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On certain proximities and preorderings on the transposition hypergroups of linear first-order partial differential operators

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Abstract

The contribution aims to create hypergroups of linear first-order partial differential operators with proximities, one of which creates a tolerance semigroup on the power set of the mentioned differential operators. Constructions of investigated hypergroups are based on the so called "Ends-Lemma" applied on ordered groups of differnetial operators. Moreover, there is also obtained a regularly preordered transpositions hypergroup of considered partial differntial operators.

1 Introduction

Proximity spaces, belonging to classical topological structures involving generalization of metric spaces and their uniformly continuous mappings, are situated out of interest of topologists since the time of the results due to E. M. Alfsen and J. E. Fenstad (1959) showing that these spaces can be considered as totally bounded uniform spaces. Nevertheless, a proximity relation seems to be very useful tool for investigation of weak hyperstructures in the sense of Vougiouklis monograph [34] and other related papers. In particular, proximities on hyperstructures yield the way for a natural generation of weak commutativity, weak associativity and weak distributivity if the incidence of

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sets in corresponding identities is changed (or generalized in fact) by nearness of the sets in question.

Moreover, the concept of similarity of various systems has its abstract mathematical expression in terms of reflexive and symmetric relation on a set. These relations are named *tolerances* and the use of these relations in connection with other structures moves corresponding mathematical theories to useful application. Many publications are devoted to systematic investigations to tolerances on algebraic structures compatible with all operations of corresponding algebras. A certain survey of important results including valuable investment of Olomouc Algebraic School can be found in [5]. In fact a proximity on a set is a tolerance on its powerset, so one can expect some interesting connection between tolerances and proximities.

There are two principal approaches to proximity structures. The classical one–used in this contribution–is based on the construction of binary relation on the power set of a set satisfying natural axioms–see below–motivated by Smirnoff theory of proximity spaces. The other approach consists in the axiomatization of the concept "to be far", where the basic role plays a proximal neighbourhood of a set. This approach has been developed in the rich theory of symtopogenous and topogenous structure by Ákos Czászár and his collaborators,see [14]. In this theory the concept of a preorder and an order is playing an important role. This shows that investigation of preordered and ordered hyperstructures is of a certain importance.

Equations expressing laws of conservation as the continuity equation, the motion equation, further Maxwells' equations of the electromagnetic field, linearized equations of acoustics, the equation of long-distance electrical line and many other equations used in physical investigations and in technical applications are all linear partial differential equations of the first order. The importance of study of those equations motivates our contribution. On the other hand algebraic (non-commutative) join spaces, called also non-commutative transposition hypergroups constitute very important and useful class of multistructures within the framework of the contemporary algebraic hyperstructures theory—cf. [3, 6, 7], [10-13], [15-20], [22, 23, 29, 33].

Principal constructions presented in this paper are based on the relationship between binary relations and hyperoperations [5-7], [12], [14-17], [19, 20, 31].

In particular there is used the "Ends-Lemma"-briefly the EL-theory [6, 10, 24], [30-32]. Hyperstructures associated with relations (binary and n-ary in genral) are developed in a series of deeply worked-out papers [13], [15-20]. Ordered hyperstructures are investigated in [1, 22]. Motivation of compatibility of orderings with hyperoperations can be found in the monography [12]. In our paper we use stronger compatibility of preorderings with the corresponding hyperstructure than there is considered in [22].

Let $\Omega \subseteq \mathrm{R}^n$ be an open connected subset (called a domain) of the n-dimensional euclidean space of n-tuples of reals. As usually, $\mathbf{C}^1(\Omega)$ stands for the ring of all continuos functions of n-variables $u\colon \Omega \to \mathrm{R}$ with continuous first-order partial derivatives $\frac{\partial u}{\partial x_k}, k = 1, 2, \ldots, n$. We will consider partial differential operators of the form

$$\mathrm{D}(a_1, \ldots, a_n, p) = \sum_{k=1}^{n} a_k(x_1, \ldots, x_n) \frac{\partial}{\partial x_k} + p(x_1, \ldots, x_n)\,\mathrm{Id},$$

where $a_k \in \mathbf{C}^1(\Omega)$ for $k = 1, 2, \ldots, n$ and $p \in \mathbf{C}^1(\Omega), p(x_1, \ldots, x_n) > 0$ for any $[x_1, \ldots, x_n] \in \Omega$. Denote by $\mathbb{L}^1\mathbb{D}(\Omega)$ the set of all such operators which are associated to linear first-order homogeneous partial differential equations

$$\sum_{k=1}^{n} a_k(x_1, \ldots, x_n) \frac{\partial u}{\partial x_k} + p(x_1, \ldots, x_n) u(x_1, \ldots, x_n) = 0,$$

with $a_k, p \in \mathbf{C}(\Omega)$.

Define a binary operation "$\cdot$" and a binary relation "$\le$" on the set $\mathbb{L}^1\mathbb{D}(\Omega)$ by the rule

$$\mathrm{D}(a_1, \ldots, a_n, p) \cdot \mathrm{D}(b_1, \ldots, b_n, q) = \mathrm{D}(c_1, \ldots, c_n, pq),$$

in short notation:

$$\mathrm{D}(\vec{a}, p) \cdot \mathrm{D}(\vec{b}, q) = \mathrm{D}(\vec{c}, pq),$$

where

$$c_k(x_1, \ldots, x_n) = a_k(x_1, \ldots, x_n) + p(x_1, \ldots, x_n) b_k(x_1, \ldots, x_n), [x_1, \ldots, x_n] \in \Omega$$

and

$$\mathrm{D}(\vec{a}, p) \le \mathrm{D}(\vec{b}, q)$$

whenever

$$p \equiv q \text{ and } a_k(x_1, \ldots, x_n) \le b_k(x_1, \ldots, x_n)$$

for any $[x_1, \ldots, x_n] \in \Omega$ and $k = 1, 2, \ldots, n$.

Evidently, the relation $\le$ on $\mathbb{L}^1\mathbb{D}(\Omega)$ is reflexive, antisymmetric and transitive hence $(\mathbb{L}^1\mathbb{D}(\Omega), \le)$ is an ordered set. Moreover, it is easy to verify that $(\mathbb{L}^1\mathbb{D}(\Omega), \cdot)$ is a non-commutative group in which any right translation and any left translation determined by arbitrary chosen operator $\mathrm{D}(a_1, \ldots, a_n, p) \in \mathbb{L}^1\mathrm{D}(\Omega)$ is an isotone selfmap of $(\mathbb{L}^1\mathbb{D}(\Omega), \le)$. Consequently the following theorem holds (see [10]):

Theorem 1.1. *Let $\Omega \subseteq \mathrm{R}^n$ be a nonempty domain. Then the system of differential operators $(\mathbb{L}^1\mathbb{D}(\Omega), \cdot, \leq)$ is an ordered (non-commutative) group.*

Now applying a simple construction from [6], chapt. IV, we get the resulting hyperstructure. The following Ends Lemma will be useful in what follows.

Lemma 1.2. [6, 31] [**Ends Lemma**] *Let a triple $(G, \cdot, \leq)$ be a quasi-ordered semigroup. Define a hyperoperation*

$$*\colon G \times G \to \mathcal{P}^*(G) \qquad by \qquad a * b = [a \cdot b)_{\leq} = \{x \in G; a \cdot b \leq x\}$$

for all pairs of elements $a, b \in G$.

i) *Then $(G, *)$ is a semihypergroup which is commutative if the semigroup $(G, \cdot)$ is commutative.*
ii) *Let $(G, *)$ be the above defined semihypergroup. Then $(G, *)$ is a hypergroup iff for any pair of elements $a, b \in G$ there exists a pair of elements $c, c' \in G$ with a property $a \cdot c \leq b$, $c' \cdot a \leq b$.*

Concerning application of the Ends lemma see also [30, 32] and [9, 10, 24, 25, 26].

We construct such an action using partial differential operators of the first order, set of which is endowed with a suitable binary multiplication turning out the set of operators into a non-commutative hypergroup. Applying the *Ends Lemma* we get then a hypergroup of linear partial differential operators acting on the ring of all continuous functions of n-variables $u\colon \Omega \to \mathrm{R}$ with continuous partial derivatives of all orders.

Recall first, that a nonempty set H endowed with a binary hyperoperation

$$\star : H \times H \to \mathcal{P}^*(H), \quad (\mathcal{P}^*(H) = \mathcal{P}(H) - \{\emptyset\})$$

is called a *hypergroupoid*. For any pair of elements $a, b \in H$ there can be defined two fractions $a/b = \{x; a \in x \star b\}$ (right extension) and $b\backslash a = \{x; a \in b \star x\}$ (left extension). If $x, y \subseteq H$ then we write $X \approx Y$ (read X meets Y) whenever X, Y are incident, i.e., $X \cap Y \neq \emptyset$. See e.g. [12, **?**, 19, 20].

Now an algebraic (non-commutative) *join space* [28] termed also a *transposition hypergroup* can be defined as an associative hypergroupoid $(H, \star)$ satisfying the reproduction axiom ($a \star H = H = H \star a$ for all $a \in H$) and the transposition axiom ($b\backslash a \approx c/d$ implies $a \star d \approx b \star c$ for all $a, b, c, d \in H$). A join space satisfying the equality $a \star a = \{a\}$ for any $a \in H$ is called as geometrical; in the opposite case we speak about algebraic join space.

A *partially order* is a binary relation R on a set X which satisfies conditions reflexivity, antisymmetry and tranzitivity. Sometimes we need to weaken the

definition of partial order as in [22]. We say that a *partial preordered* is a relation which satisfies conditions reflexivity and transitivity. An algebraic system $(G,\cdot,\le)$ is called a *partially preordered (ordered) groupoid* if $(G,\cdot)$ is a groupoid and $G,\le$ is a partially preordered (ordered) set which satisfies monotone condition as follows:
if $x\le y$ then $a\cdot x\le a\cdot y$ and $x\cdot a\le y\cdot a$ for every $x,y,a\in G$.

Defining a binary hyperoperation on $\mathbb{L}^1\mathbb{D}(\Omega)$ by

$$\begin{aligned}\mathrm{D}(\vec{a},p)\star\mathrm{D}(\vec{b},q)&=\{\mathrm{D}(\vec{c},s);\mathrm{D}(\vec{a},p)\cdot\mathrm{D}(\vec{b},q)\le\mathrm{D}(\vec{c},s)\}\\&=\{\mathrm{D}(\vec{c},pq);a_k+pb_k\le c_k,\}\end{aligned}$$

where $c_k,s\in\mathbf{C}^1(\Omega),k=1,2,\dots,n$, we obtain with respect to [6], Chpt. IV, Theorems 1.3, 1.4 and Theorem 1.1 the following result:

Theorem 1.3. *Let Let $\Omega\subseteq\mathrm{R}^n$ be a domain. The hypergroupoid $(\mathbb{L}^1\mathbb{D}(\Omega),\star)$ is a non-commutative algebraic join space.*

Let $M\subset\Omega$ be a finite subset. Denote

$$\mathbb{L}^1_M\mathbb{D}(\Omega)=\{\mathrm{D}(\vec{a},p)\}\in\mathbb{L}^1\mathbb{D}(\Omega);\ \operatorname{grad}p|_\xi=0\text{ for any }\xi\in M\}.$$

Evidently $(\mathbb{L}^1_M\mathbb{D}(\Omega),\cdot)$ is a subgroup of the group $(\mathbb{L}^1\mathbb{D}(\Omega),\cdot)$. We define a binary relation R_M on the set of operators $\mathbb{L}^1_M\mathbb{D}(\Omega)$ by the condition

$$\mathrm{D}(\vec{a},p)\ \mathrm{R}_M\ \mathrm{D}(\vec{b},q)\text{ whenever }p=q\text{ and }\operatorname{grad}a_k|_\xi=\operatorname{grad}b_k|_\xi$$

for any $\xi\in M$ and $k=1,2,\dots n$. Clearly, R_M is an equivalence relation on the set $\mathbb{L}^1_M\mathbb{D}(\Omega)$. Suppose $\mathrm{D}(c_1,\dots,c_n,s)\in\mathbb{L}^1_M\mathbb{D}(\Omega)$ is an arbitrary operator. Since

$$\begin{aligned}\operatorname{grad}(a_k+pc_k)|_\xi&=\operatorname{grad}a_k|_\xi+\operatorname{grad}p|_\xi c_k+p\operatorname{grad}c_k|_\xi\\&=\operatorname{grad}b_k|_\xi+q\operatorname{grad}c_k|_\xi\\&=\operatorname{grad}b_k|_\xi+\operatorname{grad}q|_\xi c_k+q\operatorname{grad}c_k|_\xi=\operatorname{grad}(b_k+qc_k)|_\xi\end{aligned}$$

for any $\xi\in M$ and $k=1,2,\dots n$, we have that

$$\mathrm{D}(a_1,\dots,a_n,p)\,\mathrm{R}_M\,\mathrm{D}(b_1,\dots,b_n,q)$$

implies

$$\big(\mathrm{D}(\vec{a},p)\,\mathrm{D}(\vec{c},s)\big)\ \mathrm{R}_M\ \big(\mathrm{D}(\vec{b},q)\,\mathrm{D}(\vec{c},s)\big)$$

and similarly

$$\big(\mathrm{D}(\vec{c},s)\,\mathrm{D}(\vec{a},p)\big)\ \mathrm{R}_M\ \big(\mathrm{D}(\vec{c},s)\,\mathrm{D}(\vec{b},q)\big).$$

Indeed

$$\begin{aligned}\operatorname{grad}(c_k + s a_k)|_\xi &= \operatorname{grad} c_k|_\xi + \operatorname{grad} s|_\xi a_k + s \operatorname{grad} a_k|_\xi \\ &= \operatorname{grad} c_k|_\xi + s \operatorname{grad} b_k|_\xi \\ &= \operatorname{grad} c_k|_\xi + \operatorname{grad} s|_\xi b_k + s \operatorname{grad} b_k|_\xi = \operatorname{grad}(c_k + s b_k)|_\xi\end{aligned}$$

$\xi \in M, k = 1, 2, \ldots n$. Further,

$$D^{-1}(a_1, \ldots, a_n, p) = \mathrm{D}\Big(-\frac{a_1}{p}, \ldots, -\frac{a_n}{p}, \frac{1}{p}\Big)$$

and

$$D^{-1}(b_1, \ldots, b_n, q) = \mathrm{D}\Big(-\frac{b_1}{q}, \ldots, -\frac{b_n}{q}, \frac{1}{q}\Big)$$

$$\begin{aligned}\operatorname{grad}\Big(-\frac{a_k}{p}\Big)\Big|_\xi &= -\operatorname{grad} \frac{a_k}{p}\Big|_\xi = -\operatorname{grad}\Big(\frac{1}{p}a_k\Big)\Big|_\xi = -\operatorname{grad} \frac{1}{p}\Big|_\xi a_k - \frac{1}{p}\operatorname{grad} a_k\Big|_\xi \\ &= \frac{1}{p^2}\operatorname{grad} p\Big|_\xi a_k - \frac{1}{p}\operatorname{grad} a_k\Big|_\xi = -\frac{1}{p}\operatorname{grad} a_k\Big|_\xi = -\frac{1}{q}\operatorname{grad} b_k\Big|_\xi \\ &= \frac{1}{q^2}\operatorname{grad} q\Big|_\xi - \frac{1}{p}\operatorname{grad} b_k\Big|_\xi = \operatorname{grad} -\frac{b_k}{q}\Big|_\xi\end{aligned}$$

and $\frac{1}{p} = \frac{1}{q}$, consequently

$$\mathrm{D}(\vec{a}, p)\, \mathrm{R}_M\, \mathrm{D}(\vec{b}, q) \text{ implies } \mathrm{D}^{-1}(\vec{a}, p)\ \mathrm{R}_M\ \mathrm{D}(\vec{b}, q).$$

Therefore the equivalence R_M is a congruence on the group $(\mathbb{L}^1_M\mathbb{D}(\Omega), \cdot)$.

Denote by $\mathrm{Fin}(\Omega)$ the lattice of all finite subset of the domain Ω. Using the Ends Lemma or using an union of the ends with respect to the ordering by set inclusion a hypergroup with the carrier $\mathrm{Fin}(\Omega)$ can be created. For any non-empty set $M \in \mathrm{Fin}(\Omega)$ we obtain the congruence R_M on the group $(\mathbb{L}^1\mathbb{D}(\Omega), \cdot)$ which was described above.

Recall the definition of a proximity in the sense of Čech monograph [4]:

Definition 1.4. *A binary relation* $\mathbf{p}$ *on the family of all subsets of the set* H *is called a* proximity *on the set* H *if* $\mathbf{p}$ *satisfies the following conditions:*

1. $\emptyset\, non\, \mathbf{p}\, H$
2. *The relation* $\mathbf{p}$ *is symmetric, i.e.,* $A, B \subset H$*,* $A\,\mathbf{p}\,B$ *implies* $B\,\mathbf{p}\,A$ *.*
3. *For any pair of subset* $A, B \subset H$*,* $A \cap B \neq \emptyset$ *implies* $A\,\mathbf{p}\,B$*.*
4. *If* A, B, C *are subsets of* H *then* $(A \cup B)\,\mathbf{p}\,C$ *if and only if either* $A\,\mathbf{p}\,C$ *or* $B\,\mathbf{p}\,C$*.*

Now, setting for any pair of subsets $A, B \subset \mathbb{L}^1_M\mathbb{D}(\Omega)$ that $A\,\mathbf{p}(R_M)\,B$ whenever $A \neq \emptyset \neq B$ and $\mathrm{D}(a_1,\dots,a_n,p)\ \mathrm{R}_M\,\mathrm{D}(b_1,\dots,b_n,q)$ for some pair $[\mathrm{D}(a_1,\dots,a_n,p),\,\mathrm{D}(b_1,\dots,b_n,q)] \in A \times B$, we obtain the following theorem:

Theorem 1.5. [10] *Let $(\mathbb{L}^1_M\mathbb{D}(\Omega),\star)$ be the join subspace of the join space $(\mathbb{L}^1\mathbb{D}(\Omega),\star)$ defined above. Then $(\mathbb{L}^1\mathbb{D}(\Omega),\mathbf{p}(R_M))$ is a proximity space such that for any quadruple $X, Y, U, V \subset \mathbb{L}^1_M\mathbb{D}(\Omega)$ of nonempty subsets with the property $X\,\mathbf{p}(R_M)\,Y$, $U\,\mathbf{p}(R_M)\,V$ we have*

$$(X \star U)\,\mathbf{p}(R_M)\,(Y \star V).$$

It is easy to see that any tolerance on a group compatible as for algebras must be transitive, hence it is a congruence. So, we can also treat semigroups of operators with compatible tolerances and semihypergroups with proximities induced by those.

The terminology in belowe stated constructions is overtaken from [2, 6, 8, 11, 13, 24, 26].

Let us define (as in [26]) a multiautomaton:

Definition 1.6. *Let S be a nonempty set, $(H,\odot)$ be a hypergroupoid and $\delta\colon S \times H \to S$ be a mapping satisfying the condition*

$$\delta\big(\delta(s,a),b\big) \in \delta(s, a \odot b) \qquad \text{(GMAC)}$$

for any triple $(s,a,b) \in S \times H \times H$, where $\delta(s, a\odot b) = \{\delta(s,x); x \in a \odot b\}$.

Then the triple $\mathbb{M} = (S,H,\delta)$ is called multiautomaton *with the state set S and the input hypergroupoid $(H,\odot)$. The mapping $\delta\colon S\times H \to S$ is called a* transition function *or a* next-state function *of the multiautomaton $\mathbb{M}$.*

In previous definition GMAC means Generalized Mixed Associativity Condition.

Now, we shall consider smooth functions $f \in \mathbf{C}^\infty(\Omega)$. Let $\mathrm{P}(\vec{a},p)\colon \mathbf{C}^\infty(\Omega) \to \mathbf{C}^\infty(\Omega)$ be a fixed chosen operator,

$$\mathrm{P}(\vec{a},p)f = \sum_{k=1}^{n} a_k(x_1,\dots,x_n)\frac{\partial f}{\partial x_k} + p(x_1,\dots,x_n)f(x_1,\dots,x_n).$$

Denote by $\mathrm{Ct}(\mathrm{P})$ the set of all differential operators $\mathrm{D} \in \mathbb{L}^1\mathbb{D}(\Omega)$ commuting with the operator P, i.e.,

$$\mathrm{Ct}(\mathrm{P}) = \{\mathrm{D} \in \mathbb{L}^1\mathbb{D}(\Omega); \mathrm{P}\cdot\mathrm{D} = \mathrm{D}\cdot\mathrm{P}\}.$$

Since the identity operator Id belongs to $\mathrm{Ct}(\mathrm{P})$, this set endowed with the unique operation “$\cdot$” is a monoid which is called the *centralizer* of the operator P within the group $(\mathbb{L}^1\mathbb{D}(\Omega),\cdot)$.

Lemma 1.7. ([26]) *Operators* $\mathrm{D}(\vec{a},p)$, $\mathrm{D}(\vec{b},q)$ *from the group* $\mathbb{L}^1\mathbb{D}(\Omega)$ *are commuting if and if for any* $k=1,2,\dots,n$ *and any point* $[x_1,\dots,x_n]\in\Omega$ *there holds*

$$\begin{vmatrix} 1-p(x_1,\dots,x_n) & 1-q(x_1,\dots,x_n) \\ a_k(x_1,\dots,x_n) & b_k(x_1,\dots,x_n) \end{vmatrix} = 0.$$

Now, for any pair $\mathrm{D}_\alpha,\mathrm{D}_\beta\in\mathrm{Ct}(\mathrm{P})$ define a hyperoperation "$\odot$" as follows:

$$\odot\colon\ \mathrm{Ct}(\mathrm{P})\times\mathrm{Ct}(\mathrm{P})\to\mathcal{P}^*\big(\mathrm{Ct}(\mathrm{P})\big)$$

by

$$\mathrm{D}_\alpha\odot\mathrm{D}_\beta=\{\mathrm{P}^n\cdot\mathrm{D}_\beta\cdot\mathrm{D}_\alpha;n\in\mathbb{N}\}.$$

Consider the binary relation $\rho_\mathrm{P}\subset\mathrm{Ct}(\mathrm{P})\times\mathrm{Ct}(\mathrm{P})$ defined by

$$\mathrm{D}_\alpha\ \rho_\mathrm{P}\ \mathrm{D}_\beta\ \text{ if and only if }\ \mathrm{D}_\beta=\mathrm{P}^n\cdot\mathrm{D}_\alpha$$

for some $n\in\mathbb{N}_0$. We get without any effort that $\big(\mathrm{Ct}(\mathrm{P}),\cdot,\rho_\mathrm{P}\big)$ is a quasi-ordered monoid.

Further, $\mathrm{D}_\alpha\odot\mathrm{D}_\beta=\rho_\mathrm{P}(\mathrm{D}_\beta\cdot\mathrm{D}_\alpha)=[\mathrm{D}_\beta\cdot\mathrm{D}_\alpha)_{\rho_\mathrm{P}}$ and by Ends Lemma 1.2 we obtain that $\big(\mathrm{Ct}(\mathrm{P}),\odot\big)$ is a hypergroup (non-commutative, in general).

As usually $\big(\mathrm{Ct}(\mathrm{P})\big)^+$ with the operation of concatenation means the free semigroup of finite nonempty words formed by operators from the set $\mathrm{Ct}(\mathrm{P})$.

Denote

$$S_\mathrm{P}=\Big\{(\mathrm{P}\cdot\mathrm{D}_1\cdot\dots\cdot\mathrm{D}_n)(f);f\in\mathbf{C}^\infty(\Omega),\mathrm{D}_1\cdot\dots\cdot\mathrm{D}_n\in\big(\mathrm{Ct}(\mathrm{P})\big)^+\Big\}$$

and $\mathbb{M}(S_\mathrm{P})$ the triple $\big(S_\mathrm{P},(\mathrm{Ct}(\mathrm{P}),\odot),\delta_\mathrm{P}\big)$, where the action or transition function

$$\delta_\mathrm{P}\colon S_\mathrm{P}\times\mathrm{Ct}(\mathrm{P})\to S_\mathrm{P}$$

is defined by the rule

$$\delta_\mathrm{P}\big((\mathrm{P}\cdot\mathrm{D}_1\cdot\dots\cdot\mathrm{D}_n)(f),\mathrm{D}_\alpha\big)=(\mathrm{P}\cdot\mathrm{D}_\alpha\cdot\mathrm{D}_1\cdot\dots\cdot\mathrm{D}_n)(f)$$

for any function $f\in\mathbf{C}^\infty(\Omega)$ and any operator $\mathrm{D}_\alpha\in\mathrm{Ct}(\mathrm{P})$. The transition function δ_P satisfies the Generalized Mixed Associativity Condition.

Indeed, suppose $f\in\mathbf{C}^\infty(\Omega)$, $\mathrm{D}_\alpha,\mathrm{D}_\beta,\mathrm{D}_1,\mathrm{D}_2\in\mathrm{Ct}(\mathrm{P})$ are arbitrary elements. We have

$$\begin{aligned}&\delta_\mathrm{P}\Big(\delta_\mathrm{P}\big((\mathrm{P}\cdot\mathrm{D}_1\cdot\dots\cdot\mathrm{D}_n)(f),\mathrm{D}_\alpha\big),\mathrm{D}_\beta\Big)=\delta_\mathrm{P}\big((\mathrm{P}\cdot\mathrm{D}_\alpha\cdot\mathrm{D}_1\cdot\dots\cdot\mathrm{D}_n)(f),\mathrm{D}_\beta\big)\\ &=\Big((\mathrm{P}\cdot\mathrm{D}_\beta\cdot\mathrm{D}_\alpha\cdot\mathrm{D}_1\cdot\dots\cdot\mathrm{D}_n)(f)\Big)\in\big\{(\mathrm{P}^{n+1}\cdot\mathrm{D}_\beta\cdot\mathrm{D}_\alpha\cdot\mathrm{D}_1\cdot\dots\cdot\mathrm{D}_n)(f),n\in\mathbb{N}_0\big\}\\ &=\delta_\mathrm{P}\big((\mathrm{P}\cdot\mathrm{D}_1\cdot\dots\cdot\mathrm{D}_n)(f),\mathrm{P}^n\cdot\mathrm{D}_\alpha\cdot\mathrm{D}_\beta\big)=\delta_\mathrm{P}\big((\mathrm{P}\cdot\mathrm{D}_1\cdot\dots\cdot\mathrm{D}_n)(f),\mathrm{D}_\alpha\odot\mathrm{D}_\beta\big),\end{aligned}$$

so GMAC is satisfied, i.e., the triple $\mathbb{M}(S_{\mathrm{P}}) = \big(S_{\mathrm{P}}, (\mathrm{Ct}(\mathrm{P}), \odot), \delta_{\mathrm{P}}\big)$ is a multiautomaton.

With respect to the definition of connectivity of an automaton we give the following definition:

Definition 1.8. 1. *Let $\mathbb{A} = (S, G, \delta)$ be a multiautomaton with an input semihypergroup G. If $\emptyset \neq T \subset S$ and $\delta(t, g) \in T$ for any pair $[t, g] \in T \times G$ then the triad $\mathbb{B} = (T, G, \delta_T)$ (where $\delta_T = \delta|_{(T\times G)}$) is called a* submultiautomaton *of the multiautomaton $\mathbb{A} = (S, G, \delta)$.*

2. *A submultiautomaton $\mathbb{B} = (T, G, \delta_T)$ of the multiautomaton $\mathbb{A} = (S, G, \delta)$ is said to be* separated *if $\delta(S \setminus T, G) \cap T = \emptyset$. A nonempty multiautomaton is said to be* connected *(in the sense of [2]) if it has no separated proper submultiautomaton.*

Let us define the transition function δ as follows. $\delta\colon \mathbb{L}^1\mathbb{D}(\Omega) \times \mathbb{C}^1(\Omega) \to \mathbb{L}^1\mathbb{D}(\Omega)$

$$\delta\big(\mathrm{D}(\vec{a}, p), f\big) = \mathrm{D}(\vec{a}, p) \cdot \mathrm{D}(f, \dots, f, 1) = \mathrm{D}(\vec{c}, p)$$

where $a_k + pf = c_k; k = 1, 2, \dots, n$.

If we define $f \cdot g = \bigcup\limits_{[a,b]\in\mathbb{R}^+\times\mathbb{R}^+} [af + bg)_{\le}$, similarly as in [23], we can prove that $\big(\mathbb{C}(\Omega), \cdot\big)$ is a join space. First, we will proof the GMAC, i.e.

$$\delta\Big(\delta\big(\mathrm{D}(\vec{a}, p), f\big), g\Big) \in \delta\big(\mathrm{D}(\vec{a}, p), f \cdot g\big).$$

Indeed,

$$\begin{aligned}\delta\Big(\delta\big(\mathrm{D}(\vec{a}, p), f\big), g\Big) &= \delta\big(\mathrm{D}(a_1 + pf, \dots, a_n + pf, p), g\big)\\ &= \mathrm{D}(a_1 + pf, \dots, a_n + pf, p) \cdot \mathrm{D}(g, \dots, g, 1)\\ &= \mathrm{D}(a_1 + pf + pg, \dots, a_n + pf + pg, p) = \mathcal{L}(\mathrm{D}, \Omega).\end{aligned}$$

On the other hand

$$\begin{aligned}\delta\big(\mathrm{D}(\vec{a}, p), f \cdot g\big) &= \delta\Big(\mathrm{D}(\vec{a}, p), \bigcup_{[a,b]\in\mathbb{R}^+\times\mathbb{R}^+} [af + bg)_{\le}\Big)\\ &= \bigcup_{[a,b]\in\mathbb{R}^+\times\mathbb{R}^+} \delta\Big(\mathrm{D}(\vec{a}, p), [af + bg)_{\le}\Big)\\ &= \bigcup_{[a,b]\in\mathbb{R}^+\times\mathbb{R}^+} \delta\Big(\mathrm{D}(\vec{a}, p), \big\{\varphi(x_1, \dots, x_n);\\ &\qquad af(x_1, \dots, x_n) + bg(x_1, \dots, x_n) \le \varphi(x_1, \dots, x_n)\big\}\Big).\end{aligned}$$

Now, choosing e.g. $a = 1$, $b = 1$ we have

$$\delta\big(\mathrm{D}(\vec{a},p), f+g\big) = \mathrm{D}\big(a_1 + p(f+g), \dots, a_n + p(f+g), p\big).$$

$$\mathcal{L}(\mathrm{D},\Omega) = \mathrm{D}\big(a_1 + p(f+g), \dots, a_n + p(f+g), p\big) \in \delta\big(\mathrm{D}(\vec{a},p), f \cdot g\big).$$

Hence GMAC is satisfied. So, $(\mathbb{C}(\Omega), \cdot)$ is a join space.

Creation of invariant subgroup:

Let $\mathrm{D}(\vec{a},p) \in \mathbb{L}^1\mathbb{D}(\Omega)$ be an arbitrary operator. Then

$$\begin{aligned}\mathrm{D}^{-1}(\vec{a},1)\cdot \mathrm{D}(\vec{b},1) \cdot \mathrm{D}(\vec{a},p)& \\ &= \mathrm{D}\Big(-\frac{a_1}{p}, \dots, -\frac{a_n}{p}, \frac{1}{p}\Big) \cdot \mathrm{D}(a_1 + b_1, \dots, a_n + b_n, p) \\ &= \mathrm{D}\Big(\frac{b_1}{p}, \dots, \frac{b_n}{p}, 1\Big) \in \mathbb{L}^1_1\mathbb{D}(\Omega).\end{aligned}$$

Proposition 1.9. *Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be an open domain. Then*

$$\big(\mathbb{L}^1_1\mathbb{D}(\Omega), \cdot\big) \lhd \big(\mathbb{L}^1\mathbb{D}(\Omega), \cdot\big).$$

Proof. Firstly, if $\big[\mathrm{D}(\vec{a},1), \mathrm{D}(\vec{b},1)\big] \in \mathbb{L}^1\mathbb{D}(\Omega) \times \mathbb{L}^1_1\mathbb{D}(\Omega)$ is an arbitrary pair of differential operators then

$$\mathrm{D}(\vec{a},1) \cdot \mathrm{D}^{-1}(\vec{b},1) = \mathrm{D}(a_1 - b_1, \dots, a_n - b_n, 1) \in \mathbb{L}^1_1\mathbb{D}(\Omega)$$

and $\mathrm{D}(0, \cdots, 0, 1) \in \mathbb{L}^1_1\mathbb{D}(\Omega)$ thus $\big(\mathbb{L}^1_1\mathbb{D}(\Omega), \cdot\big)$ is a subgroup of the group $\big(\mathbb{L}^1\mathbb{D}(\Omega), \cdot\big)$. Further for an arbitrary pair of operators

$$\big[\mathrm{D}(\vec{a},p), \mathrm{D}(\vec{b},q)\big] \in \mathbb{L}^1\mathbb{D}(\Omega) \times \mathbb{L}^1_1\mathbb{D}(\Omega)$$

we have, according the above calculation, $\mathrm{D}^{-1}(\vec{a},p)\cdot\mathrm{D}(\vec{b},1)\cdot\mathrm{D}(\vec{a},p) \in \mathbb{L}^1_1\mathbb{D}(\Omega)$, thus $\mathrm{D}^{-1}(\vec{a},p) \cdot \mathbb{L}^1_1\mathbb{D}(\Omega) \cdot \mathrm{D}(\vec{a},p) \subseteq \mathbb{L}^1_1\mathbb{D}(\Omega)$ hence $\big(\mathbb{L}^1_1\mathbb{D}(\Omega), \cdot\big)$ is an invariant subgroup of the group $\big(\mathbb{L}^1\mathbb{D}(\Omega), \cdot\big)$. □

Denote $\mathbb{L}^1_0\mathbb{D}(\Omega) = \big\{\mathrm{D}(\varphi, \cdots \varphi, 1); \varphi \in \mathbb{C}(\Omega)\big\}$.

Theorem 1.10. *Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be an open domain. Then*

$$\big(\mathbb{L}^1_0\mathbb{D}(\Omega), \cdot\big) \lhd \big(\mathbb{L}^1_1\mathbb{D}(\Omega), \cdot\big) \lhd \big(\mathbb{L}^1\mathbb{D}(\Omega), \cdot\big)$$

and as well

$$\big(\mathbb{L}^1_0\mathbb{D}(\Omega), \cdot\big) \lhd \big(\mathbb{L}^1\mathbb{D}(\Omega), \cdot\big).$$

Proof. Consider an arbitrary element $\mathrm{D}(\vec{a},p) \in \mathbb{L}^1\mathbb{D}(\Omega)$ which is different from the unit of this group and the corresponding inner automorphism $\Psi_{\vec{a}}$ of the group $(\mathbb{L}^1\mathbb{D}(\Omega),\cdot)$ determined by the operator $\mathrm{D}(\vec{a},p)$.

Then for arbitrary $\mathrm{D}(\vec{b},1) \in \mathbb{L}^1\mathbb{D}(\Omega)$ we have

$$\begin{aligned}\Psi_{\vec{a}}\big(\mathrm{D}(b_1,\dots,b_n,1)\big) &= \mathrm{D}^{-1}(a_1,\dots,a_n,p)\cdot\mathrm{D}(b_1,\dots,b_n,1)\cdot\mathrm{D}(a_1,\dots,a_n,p)\\ &= \mathrm{D}\Big(-\frac{a_1}{p},\dots,-\frac{a_n}{p},\frac{1}{p}\Big)\cdot\mathrm{D}(a_1+b_1,\dots,a_n+b_n,p)\\ &= \mathrm{D}\Big(\frac{a_1(p-1)}{p}+b_1,\dots,\frac{a_n(p-1)}{p}+b_n,1\Big)\in\mathbb{L}^1\mathbb{D}(\Omega),\end{aligned}$$

thus $\Psi_{\vec{a}}(\mathbb{L}^1\mathbb{D}(\Omega),\cdot) = (\mathbb{L}^1\mathbb{D}(\Omega),\cdot)$, consequently $\big(\mathbb{L}^1_1\mathbb{D}(\Omega),\cdot\big) \lhd \big(\mathbb{L}^1\mathbb{D}(\Omega),\cdot\big)$.

Similarly, denoting by $\Psi_{\vec{a}}(\mathbb{L}^1_1\mathbb{D}(\Omega),\cdot) \to (\mathbb{L}^1_1\mathbb{D}(\Omega),\cdot)$ the inner automorphism of the group $(\mathbb{L}^1_1\mathbb{D}(\Omega),\cdot)$ detemined by the element $\mathrm{D}(a_1,\dots,a_n,1) \in \mathbb{L}^1_1\mathbb{D}(\Omega)$ we have for an arbitrary operator $\mathrm{D}(\varphi,\dots,\varphi,1) \in \mathbb{L}^1_0\mathbb{D}(\Omega)$:

$$\begin{aligned}\Psi_{\vec{a}}\big(\mathrm{D}(\varphi,\dots,\varphi,1)\big) &= \mathrm{D}(-a_1,\dots,-a_n,1)\cdot\mathrm{D}(\varphi,\dots,\varphi,1)\cdot\mathrm{D}(a_1,\dots,a_n,1)\\ &= \mathrm{D}(-a_1,\dots,-a_n,1)\cdot\mathrm{D}(a_1+\varphi,\dots,a_n+\varphi,1)\\ &= \mathrm{D}(\varphi,\dots,\varphi,1)\in\mathbb{L}^1_0\mathbb{D}(\Omega),\end{aligned}$$

i.e. $\Psi_{\vec{a}}(\mathbb{L}^1_0\mathbb{D}(\Omega)) = \mathbb{L}^1_0\mathbb{D}(\Omega)$ (here $\Psi_{\vec{a}} \mid \mathbb{L}^1_0\mathbb{D}(\Omega) = \mathrm{Id}$). Thus $\big(\mathbb{L}^1_0\mathbb{D}(\Omega),\cdot\big) \lhd \big(\mathbb{L}^1_1\mathbb{D}(\Omega),\cdot\big)$.

In a similar way we obtain the third assertion. □

The just proved theorem allows us to define two proximities on the hypergroup $\big(\mathbb{L}^1\mathbb{D}(\Omega),*\big)$ which are compatible in the above mentioned sense. Denote by $\mathbb{H}$ one of carriers $\mathbb{L}^1_1\mathbb{D}(\Omega)$, $\mathbb{L}^1_0\mathbb{D}(\Omega)$ of normal subgroups of the group $\big(\mathbb{L}^1\mathbb{D}(\Omega),\cdot\big)$. Denoting shortly by $\mathbb{L}^1/\mathbb{H}$ the corresponding decomposition of the set $\mathbb{L}^1\mathbb{D}(\Omega)$, i.e. in fact one of systems $\mathbb{L}^1\mathbb{D}(\Omega)/\mathbb{L}^1_1\mathbb{D}(\Omega)$, $\mathbb{L}^1\mathbb{D}(\Omega)/\mathbb{L}^1_0\mathbb{D}(\Omega)$ of equivalence-block of operators then for any subset $U \subseteq \mathbb{L}^1\mathbb{D}(\Omega)$ its star $\mathrm{St}(U,\mathbb{L}^1/\mathbb{H})$ in the covering $\mathbb{L}^1/\mathbb{H}$ of $\mathbb{L}^1\mathbb{D}(\Omega)$ is union of all blocks from $\mathbb{L}^1/\mathbb{H}$ incident with U.

Define $U \,\mathrm{p}_{\mathbb{H}}\, V$ for $U,V \subseteq \mathbb{L}^1\mathbb{D}(\Omega)$ whenever $\mathrm{St}(U,\mathbb{L}^1/\mathbb{H}) \approx \mathrm{St}(V,\mathbb{L}^1/\mathbb{H})$ (i.e. these sets has non-empty intersection). In our considerations by a proximity (space) we mean a proximity (space) in the sense [4, p. 439], cf. Definition 1.4 above.

Theorem 1.11. *Let $\Omega \subseteq \mathbb{R}^n$ be an open domain and $\mathbb{H} \in \{\mathbb{L}^1_1\mathbb{D}(\Omega), \mathbb{L}^1_0\mathbb{D}(\Omega)\}$. The binary relation*

$$\mathrm{p}_{\mathbb{H}} \subseteq \mathcal{P}(\mathbb{L}^1\mathbb{D}(\Omega)) \times \mathcal{P}(\mathbb{L}^1\mathbb{D}(\Omega))$$

is a proximity on the set $\mathbb{L}^1\mathbb{D}(\Omega)$*, compatible in the sense* $U, V, W \subseteq \mathbb{L}^1\mathbb{D}(\Omega)$*,* $U \ \mathrm{p}_{\mathbb{H}}\ V$ *implies* $(U * V)\ \mathrm{p}_{\mathbb{H}}\ (V * W)$ *and* $(W * U)\ \mathrm{p}_{\mathbb{H}}\ (W * V)$*, consequently* $\big(\mathcal{P}(\mathbb{L}^1\mathbb{D}(\Omega)), *, \mathrm{p}_{\mathbb{H}}\big)$ *is a tolerance semigroup.*

Proof. Evidently, the above defined relation $\mathrm{p}_{\mathbb{H}}$ satisfies the condition 1 from Definition 1.4, i.e. $\emptyset \mathrm{non}\ \mathrm{p}_{\mathbb{H}}\ \mathbb{L}^1\mathbb{D}(\Omega)$ and moreover it is also symmetrical, so condition 2 is also satisfied. Further, if $U, V \subseteq \mathbb{L}^1\mathbb{D}(\Omega)$, $U \approx V$ then $\mathrm{St}(U, \mathbb{L}^1/\mathbb{H}) \approx \mathrm{St}(V, \mathbb{L}^1/\mathbb{H})$, thus $U\ \mathrm{p}_{\mathbb{H}}\ V$. (condition 3) Since $\mathrm{St}(U \cup V, \mathbb{L}^1/\mathbb{H}) = \mathrm{St}(U, \mathbb{L}^1/\mathbb{H}) \cup \mathrm{St}(V, \mathbb{L}^1/\mathbb{H})$ (in fact, the mapping $\mathrm{St}(-, \mathbb{L}^1/\mathbb{H})\colon \mathcal{P}(\mathbb{L}^1\mathbb{D}(\Omega)) \to \mathcal{P}(\mathbb{L}^1\mathbb{D}(\Omega))$ is a totally additive idempotent closure operation), we obtain that condition 4 from Definition 1.4 is also satisfied.

Now suppose $U, V, W \subseteq \mathbb{L}^1\mathbb{D}(\Omega)$ are subsets such that $U\ \mathrm{p}_{\mathbb{H}}\ V$. Then we have $\mathrm{St}(U, \mathbb{L}^1/\mathbb{H}) \approx \mathrm{St}(V, \mathbb{L}^1/\mathbb{H})$ which means that there exists a block $B \in \mathbb{L}^1/\mathbb{H}$, $B \in \mathrm{St}(U, \mathbb{L}^1/\mathbb{H})$ which is of the form $B = \mathbb{H} \cdot \mathrm{D}(\vec{a}, p)$ for a suitable operator $\mathrm{D}(\vec{a}, p) \in U \cap B$ and $B \in \mathrm{St}(V, \mathbb{L}^1/\mathbb{H})$. There exists also an operator $\mathrm{D}(\vec{b}, q) \in V \cap B$ such that $B = \mathbb{H} \cdot \mathrm{D}(\vec{b}, q)$. In fact

$$\begin{aligned}\mathrm{St}(U * W, \mathbb{L}^1/\mathbb{H}) &= \mathrm{St}\Big(\bigcup_{[D,F]\in U\times W} D * F, \mathbb{L}^1/\mathbb{H}\Big)\\ &= \bigcup_{[D,F]\in U\times W} \mathrm{St}\big(D * F, \mathbb{L}^1/\mathbb{H}\big)\end{aligned}$$

and

$$D * F\big\{\mathrm{D}(\vec{c}, s); D \cdot F \leq \mathrm{D}(\vec{c}, s)\big\}.$$

For any operator $\mathrm{D}(\vec{\theta}, \vartheta) \in \mathbb{L}^1\mathbb{D}(\Omega)$ we have

$$\mathrm{D}(\vec{a}, p) \cdot \mathrm{D}(\vec{\theta}, \vartheta) \in B \cdot \mathrm{D}(\vec{\theta}, \vartheta)$$

and

$$\mathrm{D}(\vec{b}, q) \cdot \mathrm{D}(\vec{\theta}, \vartheta) \in B \cdot \mathrm{D}(\vec{\theta}, \vartheta).$$

Since any translation of each decomposition-block from $\mathbb{L}^1/\mathbb{H}$ is a block of the same decomposition, there holds

$$B \cdot \mathrm{D}(\vec{\theta}, \vartheta) \subset \mathrm{St}\big(U * W, \mathbb{L}^1/\mathbb{H}\big)$$

and simultaneously

$$B \cdot \mathrm{D}(\vec{\theta}, \vartheta) \subset \mathrm{St}\big(V * W, \mathbb{L}^1/\mathbb{H}\big).$$

Consequently,

$$B \cdot \mathrm{D}(\vec{\theta}, \vartheta) \subset \mathrm{St}\big(U * W, \mathbb{L}^1/\mathbb{H}\big) \cap \mathrm{St}\big(V * W, \mathbb{L}^1/\mathbb{H}\big),$$

hence

$$\mathrm{St}\big(U * W, \mathbb{L}^1/\mathbb{H}\big) \approx \mathrm{St}\big(V * W, \mathbb{L}^1/\mathbb{H}\big),$$

which means $(U{*}W)\ \mathrm{p}_{\mathbb{H}}\ (V{*}W)$. In a similar way we can verify the implication: $U\ \mathrm{p}_{\mathbb{H}}\ V$ implies $(W * U)\ \mathrm{p}_{\mathbb{H}}\ (W * V)$. $\square$

In the interesting paper [22] of Heidary and B. Davvaz there is defined a partially preordered (ordered) semihypergroup–Definition 2.4, p. 87. In details: An algebraic structure $(H, \cdot, \leq)$ is called $G, \leq$ is a partially preordered (ordered) semihypergroup, if $(H, \cdot)$ is a semihypergroup and "$\leq$" is a partial preordered(ordered) relation on H such that the monotonicity condition holds as follows:

$$x \leq y \text{ then } a \cdot x \leq a \cdot y \text{ for every } x, y, a \in H,$$

where, if A and B are non-empty subsets of H, then we say that $A \leq B$ if for every $a \in A$ there exists $b \in B$ such that $a \leq b$.

In the same paper [22, p. 87], there is defined a *regular equivalence relation on the right* and *on the left* and also a *strongly regular equivalence-one-sided* and *both-sided*, as well. These concepts are overtaken from the monograph [4]. In connection with the above concept of the regularity of a binary relation we introduce the notion of a regular preorder (preordering).

Definition 1.12. *A semihypergroup $(H, \cdot)$ with a reflexive and transitive binary relation "$\leq$" on the carrier H is said to be* regularly preordered on the right (on the left) *if for any triplet $a, x, y \in H$ such that $x \leq y$ there follows*

$$x \cdot a \mathrel{\bar{\leq}} y \cdot a \qquad (a \cdot x \mathrel{\bar{\leq}} a \cdot y, \ \textit{respectively}),$$

where for $A, B \subseteq H$ the relationship $A \mathrel{\bar{\leq}} B$ means that for any $t \in A$ there exists $s \in B$ such $t \leq s$ and for any $s' \in B$ there exists $t' \in A$ such that $t' \leq s'$. The preordering "$\leq$" on H is called regular *if it is regular on the right and on the left. If both conditions are satisfied we say that a semihypergroup $(H, \cdot, \leq)$ is* regularly ordered.

Construction

Let $\delta\colon \mathbb{L}^1\mathbb{D}(\Omega) \times \mathbb{C}^1(\Omega) \to \mathbb{L}^1\mathbb{D}(\Omega)$ be the action of the join space $(\mathbb{C}^1(\Omega), \circ)$ on the transposition hypergroup $(\mathbb{L}^1\mathbb{D}(\Omega), *)$ defined by

$$\begin{aligned}\delta\big(\mathrm{D}(a_1, \ldots, a_n, p), f\big) &= \mathrm{D}(a_1, \ldots, a_n, p) \cdot \mathrm{D}(f, \ldots, f, 1)\\ &= \mathrm{D}(a_1 + pf, \ldots, a_n + pf, p)\end{aligned}$$

for any pair $\big[\mathrm{D}(a_1,\dots,a_n,p),f\big]\in \mathbb{L}^1\mathbb{D}(\Omega)\times\mathbb{C}^1(\Omega)$.

As above, we write sometimes $\mathrm{D}(\vec{a},p)$ instead of $\mathrm{D}(a_1,\dots,a_n,p)$. Now, define a binary relation "$\leq^1_\delta$" on the set $\mathbb{L}^1\mathbb{D}(\Omega)$ by the rule

$$\mathrm{D}(a_1,\dots,a_n,p)\leq^1_\delta \mathrm{D}(b_1,\dots,b_n,q)$$

it there is a function $f\in\mathbb{C}^1(\Omega)$ such that

$$\delta\big(\mathrm{D}(a_1,\dots,a_n,p),f\big)=\mathrm{D}(b_1,\dots,b_n,q),$$

i.e.

$$\mathrm{D}(a_1,\dots,a_n,p)\cdot\mathrm{D}(f,\dots,f,1)=\mathrm{D}(b_1,\dots,b_n,q),$$

where $b_k=a_k+fb_k$, $k=1,2,\dots,n$ and $p=q$.

It is easy to see that the binary relation "$\leq^1_\delta$" is reflexive and transitive, i.e. is is a preordering on the set $\mathbb{L}^1\mathbb{D}(\Omega)$. Indeed, $\mathrm{D}(\vec{a},p)\cdot\mathrm{D}(0,\dots,0,1)=\mathrm{D}(\vec{a},p)$, thus the relation is reflexive.

If $\mathrm{D}(\vec{a},p)\leq^1_\delta\mathrm{D}(\vec{b},q)$ and $\mathrm{D}(\vec{b},q)\leq^1_\delta\mathrm{D}(\vec{c},s)$ we have $\delta\big(\mathrm{D}(\vec{a},p),f\big)=\mathrm{D}(\vec{b},q)$ and $\delta\big(\mathrm{D}(\vec{b},q),g\big)=\mathrm{D}(\vec{c},s)$ for suitable functions $f,g\in\mathbb{C}^1(\Omega)$. Then

$$\begin{aligned}\mathrm{D}(\vec{c},s)&=\mathrm{D}(\vec{b},q)\cdot\mathrm{D}(g,\dots,g,1)=\mathrm{D}(\vec{a},p)\cdot\mathrm{D}(f,\dots,f,1)\cdot\mathrm{D}(g,\dots,g,1)\\&=\mathrm{D}(\vec{a},p)\cdot\mathrm{D}(f+g,\dots,f+g,1),\end{aligned}$$

hence $\mathrm{D}(\vec{a},p)\ \leq^1_\delta\ \mathrm{D}(\vec{c},s)$, so the relation "$\leq^1_\delta$" is also transitive.

Theorem 1.13. *Let $\Omega\in\mathbb{R}^n$ be an open domain. The triad $(\mathbb{L}^1\mathbb{D}(\Omega),*,\leq^1_\delta)$ is a regularly preordered transposition hypergroup, i.e. a regularly preordered noncommutative join space.*

Proof. Let $\mathrm{D}(\vec{a},p),\mathrm{D}(\vec{b},q),\mathrm{D}(\vec{c},s)\in\mathbb{L}^1\mathbb{D}(\Omega)$ be an arbitrary triplet such that $\mathrm{D}(\vec{a},p)\leq^1_\delta\mathrm{D}(\vec{b},q)$. We are going to show that

$$\mathrm{D}(\vec{a},p)*\mathrm{D}(\vec{c},s)\,\bar{\leq}^1_\delta\,\mathrm{D}(\vec{b},q)*\mathrm{D}(\vec{c},s)$$

and

$$\mathrm{D}(\vec{c},s)*\mathrm{D}(\vec{a},p)\,\bar{\leq}^1_\delta\,\mathrm{D}(\vec{c},s)*\mathrm{D}(\vec{b},q).$$

If $\mathrm{D}(\vec{a},p)\leq^1_\delta\mathrm{D}(\vec{b},q)$ there exists a function $f\in\mathbb{C}^1(\Omega)$ such that $\mathrm{D}(\vec{b},q)=\mathrm{D}(\vec{a},p)\cdot\mathrm{D}(\vec{f},1)$. Here $\vec{f}=(f_1,\cdots,f_n)=(f,\cdots,f)$ thus $\mathrm{D}(\vec{b},q)=\mathrm{D}(\overrightarrow{a+pf},p)=\mathrm{D}(a_1+pf,\cdots,a_n+pf,p)$.

Now suppose

$$\mathrm{D}(\vec{d},u)\in\mathrm{D}(\vec{a},p)*\mathrm{D}(\vec{c},s)=\Big\{\mathrm{D}(\vec{\Gamma},\gamma),\overrightarrow{a+pc}\leq\vec{\Gamma},\gamma=ps\Big\}.$$

Here $\vec{\Phi} \le \vec{\Psi}$ (for $\vec{\Phi} = (\Phi_1, \dots, \Phi_n), \vec{\Psi} = (\Psi_1, \dots, \Psi_n)$) means $\Psi_k(x_1, \dots, x_n) \le \Phi_k(x_1, \dots, x_n)$ for any points $[x_1, \dots, x_n] \in \Omega$. We have $\overrightarrow{a + pc} \le \vec{d}$ and $u = ps$. Denote $\vec{g} = \overrightarrow{d + pf}$ and $h = ps$, i.e. $\vec{g} = (g_1, \dots, g_n)$, $g_k = d_k + pf$, $k = 1, 2, \dots, n$.

Then

$$\begin{aligned} \mathrm{D}(\vec{g}, h) &= \mathrm{D}(\overrightarrow{d + pf}, ps) = \mathrm{D}(\vec{d}, ps) \cdot \mathrm{D}\big(\frac{1}{s}\vec{f}, 1\big) \\ &= \delta\Big(\mathrm{D}(\vec{d}, u), \frac{1}{s}\vec{f}\Big), \end{aligned}$$

thus $\mathrm{D}(\vec{d}, u) \le^1_\delta \mathrm{D}(\vec{g}, h)$. Further, from $\overrightarrow{a + pc} \le \vec{d}$ it follows $\overrightarrow{a + pc} + \overrightarrow{pf} \le \overrightarrow{d + pf}$ and

$$\begin{aligned} \mathrm{D}(\vec{g}, h) = \mathrm{D}(\overrightarrow{d + pf}, ps) &\in \big\{\mathrm{D}(\vec{\theta}, \vartheta); \mathrm{D}(\overrightarrow{a + pc + pf}, ps) \le \mathrm{D}(\vec{\theta}, \vartheta)\big\} \\ &= \mathrm{D}(\overrightarrow{a + pf}, p) * \mathrm{D}(\vec{c}, s) = \mathrm{D}(\vec{b}, q) * \mathrm{D}(\vec{c}, s). \end{aligned}$$

Now suppose $\mathrm{D}(\vec{g}, h) \in \mathrm{D}(\vec{b}, q) * \mathrm{D}(\vec{c}, s)$ is an arbitrary operator. Since $\overrightarrow{a + pf} = \vec{b}$, $q = p$, we have

$$\begin{aligned} \mathrm{D}(\vec{b}, q) * \mathrm{D}(\vec{c}, s) &= \big\{\mathrm{D}(\vec{\xi}, t); \mathrm{D}(\overrightarrow{b + qc}, qs) \le \mathrm{D}(\vec{\xi}, t)\big\} \\ &= \big\{\mathrm{D}(\vec{\xi}, t); \mathrm{D}(\overrightarrow{a + pf + pc}, ps) \le \mathrm{D}(\vec{\xi}, t)\big\}, \end{aligned}$$

thus $\overrightarrow{a + pf + pc} \le \vec{g}$ and $h = ps$. Then there exists a function $\lambda \in \mathbb{C}^1(\Omega)$, $\vec{\lambda} \ge \vec{0}$, which means $\vec{\lambda} = (\lambda_1, \cdots, \lambda_n, \lambda_k(x_1, \dots, x_n) \ge 0$ for $k = 1, 2, \dots, n$– such that $\overrightarrow{a + pc + pf} + \vec{\lambda} = \vec{g}$. Further,

$$\begin{aligned} \mathrm{D}(\vec{g}, h) &= \mathrm{D}(\overrightarrow{a + pc + pf} + \vec{\lambda}, ps) = \mathrm{D}(\overrightarrow{a + pc} + \vec{\lambda}, ps) \cdot \mathrm{D}(pf, \dots, pf, 1) \\ &= \delta\big(\mathrm{D}(\overrightarrow{a + pc} + \vec{\lambda}, ps), pf\big) \end{aligned}$$

i.e. $\mathrm{D}(\overrightarrow{a + pc} + \vec{\lambda}, ps) \bar{\le}^1_\delta \mathrm{D}(\vec{g}, h)$ and $\overrightarrow{a + pc} \le \overrightarrow{a + pc} + \vec{\lambda}$, which means

$$\mathrm{D}(\overrightarrow{a + pc} + \vec{\lambda}, ps) \in \mathrm{D}(\vec{a}, p) * \mathrm{D}(\vec{c}, s).$$

Consequently

$$\mathrm{D}(\vec{a}, p) * \mathrm{D}(\vec{c}, s) \bar{\le}^1_\delta \mathrm{D}(\vec{b}, q) * \mathrm{D}(\vec{c}, s).$$

Now we verify that the preordering "$\bar{\le}^1_\delta$" is the preordering of the hypergroup $(\mathbb{L}^1\mathbb{D}(\Omega), *)$ regular on the left. So, suppose $\mathrm{D}(\vec{d}, u) \in \mathrm{D}(\vec{c}, s) * \mathrm{D}(\vec{a}, p)$, thus $\mathrm{D}(c \vec{+} sa, sp) \le \mathrm{D}(\vec{d}, u)$ which means $(c \vec{+} sa \le \vec{d}$ and $sp = u$.

Then $\mathrm{D}(\vec{d},u)\cdot\mathrm{D}(\vec{f},1)=\mathrm{D}(\overrightarrow{d+uf},u)=\mathrm{D}(\overrightarrow{d+spf},sp)$, hence

$$\overrightarrow{c+sa+spf}=\overrightarrow{c+sa}+\overrightarrow{spf}\le\vec{d}+\overrightarrow{spf}. \tag{1}$$

However, $\mathrm{D}(\overrightarrow{d+spf},sp)=\delta\big(\mathrm{D}(\vec{d},u),f\big)$, i.e. $\mathrm{D}(\vec{d},u)\le^{1}_{\delta}\mathrm{D}(\overrightarrow{d+spf},sp)$. Further, since $\mathrm{D}(\vec{b},q)=\mathrm{D}(\vec{a},p)\cdot\mathrm{D}(\vec{f},1)=\mathrm{D}(\overrightarrow{a+pf},p)$, we have

$$\begin{aligned}\mathrm{D}(\vec{c},s)*\mathrm{D}(\vec{b},q)&=\mathrm{D}(\vec{c},s)*\mathrm{D}(\overrightarrow{a+pf},p)\\&=\big\{\mathrm{D}(\vec{\varphi},t);\mathrm{D}(\overrightarrow{c+sa+spf},sp)\le\mathrm{D}(\vec{\varphi},t)\big\}.\end{aligned}$$

With respect to the inequality (1) we obtain

$$\mathrm{D}(\overrightarrow{d+spf},sp)\in\mathrm{D}(\vec{c},s)*\mathrm{D}(\vec{b},q).$$

It remains to show that for any operator $\mathrm{D}(\vec{\theta},\vartheta)\in\mathrm{D}(\vec{c},s)*\mathrm{D}(\vec{b},q)$ there exists an operator $\mathrm{D}(\vec{\Gamma},\gamma)\in\mathrm{D}(\vec{c},s)*\mathrm{D}(\vec{a},p)$ such that $\mathrm{D}(\vec{\Gamma},\gamma)\le^{1}_{\delta}\mathrm{D}(\vec{\theta},\vartheta)$.

Thus, suppose $\mathrm{D}(\vec{\theta},\vartheta)\in\mathrm{D}(\vec{c},s)*\mathrm{D}(\vec{b},q)$, i.e. $\mathrm{D}(\overrightarrow{c+sb},sq)\le\mathrm{D}(\overrightarrow{\theta},\xi)$. As above, $\vec{b}=\overrightarrow{a+pf}$ and $q=p$, hence $\mathrm{D}(\overrightarrow{c+sb},sq)=\mathrm{D}(\overrightarrow{c+sa+spf},sp)$. From $\mathrm{D}(\overrightarrow{c+sb},sq)\le\mathrm{D}(\theta,\xi)$ there follows

$$\overrightarrow{c+sa+spf}\le\vec{\theta}$$

thus there exists a vector $\vec{\lambda}\ge\vec{0}$ with the property $\overrightarrow{c+sa+spf}+\vec{\lambda}=\vec{\theta}$. Denote $\Gamma=\overrightarrow{c+sa+\lambda}$, $\gamma=sp$. Then $\overrightarrow{c+sa}\le\overrightarrow{c+sa}\vec{\lambda}=\Gamma$ thus

$$\mathrm{D}(\vec{\Gamma},\gamma)\in\big\{\mathrm{D}(\vec{\psi},v);\mathrm{D}(\overrightarrow{c+sa},sp)\le\mathrm{D}(\vec{\psi},v)\big\}=\mathrm{D}(\vec{c},s)*\mathrm{D}(\vec{a},p).$$

Further,

$$\begin{aligned}\delta\big(\mathrm{D}(\Gamma,\gamma),f\big)&=\mathrm{D}(\Gamma,\gamma)\cdot\mathrm{D}(f,1)=\mathrm{D}(\overrightarrow{c+sa+\lambda},sp)\cdot(f,1)\\&=\mathrm{D}(\overrightarrow{c+sa+spf+\lambda},sp)=\mathrm{D}(\vec{\theta},\xi).\end{aligned}$$

Hence

$$\mathrm{D}(\vec{c},s)*\mathrm{D}(\vec{a},p)\le^{1}_{\delta}\mathrm{D}(\vec{c},s)*\mathrm{D}(\vec{b},q)$$

and the proof is complete. □

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References

[1] Bakhshi, M., Borzooei, R. A.: *Ordered polygroups*, Ratio Math. 24(2013), 31–40.

[2] Bavel, Z., Grzymata-Busse, Soohong, K.: *On the connectivity of the product of automata*, Annales Societatis Mathematicae Polonae, Serie IV: Fundamenta Informaticae VII.2 (1984), 225–266.

[3] Antampoufis, N., Spartalis, S., Vougiouklis, Th.: *Fundamental relations in special extensions*. In. 8th International Congress on AHA and Appl., Samothraki, Greece (2003), 81–90.

[4] Čech, E.: *Topological Spaces*, revised by Z. Frolík and M. Katětov, Academia, Prague, 1966.

[5] Chajda, I.: *Algebraic Theory of Tolerance Relations*, The Palacky University, Olomouc, 1991.

[6] Chvalina, J.: *Functional Graphs, Quasi-ordered Sets and Commutative Hypergroups*, MU Brno, Czech, 1995.

[7] Chvalina, J.: *Commutative hypergroups in the sense of Marthy*, In. Proc. Summer School, Horní Lipová, Czech Republic (1994), 19–30.

[8] Chvalina, J., Chvalinová, L.: *State hypergroups of automata*, Acta Math. et Inf. Univ. Ostraviensis 4 (1996), 105–120.

[9] Chvalina, J., Chvalinová, L.: *Modelling of join spaces by nth order linear ordinary differential operators*, In: Proc. Aplimat, Bratislava (2005), 279–285.

[10] Chvalina, J., Hošková, Š.: *Join space of first-order linear partial differential operators with compatible proximity induced by a congruence on their group*, In: Proc. Mathematical and Computer Modelling in Science and Engineering, Prague (2003), 166–170.

[11] Chvalina, J., Hošková, Š.: *Transformation Tolerance Hypergroups*, Thai Journal of Mathematics, 4(I)(2006), 63–72.

[12] Corsini, P.: *Prolegomena of Hypergroup Theory*, Aviani Editore, Tricesimo, 1993.

[13] Corsini, P., Leoreanu. V.: *Applications of Hyperstructure Theory*, Kluwer Academic Publishers, Dordrecht, Hardbound, 2003.

[14] Czászár, A.: *Fundations of General Topology*, A. P. Pergamon Press book, The Macmillan Co., New York, 1963.

[15] Cristea, I.: *Several aspects on the hypergroups associated with n-ary relations*, Analele Št. Univ. Ovidius Constanťa, 17(2009), 99–110.

[16] Cristea, I., Ştefănescu, M.: *Binary relations and reduced hypergroups*, Discrete Math. 308(2008), 3537–3544.

[17] Cristea, I., Ştefănescu, M.: *Hypergroups and n-ary relations*, European J. Combin. 31(2010), 780–789.

[18] Cristea, I., Ştefănescu, M., Angheluţă, C.: *About the fundamental relations defined on the hypergroupoids associated with binary relations*, Europian J. Combin., 32(2011), 72–81.

[19] Davvaz, B., Leoreanu Fotea, V.: *Hyperring theory and Applications*, Hadronic Press, Palm Barbor, Fl. U.S.A, 2008.

[20] Davvaz, B.: *Polygroup Theory and Related Systems,* World Scientific, 2013.

[21] Flaut, C.: *Division algebras with dimension* $2^t, t \in \mathbb{N}$, Analele Št. Univ. Ovidius Constanťa, 13(2)(2005), 31–38.

[22] Heidari, D., Davvaz, B.: *On ordered hyperstructures.* U.P.B. Sci. Bull., Series A, 73(2)(2011), 85–96.

[23] Hošková, Š, Chvalina, J., Račková, P.: *Transposition hypergroups of Fredholm integral operators and related hyperstructures*, Journal of Basic Science 41(2005), 43–51.

[24] Hošková, Š., Chvalina, J.: *Discrete transformation hypergroups and transformation hypergroups with phase tolerance space*, Discrete Math. 308(2008), 4133–4143.

[25] Hošková, Š.: *Discrete Transformation Hypergroups*, In: Proc. Aplimat, Bratislava (2005), 275–279.

[26] Hošková, Š., Chvalina, J.: *Multiautomata formed by first order partial differential operators*, Aplimat-Journal of Applied Mathematics, 1(I)(2008), 423–439.

[27] Hörmander, L.: *Analysis of Linear Partial Differential Operators I, II*, Berlin, Springer I (1983).

[28] Jantosciak, J.: *Transposition hypergroups: Noncommutative join spaces*, J. Algebra 187(1997), 97–119.

[29] Massouros, G. G., Massouros, CH. G., Mittas I. D.: *Fortified Join hypergroups*, Annales mathmatiques Blaise Pascal, 3(II)(1996), 155–169.

[30] Novák, M.: *The notions of subhyperstructure of "Ends-lemma"-based hyperstructures*, Aplimat-J. of Applied Math. 3(II)(2010), 237–247.

[31] Novák, M.: *Some basic properties of EL-hyperstructures*, European J. of Combinatorics 34(2013), 446–459.

[32] Novák, M.: *EL-hyperstructures: an overview*, Ratio Math. 23(2012), 65–80.

[33] Spartalis, S., Mamaloukas, C.: *Hyperstructures associated with binary relations*, Comput. Math. Appl. 51(2006), 41–50.

[34] Vougiouklis, T.: *Hyperstructures and their Representations*, Hadronic Press Monographs in Mathematics, Palm Harbor Florida 1994.

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