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A Fixed Point in Partial S_b-Metric Spaces

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Abstract

In this paper, we introduce an interesting extention of the partial b-metric spaces called partial S_b-metric spaces, and we show the existence of fixed point for a self mapping defined on such spaces.

1 Introduction

There exist many generalizations of the concept of metric spaces in the literature. Several papers have been published on the fixed point theory in S-metric spaces [7], [8], [9], [13], and [14]. Also, fixed point results in b-metric spaces were also studied by many authors [1], [2], [3], [4], [5] and [15].
In this work, we consider a new concept of S-metric spaces called partial S_b-metric spaces, which is an extension of the S-metric spaces, by allowing the self distance to be different from zero. We extend the results obtained by Shukla [15] in partial b-metric spaces, and we prove theorems for some contractive type mapping.

First we would like to point out three errors in the proof of Theorem 1 (on page 5) in [15]. The equation $b(Fz, Fx_l) = \lambda^{n_0} b(z, x_l)$ must be an inequality. Also, the inequality $b(Fz, x_l) \leq s[b(Fz, Fx_l) + b(Fx_l, x_l)] - b(x_l, x_l)$, should instead be written as $b(Fz, x_l) \leq s[b(Fz, Fx_l) + b(Fx_l, x_l)] - b(Fx_l, Fx_l)$. The author used a wrong argument to show that $\{x_n\}$ is Cauchy sequence by mentioning that since $x_n \in B[x_l, \frac{\epsilon}{2}]$ and $x_m \in B[x_l, \frac{\epsilon}{2}]$, then $b(x_n, x_m) < \dfrac{\epsilon}{2} + b(x_l, x_l)$ for all $n, m > l$. We suggest using the contraction principle after

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showing that $Fz \in B[x_l, \frac{\epsilon}{2}]$.

Let us recall the definitions of the b-metric spaces and the partial b-metric spaces.

Definition 1.1. [2] Let X be a nonempty set. A b-metric on X is a function $d : X^2 \to [0, \infty)$ if there exists a real number $s \geq 1$ such that the following conditions hold for all $x, y, z \in X$:

(i) $d(x, y) = 0$ if and only if $x = y$

(ii) $d(x, y) = d(y, x)$

(iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a b-metric space.

Definition 1.2. [15] A partial b-metric on a nonempty set X is a function $b : X^2 \to [0, \infty)$ such that for all $x, y, z \in X$: :

(i) $x = y$ if and only if $b(x, x) = b(x, y) = b(y, y)$

(ii) $b(x, x) \leq b(x, y)$

(iii) $b(x, y) = b(y, x)$

(iv) there exists a real number $s \geq 1$ such that $b(x, y) \leq s[b(x, z) + b(z, y)] - b(z, z)$.

The partial b-metric space is a pair (X, b) such that X is a nonempty set and b is a partial b-metric on X.

Definition 1.3. A partial S_b-metric on a empty set X is a function $S_b : X^3 \longrightarrow \mathbb{R}_+$ such that for all $x, y, z, t \in X$:

(i) $x = y = z$ if and only if $S_b(x, x, x) = S_b(y, y, y) = S_b(z, z, z) = S_b(x, y, z)$

(ii) $S_b(x, x, x) \leq S_b(x, y, z)$

(iii) $S_b(x, x, y) = S_b(y, y, x)$

(iv) there exists $s \geq 1$ such that

$$S_b(x, y, z) \leq s\left[S_b(x, x, t) + S_b(y, y, t) + S_b(z, z, t)\right] - S_b(t, t, t).$$

(X, S_b) is then called a partial S_b-metric space.

Definition 1.4. Let (X, S_b) be a partial S_b-metric space and $\{x_n\}$ be a sequence in X. Then:

1. $\{x_n\}$ is called convergent if and only if there exists $z \in X$ such that $S_b(x_n, x_n, z) \longrightarrow S_b(z, z, z)$ as $n \to \infty$.
2. $\{x_n\}$ is said to be Cauchy sequence in (X, S_b) if $\lim_{n\to\infty} S_b(x_n, x_n, x_m)$ exists and finite.
3. (X, S_b) is a complete partial S_b-metric space if for every Cauchy sequence $\{x_n\}$ there exists $x \in X$ such that:
$$\lim_{n\to\infty} S_b(x_n, x_n, x_m) = \lim_{n\to\infty} S_b(x_n, x_n, x) = S_b(x, x, x).$$

Now, we give an example of a partial S_b-metric space that is not a partial S-metric space.

Example 1.5. Let $X = \mathbb{R}_+$, and $p > 1$ be a constant and $S_b : X \times X \times X \longrightarrow \mathbb{R}_+$ defined by $S_b(x, y, z) = [max\{x, y\}]^p + |max\{x, y\} - z|^p$ for all $x, y, z \in X$. Then (X, S_b) is a partial S_b-metric space with coefficient $s = 2p > 1$, but it is not a partial S-metric space. Indeed, for $x = 5, y = 2, z = 1, t = 4$ we have $S_b(x, y, z) = 5^p + 4^p$ and $S_b(x, x, t) + S_b(y, y, t) + S_b(z, z, t) - S_b(t, t, t) = 5^p + 1 + 3^p + 1 + 1 + 3^p - 4^p = 5^p + 2 \times 3^p + 3 - 4^p$, hence $S_b(x, y, z) > S_b(x, x, t) + S_b(y, y, t) + S_b(z, z, t) - S_b(t, t, t)$ for all $p > 1$; therefore, S_b is not a partial S-metric on X.

2 Main result

Theorem 2.1. *Let (X, S_b) be a complete partial S_b-metric space with coefficient $s \geq 1$ and $T : X \longrightarrow X$ be a mapping satisfying the following condition:*

$$S_b(Tx, Ty, Tz) \leq \lambda S_b(x, y, z) \quad \forall x, y, z \in X, \quad \lambda \in [0, 1). \tag{2.1}$$

Then, T has a unique fixed point $u \in X$ and $S_b(u, u, u) = 0$.

Proof. Let's start by proving the uniqueness of the fixed point. Let $u, v \in X$ be two distinct fixed point of T, that is, $Tu = u$ and $Tv = v$.
We have

$$S_b(u, u, v) = S_b(Tu, Tu, Tv) \leq \lambda S_b(u, u, v) < S_b(u, u, v).$$

So, we must have $S_b(u, u, v) = 0 \Longrightarrow u = v$. Therefore, if T has a fixed point, then it is unique.

Let prove that $S_b(u,u,u)=0$.
Suppose that $S_b(u,u,u)>0$. From equation (2.1),

$$S_b(u,u,u)=S_b(Tu,Tu,Tu)\leq \lambda S_b(u,u,u)<S_b(u,u,u),$$

which leads to a contradiction, then $S_b(u,u,u)=0$.

For the existence of fixed point, since $\lambda\in[0,1)$, we can choose $n_0\in\mathbb{N}$ such that for given $0<\epsilon<1$, we have

$$\lambda^{n_0}<\frac{\epsilon}{8s}. \tag{2.2}$$

Let $T^{n_0}\equiv F$ and $Fx_0^k=x_k$ $\forall k\in\mathbb{N}$, where $x_0\in X$ is arbitrary. Then, $\forall x,y\in X$ we have

$$S_b(Fx,Fy,Fz)=S_b(T^{n_0}x,T^{n_0}y,T^{n_0}z)\leq \lambda^{n_0}S_b(x,y,z).$$

For any $k\in\mathbb{N}$, we have

$$\begin{aligned} S_b(x_{k+1},x_{k+1},x_k)=S_b(Fx_k,Fx_k,Fx_{k-1}) &\leq \lambda^{n_0}S_b(x_k,x_k,x_{k-1})\\ &\leq \lambda^{n_0}kS_b(x_1,x_1,x_0)\longrightarrow 0 \text{ as } k\to+\infty. \end{aligned}$$

Therefore, we can choose $l\in\mathbb{N}$ such that $S_b(x_{l+1},x_{l+1},x_l)<\dfrac{\epsilon}{8s}.(*)$
Let's define the ball

$$B_b(x_l,\frac{\epsilon}{2}):=\{y\in X/S_b(x_l,x_l,y)<\frac{\epsilon}{2}+S_b(x_l,x_l,x_l)\} \tag{2.3}$$

Now, we shall show that F maps $B_b(x_l,\frac{\epsilon}{2})$ into itself.
We have $B_b(x_l,\frac{\epsilon}{2})\neq\emptyset$ since $x_l\in B_b(x_l,\frac{\epsilon}{2})$. Let $x_z\in B_b(x_l,\frac{\epsilon}{2})$, then

$$\begin{aligned} S_b(Fx_z,Fx_z,Fx_l) &\leq \lambda^{n_0}S_b(x_z,x_z,x_l)\\ &\leq \frac{\epsilon}{8s}S_b(x_z,x_z,x_l)\\ &\leq \frac{\epsilon}{8s}[\frac{\epsilon}{2}+S_b(x_l,x_l,x_l)]\\ &\leq \frac{\epsilon}{8s}[1+S_b(x_l,x_l,x_l)]. \end{aligned} \tag{2.4}$$

Using the definition of the partial S_b-metric space, we obtain

$$\begin{aligned} S_b(Fx_z, Fx_l, Fx_l) \leq\ & s[S_b(Fx_z, Fx_z, Fx_l) + S_b(Fx_l, Fx_l, Fx_l) + S_b(Fx_l, Fx_l, Fx_l)] \\ - & \ S_b(Fx_l, Fx_l, Fx_l) \\ \leq & \ s[\frac{\epsilon}{8s}(1 + S_b(x_l, x_l, x_l)) + 2S_b(x_l, x_l, Fx_l)] \\ \leq & \ s[\frac{\epsilon}{8s}(1 + S_b(x_l, x_l, x_l)) + 2S_b(x_l, x_l, x_{l+1})] \\ \leq & \ s[\frac{\epsilon}{8s}(1 + S_b(x_l, x_l, x_l)) + 2\frac{\epsilon}{8s}] \\ \leq & \ \frac{\epsilon}{8} + \frac{\epsilon}{8}S_b(x_l, x_l, x_l) + \frac{\epsilon}{4} \\ \leq & \ \frac{3\epsilon}{8} + \frac{\epsilon}{8}S_b(x_l, x_l, x_l) \\ \leq & \ \frac{\epsilon}{2} + S_b(x_l, x_l, x_l). \end{aligned}$$

Then, $Fx_z \in B_b(x_l, \frac{\epsilon}{2})$. Thus F maps $B_b(x_l, \frac{\epsilon}{2})$ to itself.
We note that $x_l \in B_b(x_l, \frac{\epsilon}{2})$, therefore $Fx_l \in B_b(x_l, \frac{\epsilon}{2})$. By repeating this process, we obtain $F^n x_l \in B_b(x_l, \frac{\epsilon}{2})$ $\forall n \in \mathbb{N}$, that is $x_m \in B_b(x_l, \frac{\epsilon}{2})$ $\forall m \geq l$.
Therefore, we obtain for all $m > n \geq l$; let $n = l + i \Longrightarrow i = n - l$

$$\begin{aligned} S_b(x_n, x_n, x_m) &= S_b(Tx_{n-1}, Tx_{n-1}, Tx_{m-1}) \\ &\leq \lambda S_b(x_{n-1}, x_{n-1}, x_{m-1}) \\ &\leq \lambda^2 S_b(x_{n-2}, x_{n-2}, x_{m-2}) \\ &\ \vdots \\ &\leq \lambda^i S_b(x_l, x_l, x_{m-l}) \\ &< S_b(x_l, x_l, x_{m-l}) \\ &< \frac{\epsilon}{2} + S_b(x_l, x_l, x_l). \end{aligned}$$

But, $S_b(x_l, x_l, x_l) < S_b(x_l, x_l, x_{l+1}) < \dfrac{\epsilon}{8s}$.
Hence,

$$S_b(x_n, x_n, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{8s} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus, $\{x_n\}$ is a Cauchy sequence.
Since X is a complete partial S_b-metric sapce, there exists $u \in X$ such that:

$$\lim_{n\to\infty} S_b(x_n, x_n, u) = \lim_{n\to\infty} S_b(x_n, x_n, x_m) = S_b(u, u, u) = 0.$$

Let's prove that u is a fixed point of T. For all $n \in \mathbb{N}$, we have

$$\begin{aligned} S_b(u,u,Tu) &\leq s[S_b(u,u,x_{n+1}) + S_b(u,u,x_{n+1}) + S_b(Tu,Tu,x_{n+1})] \\ &\quad -S_b(x_{n+1},x_{n+1},x_{n+1}) \\ &\leq s[2S_b(u,u,x_{n+1}) + S_b(Tu,Tu,Tx_n)] \\ &\leq s[2S_b(u,u,x_{n+1}) + S_b(Tu,Tu,Tx_n)] \\ &\leq s[2S_b(u,u,x_{n+1}) + \lambda S_b(u,u,x_n)] \\ &\leq (2sS_b(u,u,x_{n+1}) + s\lambda S_b(u,u,x_n)) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus, $S_b(u,u,Tu) = 0$, that is $Tu = u$. Hence, u is a unique fixed point of T. □

Theorem 2.2. *Let (X, S_b) be a complete partial S_b-metric space with coefficient $s \geq 1$ and $T : X \longrightarrow X$ be a mapping satisfying the following condition:*

$$S_b(Tx,Ty,Tz) \leq \lambda[S_b(x,x,Tx) + S_b(y,y,Ty) + S_b(z,z,Tz)] \quad \forall x,y,z \in X. \tag{2.5}$$

where $\lambda \in [0, \frac{1}{3})$, $\lambda \neq \frac{1}{3s}$ Then, T has a unique fixed point $u \in X$ and $S_b(u,u,u) = 0$.

Proof. We first prove the uniqueness of the fixed point of T if it has. We must show that, if $u \in X$ is a fixed point of T, that is $Tu = u$ then $S_b(u,u,u) = 0$.
From(2.5), we obtain

$$\begin{aligned} S_b(u,u,u) = S_b(Tu,Tu,Tu) &\leq \lambda[S_b(u,u,Tu) + S_b(u,u,Tu) + S_b(u,u,Tu)] \\ &= 3\lambda S_b(u,u,Tu) \text{ since } \lambda \in [0, \tfrac{1}{3}), \text{ we have} \\ &< S_b(u,u,u), \end{aligned}$$

which implies that we must have $S_b(u,u,u) = 0$
Suppose $u, v \in X$ be two fixed point, that is $Tu = u$ and $Tv = v$. Then we have $S_b(u,u,u) = S_b(v,v,v) = 0$.
Equation (2.5) gives

$$\begin{aligned} S_b(u,u,v) &= S_b(Tu,Tu,Tv) \\ &\leq \lambda[S_b(u,u,Tu) + S_b(u,u,Tu) + S_b(v,v,Tv)] \\ &= 2\lambda S_b(u,u,u) + \lambda S_b(v,v,v) \\ &= 0. \end{aligned}$$

Therefore, $u = v$. Thereby, the uniqueness of the fixed point if it exists.

For the existence of the fixed point, let $x_0 \in X$ arbitrary, set $x_n = T^n x_0$ and $S_{b_n} = S(x_n, x_n, x_{n+1})$.
We can assume $S_{b_n} > 0$ for all $n \in \mathbb{N}$ otherwise x_n is a fixed point of T for at least one $n \geq 0$. For all n, we obtain from (2.5)

$$\begin{aligned} S_{b_n} &= S_b(x_n, x_n, x_{n+1}) = S_b(Tx_{n-1}, Tx_{n-1}, Tx_n) \\ &\leq \lambda[2S_b(x_{n-1}, x_{n-1}, Tx_{n-1}) + S_b(x_n, x_n, Tx_n)] \\ &= \lambda[2S_b(x_{n-1}, x_{n-1}, x_n) + S_b(x_n, x_n, x_{n+1})] \\ &= \lambda[2S_{b_{n-1}} + S_{b_n}]. \end{aligned}$$

Therefore $(1-\lambda)S_{b_n} \leq 2\lambda S_{b_{n-1}}$. Thus

$$S_{b_n} \leq \frac{2\lambda}{1-\lambda} S_{b_{n-1}}, \quad \lambda \in [0, \frac{1}{3}). \tag{2.6}$$

Let $\beta = \dfrac{2\lambda}{1-\lambda} < 1$. By repeating this process we obtain

$$S_{b_n} \leq \beta^n b_0.$$

Therefore, $\lim\limits_{n\to\infty} S_{b_n} = 0$. Let prove that $\{x_n\}$ is a Cauchy sequence. It follows from (2.5) that for $n, m \in \mathbb{N}$:

$$\begin{aligned} S_b(x_n, x_n, x_m) &= S_b(T^n x_0, T^n x_0, T^m x_0) \\ &= S_b(Tx_{n-1}, Tx_{n-1}, Tx_{m-1}) \\ &\leq \lambda[2S_b(x_{n-1}, x_{n-1}, Tx_{n-1}) + S_b(x_{m-1}, x_{m-1}, Tx_{m-1})] \\ &= \lambda[2S_b(x_{n-1}, x_{n-1}, x_n) + S_b(x_{m-1}, x_{m-1}, x_m)] \\ &= \lambda[2S_{b_{n-1}} + S_{b_{m-1}}]. \end{aligned}$$

So, for every $\epsilon > 0$, as $\lim\limits_{n\to\infty} S_{b_n} = 0$, we can find $n_0 \in \mathbb{N}$ such that $S_{b_{n-1}} < \dfrac{\epsilon}{4}$ and $S_{b_{m-1}} < \dfrac{\epsilon}{2}$ for all $n, m > n_0$. Then, we obtain $2S_{b_{n-1}} + S_{b_{m-1}} \leq 2\dfrac{\epsilon}{4} + \dfrac{\epsilon}{2} = \epsilon$.
As $\lambda < 1$ it follows that $S_b(x_n, x_n, x_m) < \epsilon \ \forall n, m > n_0$.
Thus, $\{x_n\}$ is a Cauchy sequence in X and $\lim\limits_{n\to\infty} S_b(x_n, x_n, x_m) = 0$.
By completeness of X, there exists $u \in X$ such that

$$\lim_{n\to\infty} S_b(x_n, x_n, u) = \lim_{n,m\to\infty} S_b(x_n, x_n, u) = S_b(u, u, u) = 0. \tag{2.7}$$

Now, we shall prove that $Tu = u$. For any $n \in \mathbb{N}$

$$\begin{aligned} S_b(u,u,Tu) &\leq s[2S_b(u,u,x_{n+1}) + S_b(Tu,Tu,x_{n+1})] - S_b(x_{n+1},x_{n+1},x_{n+1}) \\ &\leq s[2S_b(u,u,x_{n+1}) + S_b(Tu,Tu,Tx_n)] \\ &\leq s[2S_b(u,u,x_{n+1}) + \lambda[2S_b(u,u,Tu) + S_b(x_n,x_n,Tx_n)]. \end{aligned}$$

Therefore, $(1-2s\lambda)S_b(u,u,Tu) \leq 2sS_b(u,u,x_{n+1}) + s\lambda S_b(x_n,x_n,Tx_n)$ giving

$$S_b(u,u,Tu) \leq \frac{2s}{1-2s\lambda} S_b(u,u,x_{n+1}) + \frac{s\lambda}{1-2s\lambda} S_b(x_n,x_n,Tx_n).$$

Since $S_b(x_n,x_n,Tx_n) \longrightarrow S_b(u,u,Tu)$, $n \longrightarrow \infty$, we obtain

$$\begin{aligned} S_b(u,u,Tu) &\leq \frac{2s}{1-2s\lambda} S_b(u,u,x_{n+1}) + \frac{s\lambda}{1-2s\lambda} S_b(u,u,Tu) \\ (1-\frac{s\lambda}{1-2s\lambda})S_b(u,u,Tu) &\leq \frac{2s}{1-2s\lambda} S_b(u,u,x_{n+1}) \\ S_b(u,u,Tu) &\leq \frac{2s}{1-3s\lambda} S_b(u,u,x_{n+1}). \end{aligned}$$

As $\lambda \neq \frac{1}{3s}$ and from (2.7), we obtain $S_b(u,u,Tu) = 0$ and then $Tu = u$. □

Theorem 2.3. *Let (X, S_b) be a complete partial S_b-metric space with coefficient $s > 1$ and $T : X \longrightarrow X$ be a mapping satisfying the following condition:*

$$S_b(Tx,Ty,Tz) \leq \lambda \max[S_b(x,y,z), S_b(x,x,Tx), S_b(y,y,Ty), S_b(z,z,Tz)] \ \ \forall x,y,z \in X. \tag{2.8}$$

where $\lambda \in [0, \frac{1}{2s})$. Then, T has a unique fixed point $u \in X$ and $S_b(u,u,u) = 0$.

Proof. Let us prove that if a fixed point of T exists, then it is unique. Let $u, v \in X$ be two fixed points of T, $u \neq v$, that is $Tu = u$ and $Tv = v$. It follows from (2.8):

$$\begin{aligned} S_b(u,u,v) = S_b(Tu,Tu,Tv) &\leq \lambda \max[S_b(u,u,v), S_b(u,u,Tu), S_b(u,u,Tu), S_b(v,v,Tv)] \\ &= \lambda \max[S_b(u,u,v), S_b(u,u,u), S_b(v,v,v)] \\ &= \lambda S_b(u,u,v) \\ &< S_b(u,u,v) \text{ since } \lambda < 1. \end{aligned}$$

We obtain $S_b(u,u,v) < S_b(u,u,v)$ which gives $S_b(u,u,v) = 0$, then $u = v$. Therefore, if a fixed point of T exists, then it is unique.

Let $x_0 \in X$ and define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ $\forall n \geq 0$. For any n, we obtain from (2.8)

$$\begin{aligned} S_b(x_{n+1}, x_{n+1}, x_n) &= S_b(Tx_n, Tx_n, Tx_{n-1}) \\ &\leq \lambda \max[S_b(x_n, x_n, x_{n-1}), S_b(x_n, x_n, Tx_n), S_b(x_n, x_n, Tx_n), S_b(x_{n-1}, x_{n-1}, Tx_{n-1})] \\ &= \lambda \max[S_b(x_n, x_n, x_{n-1}), S_b(x_n, x_n, Tx_n), S_b(x_{n-1}, x_{n-1}, Tx_{n-1})]. \end{aligned}$$

Since $S_b(x_{n-1}, x_{n-1}, Tx_{n-1}) = S_b(x_{n-1}, x_{n-1}, x_n)$ and by symmetry we have $S_b(x_{n-1}, x_{n-1}, x_n) = S_b(x_n, x_n, x_{n-1})$, thus

$$S_b(x_{n+1}, x_{n+1}, x_n) \leq \lambda \max[S_b(x_n, x_n, x_{n-1}), S_b(x_n, x_n, x_{n+1})].$$

If $\max[S_b(x_n, x_n, x_{n-1}), S_b(x_n, x_n, x_{n+1})] = S_b(x_n, x_n, x_{n+1})$, then we obtain

$$\begin{aligned} S_b(x_{n+1}, x_{n+1}, x_n) &\leq \lambda S_b(x_n, x_n, x_{n+1}) \\ &= \lambda S_b(x_{n+1}, x_{n+1}, x_n) \\ &< S_b(x_{n+1}, x_{n+1}, x_n) \text{ absurd.} \end{aligned}$$

Therefore, $\max[S_b(x_n, x_n, x_{n-1}), S_b(x_n, x_n, x_{n+1})] = S_b(x_n, x_n, x_{n-1})$ and

$$S_b(x_{n+1}, x_{n+1}, x_n) \leq \lambda S_b(x_n, x_n, x_{n-1}), \tag{2.9}$$

that is

$$S_b(Tx_n, Tx_n, Tx_{n-1}) \leq \lambda S_b(x_n, x_n, x_{n-1}). \tag{2.10}$$

By repeating this process, we obtain

$$S_b(x_{n+1}, x_{n+1}, x_n) \leq \lambda^n S_b(x_1, x_1, x_0). \tag{2.11}$$

For $n, m \in \mathbb{N}$, $m > n$, we obtain

$$
\begin{aligned}
S_b(x_n,x_n,x_m) \quad \leq \quad & s\big[S_b(x_n,x_n,x_{n+1})+S_b(x_n,x_n,x_{n+1})+S_b(x_m,x_m,x_{n+1})\big] \\
& -S_b(x_{n+1},x_{n+1},x_{n+1}) \\
& \leq 2sS_b(x_n,x_n,x_{n+1})+sS_b(x_m,x_m,x_{n+1}) \\
& \leq 2sS_b(x_n,x_n,x_{n+1})+s\Bigg[s\Bigg(S_b(x_m,x_m,x_{n+2})+S_b(x_m,x_m,x_{n+2}) \\
& +S_b(x_{n+1},x_{n+1},x_{n+2})\Bigg)-S_b(x_{n+2},x_{n+2},x_{n+2})\Bigg] \\
& \leq 2sS_b(x_n,x_n,x_{n+1})+s\Bigg[s(2S_b(x_m,x_m,x_{n+2})+S_b(x_{n+1},x_{n+1},x_{n+2}))\Bigg] \\
& \leq 2sS_b(x_n,x_n,x_{n+1})+s^2S_b(x_{n+1},x_{n+1},x_{n+2})+2s^2S_b(x_m,x_m,x_{n+2}) \\
& \leq 2sS_b(x_n,x_n,x_{n+1})+s^2S_b(x_{n+1},x_{n+1},x_{n+2})+2s^2\big[s(2S_b(x_m,x_m,x_{n+3})+ \\
& +S_b(x_{n+2},x_{n+2},x_{n+3}))\big] \\
& \leq 2sS_b(x_n,x_n,x_{n+1})+s^2S_b(x_{n+1},x_{n+1},x_{n+2})+2s^3S_b(x_{n+2},x_{n+2},x_{n+3}) \\
& +2^2s^3S_b(x_m,x_m,x_{n+3}) \\
& \leq 2sS_b(x_n,x_n,x_{n+1})+s^2S_b(x_{n+1},x_{n+1},x_{n+2})+\ldots+ \\
& +2^{m-n-2}s^{m-n}S_b(x_m,x_m,x_{m-1}). \\
& =2sS_b(x_{n+1},x_{n+1},x_n)+s^2S_b(x_{n+2},x_{n+2},x_{n+1})+\ldots+ \\
& +2^{m-n-2}s^{m-n}S_b(x_m,x_m,x_{m-1}).
\end{aligned}
$$

Now, using (2.11), we obtain

$$
\begin{aligned}
& S_b(x_n,x_n,x_m) \\
& \leq\ 2s\lambda^n S_b(x_1,x_1,x_0)+s^2\lambda^{n+1}S_b(x_1,x_1,x_0)+2s^3\lambda^{n+2}S_b(x_1,x_1,x_0)+\ldots+ \\
& +\ 2^{m-n-2}s^{m-n}\lambda^{m-1}S_b(x_1,x_1,x_0) \\
& \leq\ s\lambda^n\big[2+s\lambda+2s^2\lambda^2+2s^3\lambda^3+\ldots+2^{m-n-2}s^{m-n-1}\lambda^{m-n-1}\big]S_b(x_1,x_1,x_0) \\
& \leq\ 2s\lambda^n\big[1+\frac{1}{2}s\lambda+s^2\lambda^2+s^3\lambda^3+\ldots+2^{m-n-3}s^{m-n-1}\lambda^{m-n-1}\big]S_b(x_1,x_1,x_0) \\
& <\ 2s\lambda^n\big[1+2s\lambda+(2s\lambda)^2+(2s\lambda)^3+\ldots+(2s\lambda)^{m-n-1}\big]S_b(x_1,x_1,x_0) \\
& \leq\ 2s\lambda^n\frac{1-(2s\lambda)^{m-n}}{1-2s\lambda}S_b(x_1,x_1,x_0) \\
& <\ 2s\lambda^n\frac{1}{1-2s\lambda}S_b(x_1,x_1,x_0)\longrightarrow 0 \text{ as } n\longrightarrow\infty.
\end{aligned}
$$

Hence, $\lim_{n\longrightarrow\infty} S_b(x_n,x_n,x_m)=0$.
Thus, $\{x_n\}$ is a Cauchy sequence in X. Since X is a complete partial metric

space, then there exists $u \in X$ such that

$$\lim_{n \longrightarrow \infty} S_b(x_n, x_n, u) = \lim_{n \longrightarrow \infty} S_b(x_n, x_n, x_m) = S_b(u, u, u) = 0. \qquad (2.12)$$

Let's prove that u is a fixed point of T. $\forall n \in \mathbb{N}$, we have

$$\begin{array}{rcl} S_b(u, u, Tu) & \leq & s\big[2S_b(u, u, x_{n+1}) + S_b(Tu, Tu, x_{n+1})\big] - S_b(x_{n+1}, x_{n+1}, x_{n+1}) \\ & \leq & s\big[2S_b(u, u, x_{n+1}) + S_b(Tu, Tu, Tx_n)\big]. \end{array}$$

Using (2.10), we obtain $S_b(Tu, Tu, Tx_n) \leq \lambda S_b(u, u, x_n)$, then

$$\begin{array}{rcl} S_b(u, u, Tu) & \leq & 2sS_b(u, u, x_{n+1}) + s\lambda S_b(u, u, x_n) \\ & = & 2sS_b(x_{n+1}, x_{n+1}, u) + s\lambda S_b(x_n, x_n, u). \end{array}$$

Using (2.12) in the above inequality, we obtain $S_b(u, u, Tu) = 0$, then $Tu = u$. Therefore, u is a fixed point of T and it is unique. □

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