hold:

(1) $(M, +)$ is a commutative hypergroup;

(2) $\cdot : R \times M \longrightarrow \mathcal{P}^*(M)$ is such that for all a, b of M and r, s of R we have

(*i*) $r \cdot (a + b) = (r \cdot a) + (r \cdot b)$;

(ii) $(r \uplus s) \cdot a = (r \cdot a) + (s \cdot a)$;

(iii) $(r \circ s) \cdot a = r \cdot (s \cdot a)$.

A nonempty subset N of M is called a *subhypermodule* of a hypermodule $(M, +, \cdot)$, if $(N, +)$ is a subhypergroup of $(M, +)$ and $R \cdot N \in \mathcal{P}^*(N)$.

Let $(M, +, \cdot)$ be a hypermodule over a hyperring $(R, \uplus, \circ)$ such that $(M, +)$ and $(R, \uplus)$ are canonical hypergroups. A subhypermodule N of M is said to be *normal*, if $x + N - x \subseteq N$, for all $x \in M$. According to [4] and [14], we recall the following results for normal subhypermodules:

Corollary 2.3. *Let N be a normal subhypermodule of M. Then*

(1) $x + N = y + N$, *for all* $y \in x + N$.

(2) $(x + N) + (y + N) = x + y + N$, *for all* $x, y \in M$.

(3) $N \cap K$ *is a normal subhypermodule of* K, *for all subhypermodule* K *of* M.

(4) N *is a normal subhypermodule of* $N + K$, *for all subhypermodule* K *of* M.

(5) $x + y + N = z + N$, *for all* $x, y \in M$ *and* $z \in x + y$.

Let $(M_1, +_1, \cdot_1)$ and $(M_2, +_2, \cdot_2)$ be two hypermodules over a hyperring R. The map $f : M_1 \longrightarrow M_2$ is called a (strong) *homomorphism* of hypermodules if for all $x, y \in M_1$ and $r \in R$, we have

$$f(x +_1 y) \subseteq f(x) +_2 f(y) \quad \text{and} \quad f(r \cdot_1 x) \subseteq r \cdot_2 f(x).$$

$$\big(f(x +_1 y) = f(x) +_2 f(y) \quad \text{and} \quad f(r \cdot_1 x) = r \cdot_2 f(x) \big).$$

2.2 Fuzzy hypermodules

Let S be a nonempty set. $\mathcal{F}^*(S)$ denotes the set of all nonzero fuzzy subsets of S. A *fuzzy hyperoperation* on S is a map " $\circ$ " : $S \times S \longrightarrow \mathcal{F}^*(S)$, which associates a nonzero fuzzy subset $a \circ b$ with each pair (a, b) of element of $S \times S$. The couple $(S, \circ)$ is called a *fuzzy hypergroupoid*. We say that $(S, \circ)$ is commutative if for all $a, b \in S$, we have $a \circ b = b \circ a$.

A fuzzy hypergroupoid $(S, \circ)$ is called a *fuzzy semihypergroup* if for all $a, b, c \in S$, we have $a \circ (b \circ c) = (a \circ b) \circ c$, where for all $\mu \in \mathcal{F}^*(S)$,

$$(a \circ \mu)(r) = \bigvee_{t \in S} ((a \circ t)(r) \wedge \mu(t)) \quad \text{and} \quad (\mu \circ a)(r) = \bigvee_{t \in S} (\mu(t) \wedge (t \circ a)(r))$$

for all $r \in S$. If A is a nonempty subset of S, then for all $t \in S$,

$$(x \circ A)(t) = \bigvee_{a \in A} (x \circ a)(t) \qquad \text{and} \qquad (A \circ x)(t) = \bigvee_{a \in A} (a \circ x)(t)$$

Let μ and λ be two nonzero fuzzy subsets of fuzzy hypergroupoid $(S, \circ)$. Then for all $t \in S$

$$(\mu \circ \lambda)(t) = \bigvee_{p,q \in S} (\mu(p) \wedge (p \circ q)(t) \wedge \lambda(q)).$$

A fuzzy semihypergroup $(S, \circ)$ is called a *fuzzy hypergroup* if for all $a \in S$, $a \circ S = S \circ a = \chi_S$.

In a fuzzy hypergroup $(S, \circ)$, an element $e \in S$ is called an *identity* (a scalar identity), if for all $x \in S$,

$$(e \circ x)(x) > 0 \ \text{ and } \ (x \circ e)(x) > 0.$$

If for $y \in S$ we have $(e \circ x)(y) > 0$ and $(x \circ e)(y) > 0$, then $x = y$. Also, an element $a' \in S$ is called an *inverse* of $a \in S$, if $(a \circ a')(e) > 0$ and $(a' \circ a)(e) > 0$.

Now, we recall that a fuzzy hypergroup $(S, \circ)$ is said to be *canonical* if

(1) is commutative;

(2) there exists an identity element "e" in S;

(3) for all $a \in S$ there exists a unique inverse $a' \in S$;

(4) for $a, x, y \in S$, $(a \circ x)(y) > 0$ implies that $(a' \circ y)(x) > 0$.

Leoreanu-Fotea and Davvaz in [20] and Leoreanu-Fotea in [21] introduced the notions of a fuzzy hyperring and a fuzzy hypermodule, as follows:

Definition 2.4. A triple $(R, \boxplus, \boxdot)$ is a *fuzzy hyperring* if:

(1) $(R, \boxplus)$ is a commutative fuzzy hypergroup;

(2) $(R, \boxdot)$ is a fuzzy semihypergroup;

(3) "$\boxdot$" is distributive over "$\boxplus$", i.e., for all a, b, c of R

$$a \boxdot (b \boxplus c) = (a \boxdot b) \boxplus (a \boxdot c) \quad \text{and} \quad (a \boxplus b) \boxdot c = (a \boxdot c) \boxplus (b \boxdot c).$$

Definition 2.5. Let $(R, \boxplus, \boxdot)$ be a fuzzy hyperring. A nonempty set M, endowed with a fuzzy hyperoperation "$\oplus$", and a fuzzy external hyperoperation "$\odot$" is called a left *fuzzy hypermodule* over $(R, \boxplus, \boxdot)$ if the following conditions hold:

(1) $(M, \oplus)$ is a commutative fuzzy hypergroup;

(2) $\odot : R \times M \longrightarrow \mathcal{F}^*(M)$ is a map that satisfies the following conditions, for all $m, n \in M$ and $r, s \in R$:

(a) $r \odot (m \oplus n) = (r \odot m) \oplus (r \odot n)$;

(b) $(r \boxplus s) \odot m = (r \odot m) \oplus (s \odot m)$;

(c) $(r \boxdot s) \odot m = r \odot (s \odot m)$.

Example 2.6. *([21]) Let $(M, +, \cdot)$ be a module over a ring $(R, +, \cdot)$ without unity. Define the following fuzzy hyperoperations for all $a, b \in M$ and $r, s \in R$:*

$$r \boxplus s = \chi_{\{r+s\}}, \quad r \boxdot s = \chi_{\{rs\}}, \quad a \oplus b = \chi_{\{a+b\}}, \quad (r \odot a)(t) = \begin{cases} 1/2, & \text{if } t = ra \\ 0, & \text{otherwise} \end{cases}$$

Then $(M, \oplus, \odot)$ is a fuzzy hypermodule over the fuzzy hyperring $(R, \boxplus, \boxdot)$.

A nonempty subset N of M is called a *subfuzzy hypermodule* if for all $x, y \in N$ and $r \in R$, we have:

(1) $(x \oplus y)(t) > 0$ implies that $t \in N$;

(2) $x \oplus N = \chi_N$;

(3) $(r \odot x)(t) > 0$ implies that $t \in N$.

We recall that if μ_1, μ_2 are fuzzy subsets on M, then we say that μ_1 is *smaller* than μ_2 and we denote $\mu_1 \leqslant \mu_2$ if and only if for all $x \in M$, we have $\mu_1(x) \leq \mu_2(x)$. Also, let $f : M_1 \longrightarrow M_2$ be a map and μ be a fuzzy subset on M_1. Then we define $f(\mu) : M_2 \longrightarrow [0, 1]$, as follows:

$$(f(\mu))(t) = \bigvee_{r \in f^{-1}(t)} \mu(r), \quad \text{if } f^{-1}(t) \neq \emptyset$$

otherwise we consider $(f(\mu))(t) = 0$.

Definition 2.7. Let $(M_1, \oplus_1, \odot_1)$ and $(M_2, \oplus_2, \odot_2)$ be two fuzzy hypermodules over a fuzzy hyperring R. A map $f : M_1 \longrightarrow M_2$ is called a *homomorphism* of fuzzy hypermodules, if for all $x, y \in M_1$ and $r \in R$, we have

$$f(x \oplus_1 y) \leqslant f(x) \oplus_2 f(y) \quad \text{and} \quad f(r \odot_1 x) \leqslant r \odot_2 f(x).$$

2.3 Connection between hypermodules and fuzzy hypermodules

First, we recall a connection between fuzzy hypermodules and hypermodules, using the p-cuts of fuzzy sets for $p \in [0,1]$. By [21], a structure $(M, \oplus, \odot)$ is a fuzzy hypermodule over a fuzzy hyperring $(R, \boxplus, \boxdot)$ if and only if $(M, \oplus_p, \odot_p)$ is a hypermodule over the hyperring $(R, \boxplus_p, \boxdot_p)$, for all $p \in [0,1]$, where

$$x \oplus_p y = \{t \in M \mid (x \oplus y)(t) \geq p\}\ , \quad r \boxplus_p s = \{u \in R \mid (r \boxplus s)(u) \geq p\}$$
$$r \odot_p x = \{z \in M \mid (r \odot x)(z) \geq p\}\ , \quad r \boxdot_p s = \{v \in R \mid (r \boxdot s)(v) \geq p\}$$

for all $x, y \in M$ and $r, s \in R$. Also, in [21], it is shown that for a nonempty subset S and for all $a \in S$ we have the following equivalence:

$$a \oplus S = \chi_S \iff \forall p \in [0,1],\ a \oplus_p S = S.$$

Therefore, for a subfuzzy hypermodule N of a fuzzy hypermodule $(M, \oplus, \odot)$, we have $a \oplus N = \chi_{a \oplus_p N}$, for all $a \in N$.

Also, according to [21] and [20], with every fuzzy hypermodule $(M, \oplus, \odot)$ over a fuzzy hyperring $(R, \boxplus, \boxdot)$, we can associate a hypermodule structure $(M, +, \cdot)$ over a hyperring $(R, \uplus, \circ)$, where

$$x + y = \{t \in M \mid (x \oplus y)(t) > 0\}\ , \quad r \uplus s = \{u \in R \mid (r \boxplus s)(u) > 0\}$$
$$r \cdot x = \{z \in M \mid (r \odot x)(z) > 0\}\ , \quad r \circ s = \{v \in R \mid (r \boxdot s)(v) > 0\}$$

for all $x, y \in M$ and $r, s \in R$. Hence, for every subfuzzy hypermodule N of M and $a \in N$, we have $a \oplus N = \chi_{a+N}$.

Moreover, let $(M_1, \oplus_1, \odot_1)$ and $(M_2, \oplus_2, \odot_2)$ be fuzzy hypermodules and $(M_1, +_1, \cdot_1)$ and $(M_2, +_2, \cdot_2)$ the associated hypermodules. By [21], if a map $f : M_1 \longrightarrow M_2$ is a homomorphism of fuzzy hypermodules, then f is a homomorphism of hypermodules, too.

3 Normal subfuzzy hypermodules

In this section, we define the concept of normal subfuzzy hypermodules for fuzzy hypermodules and obtain some basic results about them. Using the connection with normal subhypermodules of the associated hypermodules, we obtain three isomorphism theorems for fuzzy hypermodules.

In what follows, $(M, \oplus, \odot)$ is a fuzzy hypermodule over a fuzzy hyperring $(R, \boxplus, \boxdot)$, where $(M, \oplus)$ is a canonical fuzzy hypergroup with a scalar identity 0, which belongs to all subfuzzy hypermodules of M.

Definition 3.1. A subfuzzy hypermodule N of $(M, \oplus, \odot)$ is said to be *normal*, if

$$x \oplus N \ominus x \leqslant \chi_N$$

for all $x \in M$.

Example 3.2. *Consider the fuzzy hypermodule $(M, \oplus, \odot)$ defined in Example 2.6. Let N be a submodule of the module $(M, +, \cdot)$. Then N is a subfuzzy hypermodule of $(M, \oplus, \odot)$. Also, for all $x, t \in M$*

$$\begin{aligned}
(x \oplus N \ominus x)(t) &= \bigvee_{z \in M} \big((x \oplus N)(z) \wedge (z \ominus x)(t)\big) \\
&= \bigvee_{z \in M} \Big(\bigvee_{n \in N} \chi_{\{x+n\}}(z) \wedge \chi_{\{z-x\}}(t) \Big) \\
&= \begin{cases} 1, & \text{if } z = x + n,\ t = z - x \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} 1, & \text{if } t \in N \\ 0, & \text{otherwise} \end{cases} \\
&= \chi_N(t)
\end{aligned}$$

Therefore, N is a normal subfuzzy hypermodule of M.

Example 3.3. *Consider a module $(M, +, \cdot)$ over a ring $(R, +, \cdot)$ and define the following fuzzy hyperoperations for all $a, b \in M$ and $r, s \in R$:*

$$r \boxplus s = \chi_{\{r,s\}}, \quad r \boxdot s = \chi_{\{rs\}}, \quad a \oplus b = \chi_{\{a,b\}}, \quad r \odot a = \chi_{\{ra\}}.$$

Then $(M, \oplus, \odot)$ is a fuzzy hypermodule over the fuzzy hyperring $(R, \boxplus, \boxdot)$, by [21]. Let N be a subfuzzy hypermodule of the fuzzy hypermodule M. Then for all $x, t \in M$ we have

$$\begin{aligned}
(x \oplus N \ominus x)(t) &= \bigvee_{z \in M} \big((x \oplus N)(z) \wedge (z \ominus x)(t)\big) \\
&= \bigvee_{z \in M} \Big(\bigvee_{n \in N} \chi_{\{x,n\}}(z) \wedge \chi_{\{z,-x\}}(t) \Big) \\
&= \begin{cases} 1, & \text{if } z \in \{x, n\},\ t \in \{z, -x\} \\ 0, & \text{otherwise} \end{cases} \\
&\geq \begin{cases} 1, & \text{if } t \in N \\ 0, & \text{otherwise} \end{cases} \\
&= \chi_N(t)
\end{aligned}$$

This implies that N is not a normal subfuzzy hypermodule of M in general.

Corollary 3.4. *N is a normal subfuzzy hypermodule of fuzzy hypermodule $(M, \oplus, \odot)$ if and only if N is a normal subhypermodule of the hypermodule $(M, \oplus_p, \odot_p)$, for all $p \in (0, 1]$.*

Proof. Let N be a normal subfuzzy hypermodule of the fuzzy hypermodule $(M,\oplus,\odot)$ and $t \in x \oplus_p N \ominus_p x$ for $x \in M$ and $p \in (0,1]$. Then there exists $a \in x \oplus_p N$ such that $t \in a \ominus_p x$. Hence, $(x \oplus N)(a) \geq p$ and $(a \ominus x)(t) \geq p$. Thus, since N is normal, it follows that

$$\chi_{_N}(t) \geq (x \oplus N \ominus x)(t) = \bigvee_{a\in M} \Big((x \oplus N)(a) \wedge (a \ominus x)(t)\Big) \geq p.$$

Then $t \in N$ and so $x \oplus_p N \ominus_p x \subseteq N$.
Conversely, let N be a normal subhypermodule of $(M,\oplus_p,\odot_p)$ for all $p \in (0,1]$. Also, let $(x \oplus N \ominus x)(t) \geq q$ for $x \in M$ and $q \in (0,1]$. Then there exists $a \in M$ such that $(x\oplus N)(a) \geq q$ and $(a\ominus x)(t) \geq q$. This implies that $t \in x \oplus_q N \ominus_q x$. Since N is normal, then $t \in N$ and so $\chi_{_N}(t) = 1 \geq q$. Thus, $x\oplus N\ominus x \leqslant \chi_{_N}$. □

Corollary 3.5. *Let N be a normal subfuzzy hypermodule of $(M,\oplus,\odot)$. Then, for all $x,y,z \in M$, we have*

(1) $(x \oplus N) \oplus (y \oplus N) = (x \oplus y) \oplus N$.

(2) $(x \oplus N)(y) = 1$ *implies that* $x \oplus N = y \oplus N$.

(3) *if* $(x \oplus y)(z) = 1$, *then* $x \oplus y \oplus N = z \oplus N$.

Proof. Consider the hypermodule $(M,\oplus_u,\odot_u)$ for all $u \in (0,1]$. By Corollary 3.4, N is a normal subhypermodule of $(M,\oplus_u,\odot_u)$ for all $u \in (0,1]$.
(1) Let $((x \oplus y) \oplus N)(t) \geq u$ for $t \in M$ and $u \in (0,1]$. It implies that $t \in x \oplus_u y \oplus_u N = (x \oplus_u N) \oplus_u (y \oplus_u N)$. Then, there exist $p \in x \oplus_u N$ and $q \in y \oplus_u N$ such that $t \in p \oplus_u q$. Hence, $(x \oplus N)(p) \geq u$, $(y \oplus N)(q) \geq u$ and $(p \oplus q)(t) \geq u$, and so

$$\big((x \oplus N) \oplus (y \oplus N)\big)(t) = \bigvee_{p,q\in M} \Big((x \oplus N)(p) \wedge (p \ominus q)(t) \wedge (y \oplus N)(q)\Big) \geq u.$$

Thus, $\big((x \oplus y) \oplus N\big)(t) \leq \big((x \oplus N) \oplus (y \oplus N)\big)(t)$ for all $t \in M$. Now, let $\big((x \oplus N) \oplus (y \oplus N)\big)(t) \geq u$ for $u \in (0,1]$ and $t \in M$. Since N is a subfuzzy hypermodule and $(M,\oplus)$ is commutative, it follows that

$$\big((x \oplus y) \oplus \chi_{_N}\big)(t) = (x \oplus y \oplus N \oplus N)(t) = \big((x \oplus N) \oplus (y \oplus N)\big)(t) \geq u.$$

Then, there exist $p,q \in M$ such that $(x \oplus y)(p) \geq u$, $(p \oplus q)(t) \geq u$ and $\chi_{_N}(q) \geq u$. This implies that $t \in x \oplus_u y \oplus_u N$. Hence, for $d \in x \oplus_u y$ we have $t \in d \oplus_u N$. Thus, $(x \oplus y)(d) \geq u$ and $(d \oplus N)(t) \geq u$. Then

$$((x \oplus y) \oplus N)(t) = \bigvee_{d\in M} \Big((x \oplus y)(d) \wedge (d \oplus N)(t)\Big) \geq u.$$

Now, consider the associated hypermodule $(M, +, \cdot)$. Then we have,

$$\begin{aligned}\big((x \oplus N) \oplus (y \oplus N)\big)(t) = 0 &\Leftrightarrow t \notin (x + N) + (y + N) = x + y + N \\ &\Leftrightarrow \big((x \oplus y) \oplus N\big)(t) = 0.\end{aligned}$$

This complete the proof of (1).
(2) Let $(x \oplus N)(y) = 1$. Then, $(x \oplus N)(y) \geq u$ for all $u \in (0, 1]$. Hence, we have $y \in x \oplus_u N$ and so $x \oplus_u N = y \oplus_u N$, since N is a normal subhypermodule of $(M, \oplus_u, \odot_u)$. Thus, $x \oplus N = \chi_{x\oplus_u N} = \chi_{y\oplus_u N} = y \oplus N$.
(3) Since N is a normal subhypermodule of the hypermodule $(M, \oplus_u, \odot_u)$, we have $x \oplus_u y \oplus_u N = z \oplus_u N$ for all $z \in x \oplus_u y$. Now, let $(x \oplus y)(z) = 1$. Then $z \in x \oplus_u y$, for all $u \in (0, 1]$. Hence, for all $t \in M$,

$$\begin{aligned}(z \oplus N)(t) \geq u &\implies t \in z \oplus_u N = x \oplus_u y \oplus_u N \\ &\implies \exists q \in x \oplus_u y \ ; \ t \in q \oplus_u N \\ &\implies (x \oplus y)(q) \geq u \ ; \ (q \oplus N)(t) \geq u \\ &\implies ((x \oplus y) \oplus N)(t) \geq u.\end{aligned}$$

Similarly, $((x \oplus y) \oplus N)(t) \geq u$ implies that $(z \oplus N)(t) \geq u$, for all $u \in (0, 1]$. Consider the associated hypermodule $(M, +, \cdot)$. Clearly, $(x\oplus y)(z) = 1$ implies that $z \in x + y$. Hence, we have

$$(z \oplus N)(t) = 0 \Leftrightarrow t \notin z + N = x + y + N \Leftrightarrow (x \oplus y \oplus N)(t) = 0.$$

Therefore, $x \oplus y \oplus N = z \oplus N$, for all $(x \oplus y)(z) = 1$. □

Corollary 3.6. *Let N and L be subfuzzy hypermodules of M such that L is normal in M. Then*

(1) *$N \cap L$ is a normal subfuzzy hypermodule of N.*

(2) *L is a normal subfuzzy hypermodule of*

$$N \oplus L = \{x \in M \mid \exists a \in N, b \in L;\ (a \oplus b)(x) = 1\}.$$

Proof. By Corollary 3.4, N and L are subhypermodules of the hypermodule $(M, \oplus_u, \odot_u)$ for all $u \in (0, 1]$ such that L is a normal subhypermodule of M. Thus, $N \cap L$ is a normal subhypermodule of N, by Corollary 2.3. Now, let $(x\oplus(N\cap L)\ominus x)(t) \geq u$ for $t \in M$. Then, for $p \in M$, we have $(x\oplus(N\cap L))(p) \geq u$ and $(p \ominus x)(t) \geq u$. It implies that $t \in x \oplus_u (N \cap L) \ominus_u x$. Since $N \cap L$ is a normal subhypermodule, then $t \in N \cap L$ and so $\chi_{N\cap L}(t) = 1 \geq u$. Therefore, $x \oplus (N \cap L) \ominus x \leqslant \chi_{N\cap L}$.
(2) Similarly, since L is a normal subhypermodule of $N + L$, it follows that L is a normal subfuzzy hypermodule of $N \oplus L$. □

Let N be a normal subfuzzy hypermodule of M. For all $x, y \in M$, we define the following relation on M:

$$xN^*y \iff \exists t \in M;\ \big((x \ominus y) \wedge \chi_{_N}\big)(t) \neq 0.$$

Lemma 3.7. *N^* is an equivalence relation on the fuzzy hypermodule M.*

Proof. Consider the associated hypermodule $(M, +, \cdot)$. Since $0 \in (x - x) \cap N$ for all $x \in M$, it follows that $(x \ominus x)(0) > 0$ and $\chi_{_N}(0) > 0$. Thus, xN^*x, for all $x \in M$ and so N^* is reflexive. Now, let xN^*y, for $x, y \in M$. Then there exists $t \in M$ such that $(x \ominus y)(t) > 0$ and $t \in N$. Thus, $t \in x - y$ and $t \in N$. Since $(M, +)$ is canonical, we have $-t \in y - x$ and $-t \in N$, and so $(y \ominus x)(-t) > 0$ and $\chi_{_N}(-t) > 0$. This implies that yN^*x, that is N^* is symmetric. Moreover, if xN^*y and yN^*z for $x, y, z \in M$, then there exist $a, b \in N$ such that $(x \ominus y)(a) > 0$ and $(y \ominus z)(b) > 0$. Since N is a normal subhypermodule of the associated hypermodule $(M, +, \cdot)$ such that $(N, +)$ is canonical, it follows that

$$x - z \subseteq a + y + b - y = y + (a + b) - y \subseteq y + N - y \subseteq N$$

Hence, there exists $t \in x - z$ such that $t \in N$. Thus, we have $\big((x \ominus z) \wedge \chi_{_N}\big)(t) > 0$. Then xN^*z and so N^* is transitive. □

Let $N^*[x]$ be the equivalence class of element $x \in M$. Then

Lemma 3.8. *If N is a normal subfuzzy hypermodule of M, then $x \oplus N = \chi_{_{N^*[x]}}$ for all $x \in M$.*

Proof. Consider the hypermodule $(M, \oplus_p, \odot_p)$ for all $p \in [0, 1]$. Set

$$N_p^*[x] = \{t \in M \mid (t \ominus_p x) \cap N \neq \emptyset\}$$

for all $p \in (0, 1]$. Then, there exists $p \in (0, 1]$ such that $N^*[x] = N_p^*[x]$. Also, by [4], for a normal subhypermodule N of the hypermodule $(M, \oplus_p, \odot_p)$, we have $x \oplus_p N = N_p^*[x]$ for all $x \in M$. Therefore, $\chi_{N^*[x]} = \chi_{N_p^*[x]} = \chi_{x \oplus_p N} = x \oplus N$. □

Lemma 3.9. *Let N be a normal subfuzzy hypermodule of M. Then, for all $r \in R$ and $x \in M$ we have $(r \odot x) \oplus N = (r \odot x) \oplus \chi_{_N}$.*

Proof. Let $r \in R$, $t, x \in M$, $u \in [0, 1]$ and $\big((r \odot x) \oplus N\big)(t) \geq u$. Then there exists $p \in M$ such that $(r \odot x)(p) \geq u$ and $(p \oplus N)(t) \geq u$. Since, $(N, \oplus_u)$ is a canonical subhypergroup of $(M, \oplus_u)$, then

$$t \in p \oplus_u N \subseteq (r \odot_u x) \oplus_u N = (r \odot_u x) \oplus_u N \oplus_u 0 \subseteq (r \odot_u x) \oplus_u N \oplus_u N.$$

It follows that $\big((r \odot x) \oplus \chi_{_N}\big)(t) = \big((r \odot x) \oplus N \oplus N\big)(t) \geq u$, since N is a subfuzzy hypermodule. Now, let $\big((r \odot x) \oplus \chi_{_N}\big)(t) \geq u$. Then, for $p, q \in M$ we have $(r \odot x)(p) \geq u$, $(p \oplus q)(t) \geq u$ and $\chi_{_N}(q) \geq u$. Thus, $p \in r \odot_u x$, $t \in p \oplus_u q$ and $q \in N$. This implies that $t \in (r \odot_u x) \oplus_u N$ and so $\big((r \odot x) \oplus N\big)(t) \geq u$. Therefore, $(r \odot x) \oplus N = (r \odot x) \oplus \chi_{_N}$. □

According to [24], if $(M, \oplus, \odot)$ is a fuzzy hypermodule and N is a subfuzzy hypermodule of M, then $M/N = \{x \oplus N \mid x \in M\}$ is a fuzzy hypermodule over a fuzzy hyperring $(R, \boxplus, \boxdot)$, where the fuzzy hyperoperations $\oplus_{_N}$ and $\odot_{_N}$ are defined as follows:

$$\big((x \oplus N) \oplus_N (y \oplus N)\big)(t \oplus N) = \big((x \oplus y) \oplus \chi_{_N}\big)(t)$$
and
$$\big(r \odot_N (x \oplus N)\big)(t \oplus N) = \big((r \odot x) \oplus \chi_{_N}\big)(t)$$

for all $r \in R$ and $x \oplus N, y \oplus N, t \oplus N \in M/N$. Therefore, we obtain the next theorem:

Theorem 3.10. *Let $(M, \oplus, \odot)$ be a fuzzy hypermodule over a fuzzy hyperring $(R, \boxplus, \boxdot)$ and N be a normal subfuzzy hypermodule of M. Then*

$$M/N = [M : N^*] = \{x \oplus N = \chi_{N^*[x]} \mid x \in M\}$$

is a fuzzy hypermodule over R, where:

$$\big(\chi_{N^*[x]} \oplus_N \chi_{N^*[y]}\big)(\chi_{N^*[t]}) = \big(x \oplus y \oplus N\big)(t)$$
and
$$\big(r \odot_N \chi_{N^*[x]}\big)(\chi_{N^*[t]}) = \big((r \odot x) \oplus N\big)(t).$$

Proof. It follows by Lemma 3.8, Lemma 3.9 and Corollary 3.5. □

4 Isomorphism theorems of fuzzy hypermodules

We investigate isomorphism theorems for fuzzy hypermodules by using normal subfuzzy hypermodules.

Let $(M_1, \oplus_1, \odot_1)$ and $(M_2, \oplus_2, \odot_2)$ be fuzzy hypermodules over a fuzzy hyperring $(R, \boxplus, \boxdot)$. We say that a fuzzy homomorphism $f : M_1 \longrightarrow M_2$ is *strong*, if

$$f(x \oplus_1 y) = f(x) \oplus_2 f(y) \quad \text{and} \quad f(r \odot_1 x) = r \odot_2 f(x),$$

for all $x, y \in M_1$ and $r \in R$ set $\ker f = \{x \in M_1 \mid f(x) = 0_{M_2}\}$.Then $\ker f$ is a subfuzzy hypermodule of M_1, but in general it is not normal in M_1.

Theorem 4.1. *(The First Isomorphism Theorem) Let $f : M_1 \longrightarrow M_2$ be a strong fuzzy homomorphism with kernel K which is a normal subfuzzy hypermodule of M_1. Then $[M_1 : K^*] \cong Imf$.*

Proof. Define $\varphi : [M_1 : K^*] \longrightarrow Imf$ by $\varphi(x \oplus K) = f(x)$ for all $x \in M_1$. Let $x \oplus K = y \oplus K$. Then xK^*y and so there exists $z \in M_1$ such that $(x \ominus y)(z) > 0$ and $\chi_K(z) > 0$. Consider the associated hypermodule $(M_1, +_1, \cdot_1)$ and the associated homomorphism f. Hence, we have $z \in x - y$ and $f(z) = 0_{M_2}$. Thus $0_{M_2} = f(z) \in f(x - y) = f(x) - f(y)$. Then, $f(x) = f(y)$ and so φ is well-defined. It is clear that φ is onto. Now, let $\varphi(x \oplus K) = \varphi(y \oplus K)$. Then $f(x) = f(y)$ and so $0_{M_2} \in f(x - y)$. Hence, there exists $z \in x - y$ such that $z \in K$. Thus $(x \ominus y)(z) > 0$ and $\chi_K(z) > 0$ and so there exists $t \in M_1$ such that $\big((x \ominus y) \wedge \chi_K\big)(t) \neq 0$. Therefore, xK^*y and

$$x \oplus K = \chi_{K^*[x]} = \chi_{K^*[y]} = y \oplus K.$$

Then φ is one to one. Moreover, for $x, y \in M_1$ and $t \in Imf$,

$$\begin{aligned}
\varphi\big(\chi_{K^*[x]} \oplus_K \chi_{K^*[y]}\big)(t) &= \bigvee_{z \oplus K \in \varphi^{-1}(t)} \big((x \oplus K) \oplus_K (y \oplus K)\big)(z \oplus K) \\
&= \bigvee_{z \in f^{-1}(t)} (x \oplus y \oplus K)(z) \\
&= \bigvee_{z \in f^{-1}(t)} \big((x \oplus K) \oplus (y \oplus K)\big)(z) \\
&= \bigvee_{z \in f^{-1}(t)} (\chi_{K^*[x]} \oplus \chi_{K^*[y]})(z) \\
&= \bigvee_{z \in f^{-1}(t)} \Big(\bigvee_{p,q \in M_1} \big(\chi_{K^*[x]}(p) \wedge \chi_{K^*[y]}(q) \wedge (p \oplus q)(z)\big)\Big) \\
&= \bigvee_{z \in f^{-1}(t)} (p \oplus q)(z) \quad \Big(\exists p, q \in M_1;\ f(x) = f(p),\ f(y) = f(q)\Big) \\
&= f(p \oplus q)(t) = \big(f(p) \oplus f(q)\big)(t) = \big(f(x) \oplus f(y)\big)(t) \\
&= \Big(\varphi(\chi_{K^*[x]}) \oplus \varphi(\chi_{K^*[y]})\Big)(t)
\end{aligned}$$

Similarly, we can show that $\varphi\big(r \odot_K \chi_{K^*[x]}\big) = r \odot \varphi(\chi_{K^*[x]})$, for all $r \in R$. Consequently, φ is an isomorphism and the proof is complete. □

Theorem 4.2. *(The Second Isomorphism Theorem) If N and K are subfuzzy hypermodules of $(M, \oplus, \odot)$ such that K is normal in M, then $[N : (N \cap K)^*] \cong [N \oplus K : K^*]$.*

Proof. By Corollary 3.6, K is a normal subfuzzy hypermodule of $N \oplus K$. Define the map $\varphi : N \longrightarrow [N \oplus K : K^*]$ by $\varphi(x) = x \oplus K$. Clearly, φ is well-defined.

Now, let $y \oplus K \in [N \oplus K : K^*]$ such that $y \in N \oplus K$. Hence, for $n \in N$ and $k \in K$ we have $(n \oplus k)(y) = 1$. Consider the associated hypermodule $(M, +, \cdot)$. Since K is a normal subhypermodule of the associated hypermodule of M, by Corollary 3.4, we obtain

$$y \oplus K = \chi_{y+K} = \chi_{n+k+K} = \chi_{n+K} = n \oplus K = \varphi(n).$$

Hence, φ is onto. Also,

$$\begin{aligned} \ker \varphi = \{x \in N \mid \varphi(x) = 0 \oplus K\} &= \{x \in N \mid \chi_{K^*[x]} = \chi_{K^*[0]}\} \\ &= \{x \in N \mid xK^*0\} \\ &= \{x \in N \mid x \in K\} \\ &= N \cap K, \end{aligned}$$

which is a normal subfuzzy hypermodule of N, by Corollary 3.6. Moreover, let $x, y \in N$, $t \oplus K \in [N \oplus K : K^*]$ and $u \in (0, 1]$ and let $\varphi(x \oplus y)(t \oplus K) \geq u$. Hence, there exists $r \oplus K = t \oplus K$ such that $(x \oplus y)(r) \geq u$. Consider the associated hypermodule $(M, \oplus_u, \odot_u)$. By Corollary 2.3, we obtain

$$t \in t \oplus_u 0 \subseteq t \oplus_u K = r \oplus_u K = x \oplus_u y \oplus_u K.$$

It follows that $\big(\varphi(x) \oplus_K \varphi(y)\big)(t \oplus K) = \big((x \oplus K) \oplus_K (y \oplus K)\big)(t \oplus K) = (x \oplus y \oplus K)(t) \geq u$. Similarly, $\big(\varphi(x) \oplus_K \varphi(y)\big)(t \oplus K) \geq u$ implies that $\varphi(x \oplus y)(t \oplus K) \geq u$. Also, we can obtain that $\varphi(x \oplus y)(t \oplus K) = 0$ if and only if $\big(\varphi(x) \oplus_K \varphi(y)\big)(t \oplus K) = 0$. Then, $\varphi(x \oplus y) = \varphi(x) \oplus_K \varphi(y)$ and similarly we have $\varphi(r \odot x) = r \odot_K \varphi(x)$, for all $r \in R$. Hence, φ is a strong fuzzy homomorphism. According to Theorem 4.1, the proof is complete. □

Theorem 4.3. *(The Third Isomorphism Theorem) Let N and K be normal subfuzzy hypermodules of M such that $N \subseteq K$. Then $[K : N^*]$ is a normal subfuzzy hypermodule of $[M : N^*]$ and $[[M : N^*] : [K : N^*]] \cong [M : K^*]$.*

Proof. Notice that $[K : N^*]$ is a normal subfuzzy hypermodule of $[M : N^*]$. Define $\alpha : [M : N^*] \longrightarrow [M : K^*]$ by $\alpha(x \oplus N) = x \oplus K$ for all $x \in M$. Let $x \oplus N = y \oplus N$. Hence, for all $u \in [0, 1]$ we have $x \oplus_u N = y \oplus_u N$. Then

$$x \oplus K = \chi_{x \oplus_u K} = \chi_{x \oplus_u N \oplus_u K} = \chi_{y \oplus_u N \oplus_u K} = \chi_{y \oplus_u K} = y \oplus K,$$

and so α is well-defined. Moreover, α is onto and $\ker \alpha = [K : N^*]$. Now, let

$x, y, t \in M$ and $u \in (0, 1]$. Then

$$\begin{aligned}
\alpha(\chi_{K^*[x]} \oplus_N \chi_{K^*[y]})(\chi_{K^*[t]}) \geq u &\Longrightarrow \bigvee_{r\oplus N \in \alpha^{-1}(t \oplus K)} \big((x \oplus N) \oplus_N (y \oplus N)\big)(r \oplus N) \geq u \\
&\Longrightarrow \bigvee_{r \oplus K = t \oplus K} (x \oplus y \oplus N)(r) \geq u \\
&\Longrightarrow \exists r \oplus K = t \oplus K;\ r \in x \oplus_u y \oplus_u N \\
&\Longrightarrow \exists r \oplus K = t \oplus K;\ r \in x \oplus_u y \oplus_u K \quad (N \subseteq K) \\
&\Longrightarrow \exists r \oplus K = t \oplus K;\ (x \oplus y \oplus K)(r) \geq u \\
&\Longrightarrow \big((x \oplus K) \oplus_K (y \oplus K)\big)(r \oplus K) \geq u \\
&\Longrightarrow \big((x \oplus K) \oplus_K (y \oplus K)\big)(t \oplus K) \geq u \\
&\Longrightarrow \big(\alpha(\chi_{K^*[x]}) \oplus_K \alpha(\chi_{K^*[y]})\big)(\chi_{K^*[t]}) \geq u.
\end{aligned}$$

Also, $\big(\alpha(\chi_{K^*[x]}) \oplus_K \alpha(\chi_{K^*[y]})\big)(\chi_{K^*[t]}) \geq u$ implies that $t \in x \oplus_u y \oplus_u K$ and so there exists $r \in x \oplus_u y$ such that $r \oplus_u K = t \oplus_u K$. Since K is normal, it follows that $r \oplus_u N = x \oplus_u y \oplus_u N$. Also, we have $r \in r \oplus_u 0 \subseteq r \oplus_u N = x \oplus_u y \oplus_u N$. Then $(x \oplus y \oplus N)(r) \geq u$. Thus

$$\begin{aligned}
\alpha\big(\chi_{K^*[x]} \oplus_N \chi_{K^*[y]}\big)(\chi_{K^*[t]}) &= \bigvee_{r \oplus N \in \alpha^{-1}(t \oplus K)} \big((x \oplus N) \oplus_N (y \oplus N)\big)(r \oplus N) \\
&= \bigvee_{r \oplus K = t \oplus K} (x \oplus y \oplus N)(r) \geq u.
\end{aligned}$$

Moreover, $\alpha(\chi_{K^*[x]} \oplus_N \chi_{K^*[y]})(\chi_{K^*[t]}) = 0$ if and only if

$$\big(\alpha(\chi_{K^*[x]}) \oplus_K \alpha(\chi_{K^*[y]})\big)(\chi_{K^*[t]}) = 0.$$

Also, we have $\alpha\big(r \odot_N \chi_{K^*[x]}\big) = r \odot_K \alpha\big(\chi_{K^*[x]}\big)$. Therefore, α is a strong fuzzy homomorphism. By Theorem 4.1, we obtain

$$[[M : N^*] : [K : N^*]] \cong [M : K^*]$$

□

5 Conclusion and future work

Using a connection between hyperstructures and fuzzy hyperstructures, presented in [21], we extend the context of researches in fuzzy algebraic hyperstructures, in particular the study of fuzzy hypermodules and their quotients. This study can be continued for other classes of fuzzy hyperstructures.

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