On the torsion group of elliptic curves induced by $D(4)$ -triples

Andrej Dujella and Miljen Mikić

Abstract

A $D(4)$ - m -tuple is a set of m integers such that the product of any two of them increased by 4 is a perfect square. A problem of extendibility of $D(4)$ - m -tuples is closely connected with the properties of elliptic curves associated with them. In this paper we prove that the torsion group of an elliptic curve associated with a $D(4)$ -triple can be either $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$, except for the $D(4)$ -triple $\{-1, 3, 4\}$ when the torsion group is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

1 Introduction

Let n be a given nonzero integer. A set of m nonzero integers $\{a_1, a_2, \dots, a_m\}$ is called a $D(n)$ - m -tuple (or a Diophantine m -tuple with the property $D(n)$) if $a_i a_j + n$ is a perfect square for all $1 \leq i < j \leq m$. Diophantus found the $D(256)$ -quadruple $\{1, 33, 68, 105\}$, while the first $D(1)$ -quadruple, the set $\{1, 3, 8, 120\}$, was found by Fermat (see [1], [2]).

One of the most interesting questions in the study of $D(n)$ - m -tuples is how large these sets can be. In this paper we will examine sets with the property $D(4)$. Mohanty and Ramasamy [17] were first to achieve a significant result on the nonextendibility of $D(4)$ - m -tuples. They proved that a $D(4)$ -quadruple $\{1, 5, 12, 96\}$ cannot be extended to a $D(4)$ -quintuple. Kedlaya [14]

Key Words: $D(4)$ -triples, elliptic curves, torsion group.
2010 Mathematics Subject Classification: Primary 11G05.
Received: April, 2013.
Revised: June, 2013
Accepted: June, 2013

later proved that if $\{1, 5, 12, d\}$ is a $D(4)$ -quadruple, then d has to be 96. Dujella and Ramasamy [9] generalized this result to the parametric family of $D(4)$ -quadruples $\{F_{2k}, 5F_{2k}, 4F_{2k+2}, 4L_{2k}F_{4k+2}\}$ involving Fibonacci and Lucas numbers. Other generalization to a two-parametric family of $D(4)$ -triples can be found in [13]. Dujella [6] proved that there does not exist a $D(1)$ -sextuple and that there are only finitely many $D(1)$ -quintuples. By observing congruences modulo 8, it is not hard to conclude that a $D(4)$ - m -tuple can contain at most two odd numbers (see [9, Lemma 1]). Thus, the results from [6] imply that there does not exist a $D(4)$ -8-tuple and that there are only finitely many $D(4)$ -7-tuples. Filipin [10, 11] significantly improved these results by proving that there does not exist a $D(4)$ -sextuple and that there are only finitely many $D(4)$ -quintuples.

Let $\{a, b, c\}$ be a $D(4)$ -triple. Then there exist nonnegative integers r, s, t such that

$$ab + 4 = r^2, \quad ac + 4 = s^2, \quad bc + 4 = t^2. \quad (1)$$

In order to extend this triple to a quadruple, we have to solve the system

$$ax + 4 = \square, \quad bx + 4 = \square, \quad cx + 4 = \square. \quad (2)$$

We assign to the system (2) the elliptic curve

$$E : y^2 = (ax + 4)(bx + 4)(cx + 4). \quad (3)$$

The purpose of this paper is to examine possible forms of torsion groups of elliptic curves obtained in this manner. Additional motivation for this paper is a gap found in the proof of [4, Lemma 1] concerning torsion groups of elliptic curves induced by $D(1)$ -triples. Namely, if $\{a', b', c'\}$ is a $D(1)$ -triple, then $\{2a', 2b', 2c'\}$ is a $D(4)$ -triple. Thus, the proof of Lemma 2 in present paper also provides a valid proof of [4, Lemma 1].

2 Torsion group of E

The coordinate transformation

$$x \mapsto \frac{x}{abc}, \quad y \mapsto \frac{y}{abc}$$

applied on the curve E leads to the elliptic curve

$$E' : y^2 = (x + 4bc)(x + 4ac)(x + 4ab).$$

There are three rational points on E' of order 2:

$$A' = (-4bc, 0), \quad B' = (-4ac, 0), \quad C' = (-4ab, 0),$$

and also other obvious rational points

$$P' = (0, 8abc), S' = (16, 8rst).$$

It is not so obvious, but it is easy to verify that $S' \in 2E'(\mathbb{Q})$. Namely, $S' = 2R'$, where

$$R' = (4rs + 4rt + 4st + 16, 8(r + s)(r + t)(s + t)).$$

In this section we will first examine one special case and after that we may assume without the loss of generality that a, b, c are positive integers such that $a < b < c$. Since $\{-a, -b, -c\}$ induces the same curve as $\{a, b, c\}$, a problem may arise only when there are mixed signs. It is easily seen that the only such possible $D(4)$ -triple is $\{-1, 3, 4\}$ (and the equivalent one $\{-4, -3, 1\}$). The elliptic curve associated with this $D(4)$ -triple has rank 0 and the torsion group isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. In this special case $B' \in 2E'(\mathbb{Q})$, more precisely $B' = 2P'$, so the point P' is of order 4. Note that in this case the point R' is also of order 4 since $R' = P' + A'$ and thus $2R' = 2P'$.

Thus, we assume from now on that a, b, c are positive integers such that $a < b < c$.

Lemma 1. *If $\{a, b, c\}$ is $D(4)$ -triple, then $c = a + b + 2r$ or $c > ab + a + b + 1 > ab$.*

Proof. By [5, Lemma 3], there exists an integer

$$e = 4(a + b + c) + 2(abc - rst) \tag{4}$$

and nonnegative integers x, y, z such that

$$ae + 16 = x^2, \tag{5}$$

$$be + 16 = y^2, \tag{6}$$

$$ce + 16 = z^2 \tag{7}$$

and $c = a + b + \frac{e}{4} + \frac{1}{8}(abe + rxy)$. From (7), it follows that $e \geq 0$ (the case $e = -1$ implies $c \leq 16$, but the only such $D(4)$ -triple $\{1, 5, 12\}$ does not satisfy (5) and (6)). For $e = 0$ we get $c = a + b + 2r$, while for $e \geq 1$ we have $c > \frac{1}{4}abe + a + b + \frac{e}{4}$. By observing congruences modulo 8, we can easily prove that at most two of the integers a, b, c are odd, which implies that $abc - rst$ is even. Hence, from (4) we conclude that $e \equiv 0 \pmod{4}$. It follows $e \geq 4$ and thus $c > ab + a + b + 1$. $\square$

Remark 1. Filipin (see [12, Lemma 4]) proved that $c = a + b + 2r$ or $c > \frac{1}{4}abe$. Lemma 1 may be considered as a slight improvement of that result.

Remark 2. Lemma 1 implies $c \geq a + b + 2r$. Indeed, the inequality $ab + a + b + 1 \geq a + b + 2r$ is equivalent to $(r - 3)(r + 1) \geq 0$, and this is satisfied for all $D(4)$ -triples with positive elements.

Remark 3. The statement of Lemma 1 is sharp in the sense that the inequality $c > ab$ cannot be replaced by $c > (1 + \varepsilon)ab$ for any fixed $\varepsilon > 0$. Indeed, for an integer $k \geq 3$, if we put $a = k^2 - 4$, $b = k^2 + 2k - 3$, $c = k^4 + 2k^3 - 3k^2 - 4k$, then $\{a, b, c\}$ is a $D(4)$ -triple and $\lim_{k \rightarrow \infty} \frac{c}{ab} = 1$.

In the next lemma we show that E' cannot have a point of order 4. We follow the strategy of the proof of an analogous result for $D(1)$ -triples [4, Lemma 1]. However, we have noted a serious gap in the proof of [4, Lemma 1]. Namely, [4, formula (7)] should be $(\beta^2 - 1)^2 = b(4c\beta^2 - a^2b - 2a(1 + \beta^2))$, instead of $(\beta^2 - 1)^2 = b(4c - a^2b - 2a(1 + \beta^2))$, so later arguments are not accurate in the case $\beta \neq 1$. Here we will prove more general result, but by taking a, b, c to be even, in the same time we fill the mentioned gap in the proof of [4, Lemma 1].

Lemma 2. $A', B', C' \notin 2E'(\mathbb{Q})$

Proof. If $A' \in 2E'(\mathbb{Q})$, then the 2-descent Proposition [15, 4.2, p.85] implies that $c(a - b)$ is a square. But $c(a - b) < 0$, a contradiction. Similarly, $B' \notin 2E'(\mathbb{Q})$. If $C' \in 2E'(\mathbb{Q})$, then

$$a(c - b) = X^2, \quad (8)$$

$$b(c - a) = Y^2, \quad (9)$$

for integers X and Y .

If $\{a, b, c\}$ is a $D(4)$ -triple where $a < b < c$, then $c = a + b + 2r$ or $c > ab + a + b + 1$ by Lemma 1.

Assume first that $c = a + b + 2r$. From (8) and (9), we get that $a = kx^2$, $c - b = ky^2$, $b = lz^2$, $c - a = lu^2$, where k, l, x, y, z, u are positive integers. We have $c = kx^2 + lu^2 = ky^2 + lz^2$, and from $c = a + b + 2r$ we get

$$2r = k(y^2 - x^2) = l(u^2 - z^2). \quad (10)$$

By squaring (10), we obtain

$$4r^2 = 16 + 4ab = 16 + 4klx^2z^2 = k^2(y^2 - x^2)^2 = l^2(u^2 - z^2)^2,$$

which implies that $k \in \{1, 2, 4\}$ and $l \in \{1, 2, 4\}$. Since kl is not a perfect square (otherwise $(2r)^2 = 16 + (2xz\sqrt{kl})^2$ which implies $2r = 5$), we may take without loss of generality $k = 1, l = 2$ or $k = 2, l = 4$. For $k = 1, l = 2$, we have $4r^2 = 16 + 8x^2z^2$, which implies $r^2 = 4 + 2x^2z^2$, which leads to the conclusion that r is even and xz is even. Therefore, $r^2 \equiv 4 \pmod{8}$ and $r \equiv 2 \pmod{4}$. But from $2r = 2(u^2 - z^2)$ we conclude $u^2 - z^2 \equiv 2 \pmod{4}$, and that is impossible. If $k = 2, l = 4$, then $4r^2 = 16 + 32x^2z^2$, which implies $r^2 = 4 + 8x^2z^2$, thus $r^2 \equiv 4 \pmod{8}$ and $r \equiv 2 \pmod{4}$. But from $2r = 2(y^2 - x^2)$ we conclude $y^2 - x^2 \equiv 2 \pmod{4}$, and that is impossible.

Assume now that $c > ab + a + b + 1 > ab$.

Let us write the conditions (8) and (9) in the form

$$ac - ab = s^2 - r^2 = (s - \alpha)^2, \quad (11)$$

$$bc - ab = t^2 - r^2 = (t - \beta)^2, \quad (12)$$

where $0 < \alpha < s, 0 < \beta < t$. Then we have

$$r^2 = 2s\alpha - \alpha^2 = 2t\beta - \beta^2. \quad (13)$$

From (13) we get

$$4(bc + 4)\beta^2 = (ab + 4 + \beta^2)^2$$

and

$$(\beta^2 - 4)^2 = b(4c\beta^2 - a^2b - 2a(4 + \beta^2)). \quad (14)$$

From (14) we conclude that either $\beta = 1$ or $\beta = 2$ or $\beta^2 \geq \sqrt{b} + 4$.

If $\beta = 1$, then

$$b(4c - a^2b - 10a) = 9 \quad (15)$$

which implies $b \mid 9$, but that is possible only for $b = 9$ (there are no $D(4)$ -triples with $b < 4$). This implies $a = 5$, but (15) then gives $c = 69$ and $\{5, 9, 69\}$ is not a $D(4)$ -triple.

If $\beta = 2$, then from (14) we find that

$$c = \frac{a^2b + 16a}{16}. \quad (16)$$

Now we have

$$s^2 = ac + 4 = \frac{1}{16}(a^3b + 16a^2 + 64) = \frac{1}{16}(a^2r^2 + 12a^2 + 64).$$

Hence $s^2 > \left(\frac{ar}{4}\right)^2$ and $s^2 < \left(\frac{ar+8}{4}\right)^2$. Therefore we have to consider several cases:

1. $s^2 = \left(\frac{ar+n}{4}\right)^2$, where n is odd. That is equivalent to

$$2a(rn - 6a) = 64 - n^2. \quad (17)$$

The left hand side of (17) is even and the right hand side is odd, a contradiction.

2. $s^2 = \left(\frac{ar+2}{4}\right)^2$, or equivalently $a(r - 3a) = 15$. The cases $a \leq 3$ and (16) imply that $c < b$. The case $a = 5$ gives the triple $\{5, 64, 105\}$ that does not satisfy $c > ab$ (c equals $a + b + 2r$), and $a = 15$ leads to $15b + 4 = 46^2$ which has no integer solutions.
3. $s^2 = \left(\frac{ar+4}{4}\right)^2$, or equivalently $a(2r - 3a) = 12$. We conclude that a must be even and we get triples: $\{2, 16, 6\}$ (with $c < b$) and $\{6, 16, 42\}$ (with $c = a + b + 2r$), so we can eliminate this case.
4. $s^2 = \left(\frac{ar+6}{4}\right)^2$ is equivalent to $3a(r - a) = 7$, which is clearly impossible.

Thus, we may assume that $\beta^2 \geq \sqrt{b} + 4$, which implies

$$\beta > \max\{\sqrt[4]{b}, 2\} \quad (18)$$

The function $f(\beta) = t^2 - (t - \beta)^2$ is increasing for $0 < \beta < t$. Thus we have

$$ab = t^2 - (t - \beta)^2 - 4 > 2t\sqrt[4]{b} - \sqrt{b} - 4 > 2\sqrt{bc}\sqrt[4]{b} - \sqrt{b} - 4,$$

which implies $ab > \sqrt{bc}\sqrt[4]{b}$, because $\sqrt{b}(\sqrt{c}\sqrt[4]{b} - 1) > 4$ (since $b \geq 4$ and $c \geq 12$, which follows from the fact that $\{3, 4, 15\}$ and $\{1, 5, 12\}$ are $D(4)$ -triples with smallest b and c respectively). This further gives

$$c < a^2\sqrt{b}. \quad (19)$$

We will use (4) to define the integer d_- as

$$d_- = \frac{e}{4} = a + b + c + \frac{abc - rst}{2}$$

Then $d_- \neq 0$ (since $c \neq a + b + 2r$) and $\{a, b, c, d_-\}$ is a $D(4)$ -quadruple. In particular,

$$ad_- + 4 = \left(\frac{rs - at}{2}\right)^2. \quad (20)$$

Moreover,

$$c = a + b + d_- + \frac{1}{2}(abd_- + \sqrt{(ab+4)(ad_-+4)(bd_-+4)}) > abd_- \quad (21)$$

(see the proof of Lemma 1). By comparing this with (19), we get

$$d_- < \frac{a}{\sqrt{b}}. \quad (22)$$

Therefore, we have $d_- < a < b$ which implies that b is the largest element in the $D(4)$ -triple $\{a, b, d_-\}$. Thus, by Remark 2, $b \geq a + d_- + 2\sqrt{ad_- + 4}$ or equivalently $d_- \leq a + b - 2r$. Let us define also

$$c' = a + b + d_- + \frac{1}{2}(abd_- - \sqrt{(ab+4)(ad_-+4)(bd_-+4)}).$$

We have

$$\begin{aligned} cc' &= (a + b + d_- + \frac{1}{2}abd_-)^2 - \frac{1}{4}(ab+4)(ad_-+4)(bd_-+4) \\ &= (a + b + d_-)^2 - 4ab - 4ad_- - 4bd_- - 16 \\ &= (a + b - d_-)^2 - 4r^2 = (a + b + 2r - d_-)(a + b - 2r - d_-) \geq 0. \end{aligned}$$

This implies

$$c < 2(a + b + d_- + \frac{1}{2}abd_-) < 4b + abd_- < 2abd_-. \quad (23)$$

(we use here $ad_- > 4$ which is true because $\{a, d_-\}$ is a $D(4)$ -pair). Let us denote $p = \frac{rs-at}{2}$. Then $p > 0$ and, by (20), we have $ad_- + 4 = p^2$. In order to estimate the size of p , we also define $p' = \frac{rs+at}{2}$. Then

$$pp' = \frac{1}{4}(a^2bc + 4ac + 4ab + 16 - a^2bc - 4a^2) = a(b + c - a) + 4,$$

and

$$\begin{aligned} p &< \frac{2a(c+b)}{2at} < \frac{c+b}{\sqrt{bc}} = \frac{\sqrt{c}}{\sqrt{b}} + \frac{\sqrt{b}}{\sqrt{c}}, \\ p &> \frac{2(ac+4)}{2rs} = \frac{s}{r}. \end{aligned}$$

Furthermore, we have

$$\frac{\sqrt{c}}{\sqrt{b}} - \frac{s}{r} = \frac{r\sqrt{c} - s\sqrt{b}}{r\sqrt{b}} = \frac{4c - 4b}{r\sqrt{b}(r\sqrt{c} + s\sqrt{b})} < \frac{4c}{2rsb} < \frac{2\sqrt{c}}{ab\sqrt{b}},$$

and thus

$$p > \frac{\sqrt{c}}{\sqrt{b}} - \frac{2\sqrt{c}}{ab\sqrt{b}}. \quad (24)$$

The inequality (19) implies that $c < \frac{ab^2}{2}$, and this is equivalent to

$$\frac{\sqrt{b}}{\sqrt{c}} > \frac{2\sqrt{c}}{ab\sqrt{b}}$$

which gives

$$p > \frac{\sqrt{c}}{\sqrt{b}} - \frac{\sqrt{b}}{\sqrt{c}}. \quad (25)$$

By comparing both estimates for p , we get

$$\left| p - \frac{\sqrt{c}}{\sqrt{b}} \right| < \frac{\sqrt{b}}{\sqrt{c}}. \quad (26)$$

Let us now define an integer α by

$$2d_-\beta = p + \alpha.$$

Assume that $\alpha = 0$. Then (20) implies that $d_-(4\beta^2d_- - a) = 4$, thus $d_- \in \{1, 2, 4\}$. We have three cases:

1. $d_- = 1$, which implies $2\beta = p$. With this assumption, (12) gives

$$r^2 + \frac{p^2}{4} = tp, \quad (27)$$

while c satisfies the inequalities

$$ab < ab + a + b + 1 < c < ab + 2a + 2b + 2 < ab + 4b < 2ab$$

(see Lemma 1 and (23) with $d_- = 1$). The left hand side of (27) is

$$< ab + 4 + \frac{c^2 + 2bc + b^2}{4bc} < ab + 4 + \frac{a}{4} + 1 + \frac{1}{2} + \frac{1}{4a} < ab + \frac{a}{4} + 6.$$

On the other hand, by (24), the right hand side of (27) is

$$> \sqrt{bc} \left(\frac{\sqrt{c}}{\sqrt{b}} - \frac{2\sqrt{c}}{ab\sqrt{b}} \right) = c - \frac{2c}{ab} > ab + a + b + 1 - 4 = ab + a + b - 3.$$

By comparing these two estimates for (27), we get

$$b + \frac{3}{4}a < 9,$$

but this is in contradiction with $b \geq 12$ (b is the largest element in the $D(4)$ -triple $\{d_-, a, b\}$).

We treat similarly the other two cases.

2. $d_- = 2$, which implies $4\beta = p$, and this leads to

$$\frac{b}{2} + \frac{3}{8}a < 8,$$

which is in contradiction with $b \geq 16$ ($D(4)$ -triple of the form $\{2, a, b\}$ with the smallest b is $\{2, 6, 16\}$).

3. $d_- = 4$ is equivalent to $8\beta = p$, which leads to

$$\frac{b}{4} + \frac{3}{16}a < 8,$$

but the only $D(4)$ -triple of the form $\{4, a, b\}$ with $b < 35$ is $\{4, 8, 24\}$, which does not satisfy (22), so we have a contradiction here as well.

Therefore, we may now assume that $\alpha \neq 0$. We will estimate $2d_-t\beta$ and compare it with c . First we will prove

$$\beta^2 < \frac{a^2b}{c}. \quad (28)$$

Since $\beta < t$, and the case $\beta = t - 1$ gives $b(c - a) = 1$, which is impossible, we conclude that $t \geq \beta + 2$. This implies $t\beta \geq \beta^2 + 2\beta$, and $ab - t\beta \geq 2\beta - 4 > 0$ because of (18). Hence, we get $t\beta < ab$, and this clearly implies (28).

Therefore,

$$0 < d_- \beta^2 < \frac{d_- a^2 b}{c} < a.$$

From $2t\beta = r^2 + \beta^2 > ab + 4$, we get $2d_-t\beta > abd_- + 4d_-$. On the other hand,

$$d_- \beta^2 < \frac{d_- a^2 b}{c} \Leftrightarrow 2d_-t\beta < abd_- + 4d_- + \frac{d_- a^2 b}{c} < abd_- + 4d_- + a.$$

By combining these two estimates, we get

$$abd_- + 4d_- < 2d_-t\beta < abd_- + 4d_- + a. \quad (29)$$

By comparing (29) with (21) and (23), we conclude that

$$|2d_-t\beta - c| < 4b. \quad (30)$$

By combining the estimate (26) for p with the trivial estimate for α , namely $|\alpha| \geq 1$, we get

$$\left| 2d_- \beta - \frac{\sqrt{c}}{\sqrt{b}} \right| = \left| p + \alpha - \frac{\sqrt{c}}{\sqrt{b}} \right| \geq 1 - \frac{\sqrt{b}}{\sqrt{c}}.$$

Note that $ad_- > 26$. Namely, only $D(4)$ -pairs such that $ad_- \leq 26$ are $\{1, 5\}, \{1, 12\}, \{1, 21\}, \{2, 6\}, \{3, 4\}$ and $\{3, 7\}$. From first three pairs, respecting (21) and (22), we find triples

$$\{5, 12, 96\}, \{12, 21, 320\}, \{12, 96, 1365\}, \{21, 32, 780\}, \{21, 320, 7392\}$$

that do not satisfy (8) nor (9). From the last three pairs we cannot obtain a $D(4)$ -triple because of (22).

Finally, we obtain

$$\begin{aligned} |2d_-t\beta - c| &= |2d_-t\beta - t\frac{\sqrt{c}}{\sqrt{b}} + t\frac{\sqrt{c}}{\sqrt{b}} - c| \geq t \left| 2d_- \beta - \frac{\sqrt{c}}{\sqrt{b}} \right| - \left| t\frac{\sqrt{c}}{\sqrt{b}} - c \right| \\ &= t \left| 2d_- \beta - \frac{\sqrt{c}}{\sqrt{b}} \right| - \left(t\frac{\sqrt{c}}{\sqrt{b}} - c \right) \geq t \left(1 - \frac{\sqrt{b}}{\sqrt{c}} \right) - \left(t\frac{\sqrt{c}}{\sqrt{b}} - c \right) \\ &= t \left(1 - \frac{\sqrt{b}}{\sqrt{c}} \right) - c \left(\sqrt{1 + \frac{4}{bc}} - 1 \right) > \sqrt{bc} - b - c \left(\sqrt{1 + \frac{4}{bc}} - 1 \right) \\ &> \sqrt{ab^2d_-} - b - \frac{2}{b} \geq b(\sqrt{ad_-} - 1 - \frac{1}{72}) > 4b \end{aligned}$$

which contradicts (30). $\square$

Theorem 3. $E'(\mathbb{Q})_{tors} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$.

Proof. By Mazur's theorem [16] which characterizes all possible torsion groups for elliptic curves over $\mathbb{Q}$, since E' has three points of order 2, the only possibilities for $E'(\mathbb{Q})_{tors}$ are $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2k\mathbb{Z}$ with $k = 1, 2, 3, 4$. But Lemma 2 shows that the cases $k = 2, 4$ are not possible for an elliptic curve induced by a $D(4)$ -triple with positive elements. $\square$

Corollary 4. Let $\{a, b, c\}$ be a $D(1)$ -triple. Then the torsion group of the elliptic curve $y^2 = (ax + 1)(bx + 1)(cx + 1)$ is either $\simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$.

Remark 4. We note that an analogue of Theorem 3 and Corollary 4 is not valid for general $D(n^2)$ -triples and their induced elliptic curves

$$y^2 = (ax + n^2)(bx + n^2)(cx + n^2).$$

For example, for the $D(9)$ -triple $\{8, 54, 104\}$ the torsion group of the induced elliptic curve is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. Also, there are examples with torsion group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$, e.g. for the $D(52208405404435206419201940^2)$ -triple

$$\{3871249317729019929807383, 101862056999203416732147408, \\ 217448139952121636379025175\}$$

(there are much simpler examples with triples with mixed signs, see e.g. [7]).

We should also mention that we do not know any example of $D(1)$ or $D(4)$ -triples inducing elliptic curves with torsion group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$. Indeed, it is known that this torsion group cannot appear for certain families of $D(1)$ -triples (see [3, 4, 8, 18]). Again, there are examples of such curves for general $D(n^2)$ -triples. For example, the $D(294^2)$ -triple $\{32, 539, 1215\}$ induces an elliptic curve with torsion group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$.

References

- [1] L. E. Dickson, *A history of the Theory of numbers*, Vol. 2, Chelsea, New York, 1966., pp. 513–520.
- [2] Diophantus of Alexandria, *Arithmetics and the Book of Polygonal Numbers* (I. G. Bashmakova, Ed.) , Nauka, 1974, (in Russian), pp. 103–104, 232.
- [3] A. Dujella, *A parametric family of elliptic curves*, Acta Arith. **94** (2000), 87–101.
- [4] A. Dujella, *Diophantine m -tuples and elliptic curves*, J. Theor. Nombres Bordeaux **13** (2001), 111–124.
- [5] A. Dujella, *On the size of Diophantine m -tuples*, Math. Proc. Cambridge Philos. Soc. **132** (2002), 23–33.
- [6] A. Dujella, *There are only finitely many Diophantine quintuples*, J. Reine Angew. Math. **566** (2004), 183–214.
- [7] A. Dujella, *On Mordell-Weil groups of elliptic curves induced by Diophantine triples*, Glas. Mat. Ser. III **42** (2007), 3–18.
- [8] A. Dujella and A. Pethő, *Integer points on a family of elliptic curves*, Publ. Math. Debrecen **56** (2000), 321–335.
- [9] A. Dujella and A. M. S. Ramasamy, *Fibonacci numbers and sets with the property $D(4)$* , Simon Stevin **12** (2005), 401–412.
- [10] A. Filipin, *There does not exist a $D(4)$ -sextuple*, J. Number Theory **128** (2008), 1555–1565.
- [11] A. Filipin, *There are only finitely many $D(4)$ -quintuples*, Rocky Mountain J. Math. **41** (2011), 1847–1859.
- [12] A. Filipin, *An irregular $D(4)$ -quadruple cannot be extended to a quintuple*, Acta Arith. **136** (2009), 167–176.

- [13] A. Filipin, Bo He and A. Togbé, *On a family of two-parametric $D(4)$ -triples*, Glas. Mat. Ser. III **47** (2012), 31–51.
- [14] K. S. Kedlaya, *Solving constrained Pell equations*, Math. Comp. **67** (1998), 833–842.
- [15] A. Knapp, *Elliptic Curves*, Princeton Univ. Press, 1992.
- [16] B. Mazur, *Rational isogenies of prime degree*, Invent. Math. **44** (1978), 129–162.
- [17] S. P. Mohanty and A. M. S. Ramasamy, *The characteristic number of two simultaneous Pell's equations and its applications*, Simon Stevin **59** (1985), 203–214.
- [18] F. Najman, *Integer points on two families of elliptic curves*, Publ. Math. Debrecen **75** (2009), 401–418.

Andrej DUJELLA,
Department of Mathematics,
University of Zagreb,
Bijenička cesta 30, 10000 Zagreb, Croatia.
Email: duje@math.hr

Miljen MIKIĆ,
Kumičićeva 20, 51000 Rijeka, Croatia
Email: miljen.mikic@gmail.com