Stanley depth of squarefree Veronese ideals

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Abstract

We compute the Stanley depth for the quotient ring of a square free Veronese ideal and we give some bounds for the Stanley depth of a square free Veronese ideal. In particular, it follows that both satisfy the Stanley's conjecture.

Introduction

Let K be a field and $S = K[x_1, \dots, x_n]$ the polynomial ring over K . Let M be a $\mathbb{Z}^n$ -graded S -module. A *Stanley decomposition* of M is a direct sum $\mathcal{D} : M = \bigoplus_{i=1}^r m_i K[Z_i]$ as K -vector space, where $m_i \in M$, $Z_i \subset \{x_1, \dots, x_n\}$ such that $m_i K[Z_i]$ is a free $K[Z_i]$ -module. We define $\text{sdepth}(\mathcal{D}) = \min_{i=1}^r |Z_i|$ and $\text{sdepth}(M) = \max\{\text{sdepth}(\mathcal{D}) \mid \mathcal{D} \text{ is a Stanley decomposition of } M\}$. The number $\text{sdepth}(M)$ is called the *Stanley depth* of M . Stanley conjecture [1] says that $\text{sdepth}(M) \geq \text{depth}(M)$.

Herzog, Vladoiu and Zheng show in [5] that $\text{sdepth}(M)$ can be computed in a finite number of steps if $M = I/J$, where $J \subset I \subset S$ are monomial ideals. There are two important particular cases, I and S/I . The Stanley conjecture for S/I and I was proved for $n \leq 5$ and in other special cases, but it remains open in the general case. See for example, [6]. Also, the explicit computation of the Stanley depth turns out to be a difficult problem, even for simpler monomial ideals, or quotient of monomial ideals. See for instance

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[2], where the authors compute the Stanley depth for the monomial maximal ideal $(x_1, \dots, x_n) \subset S$.

For any $d \in [n]$, we denote $I_{n,d} := (u \in S \text{ square free monomial} : \deg(u) = d)$. It is well known that $\dim(S/I_{n,d}) = \text{depth}(S/I_{n,d}) = d - 1$. Let $\mathbf{m} = (x_1, \dots, x_n) \subset S = K[x_1, \dots, x_n]$ be the maximal monomial ideal of S . We showed in [4] that $\text{sdepth}(\mathbf{m}^k) \leq \left\lceil \frac{n}{k+1} \right\rceil$, for any positive integer k . In this paper, we use similar techniques to give an upper bound for $\text{sdepth}(I_{n,d})$. More precisely, we show that $\text{sdepth}(S/I_{n,d}) = d - 1$ and $d \geq \text{sdepth}(I_{n,d}) \frac{n-d}{d+1} + d$, see Theorem 1.1. As a consequence, it follows that $I_{n,d}$ and $S/I_{n,d}$ satisfy the Stanley conjecture, see Corollary 1.2. Also, we prove that $\text{sdepth}(I_{n,d}) = d + 1$, if $2d + 1 \leq n \leq 3d$, see Corollary 1.5. In order to do so, we use some combinatorics results, see Theorem 1.3 and Corollary 1.4. Finally, we conjecture that $\text{sdepth}(I_{n,d}) = \left\lceil \frac{n-d}{d+1} \right\rceil + d$.

1 Main results

Theorem 1.1. (1) $\text{sdepth}(S/I_{n,d}) = d - 1$.

(2) $d \leq \text{sdepth}(I_{n,d}) \leq \frac{n-d}{d+1} + d$.

Proof. (1) Firstly, note that $\text{sdepth}(S/I_{n,d}) \leq d - 1 = \dim(S/I_{n,d})$. We use induction on n and d . If $n = 1$, there is nothing to prove. If $d = 1$, it follows that $I_{n,1} = (x_1, \dots, x_n)$ and thus $\text{sdepth}(S/I_{n,1}) = 0$, as required. If $d = n$, it follows that $I_{n,n} = (x_1 \cdots x_n)$ and therefore $\text{sdepth}(S/I_{n,n}) = n - 1$, as required. Now, assume $n > 1$ and $1 < d < n$. Note that

$$S/I_{n,d} = \bigoplus_{|supp(u)| < d} u \cdot K = \sum_{Z \subset \{x_1, \dots, x_n\}, |Z|=d-1} K[Z].$$

We denote $S' = K[x_1, \dots, x_{n-1}]$. By previous equality, we get

$$\begin{aligned} S/I_{n,d} &= \sum_{Z \subset \{x_1, \dots, x_{n-1}\}, |Z|=d-1} K[Z] \oplus x_n \left(\sum_{Z \subset \{x_1, \dots, x_{n-1}\}, |Z|=d-1} K[Z] \right) [x_n] = \\ &= S'/I_{n-1,d} \oplus x_n (S'/I_{n-1,d-1}) [x_n]. \end{aligned}$$

By induction hypothesis, it follows that $\text{sdepth}(S/I_{n,d}) = d - 1$.

(2) We consider the following simplicial complex, associated to $I_{n,d}$,

$$\Delta_{n,d} := \{supp(u) : u \in I_{n,d} \text{ monomial}\}.$$

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Note that, by [5, Theorem 2.4], there exists a partition of $\Delta_{n,d} = \bigcup_{i=1}^r [F_i, G_i]$, such that $\min_{i=1}^r |G_i| = \text{sdepth}(I_{n,d}) := s$. Note that $\Delta_{n,d} = \{F \subset [n] : |F| \geq d\}$. It follows that $\text{sdepth}(I_{n,d}) \geq d$.

We consider an interval $[F_i, G_i]$ with $|F_i| = d$. Since $|G_i| \geq s$, it follows that there exists at least $(s - d)$ distinct sets in $[F_i, G_i]$ of cardinality $d + 1$. Since $\Delta_{n,d} = \bigcup_{i=1}^r [F_i, G_i]$ is a partition, it follows that $\binom{n}{d+1} = \frac{n-d}{d+1} \binom{n}{d} \geq (s-d) \binom{n}{d}$. Thus, $s \leq d + \frac{n-d}{d+1}$. $\square$

Corollary 1.2. *$I_{n,d}$ and $S/I_{n,d}$ satisfy the Stanley's conjecture. Also,*

$$\text{sdepth}(I_{n,d}) \geq \text{sdepth}(S/I_{n,d}) + 1.$$

Let $k \leq n$ be two positive integers. We denote $A_{n,k} = \{F \subset [n] \mid |F| = k\}$. We present the following well known result from combinatorics. In order of completeness, we give also a sketch of the proof.

Theorem 1.3. *For any positive integers $d \leq n$ such that $d \leq n/2$, there exists a bijective map $\Phi_{n,d} : A_{n,d} \rightarrow A_{n,d}$, such that $\Phi_{n,d}(F) \cap F = \emptyset$ for any $F \in A_{n,d}$.*

Proof. We use induction on n and d . If $n \leq 2$ the statement is obvious. If $d = 1$, for any $i \in [n]$, we define $\Phi_{n,1}(\{i\}) = \{j\}$, where $j = \max([n] \setminus \{\Phi_{n,1}(\{1\}), \dots, \Phi_{n,1}(\{i-1\})\})$. $\Phi_{n,1}$ is well defined and satisfy the required conditions. Now, assume $n \geq 3$ and $d \geq 2$. If $n = 2d$ we define $\Phi_{n,d}(F) = [n] \setminus F$. Obviously, $\Phi_{n,d}$ satisfy the required conditions. Thus, we may also assume $d < n/2$.

On $A_{n,d}$, we consider the lexicographic order, recursively defined by $F < G$ if and only if $\max\{F\} < \max\{G\}$ or $\max\{F\} = \max\{G\} = k$ and $F \setminus \{k\} < G \setminus \{k\}$ on $A_{n,d-1}$. For any $F \in A_{n,d}$, we define $G := \Phi_{n,d}(F)$ to be the maximum set, with respect to " $<$ ", such that $G \cap F = \emptyset$ and $G \neq \Phi_{n,d}(H)$ for all $H < F$. In order to complete the proof, it is enough to show that each collection of sets

$$\mathcal{M}_F^n = \{G \subset [n] : |G| = d, G \cap F = \emptyset, G \neq \Phi_{n,d}(H) \ (\forall) H < F\}$$

is nonempty, for all $F \subset [n]$. Assume there exists some $F \subset [n-1]$ such that $\mathcal{M}_F^n = \emptyset$. It obviously follows that $\mathcal{M}_F^{n-1} = \emptyset$ and thus $\Phi_{n-1,d}$ is not well defined, a contradiction. Also, if $\mathcal{M}_F^n = \emptyset$ for some $F \subset [n]$ with $n \in F$, it follows similarly that $\Phi_{n-1,d-1}$ is not well defined, again a contradiction. Therefore, the required conclusion follows. $\square$

Corollary 1.4. *For any positive integers d and n such that $d < n/2$, there exists an injective map $\Psi_{n,d} : A_{n,d} \rightarrow A_{n,d+1}$, such that $F \subset \Psi_{n,d}(F)$ for any $F \in A_{n,d}$.*

Proof. We use induction on n . If $n \leq 2$ there is nothing to prove. If $d = 1$, we define $\Psi_{n,1} : A_{n,1} \rightarrow A_{n,2}$ by $\Psi_{n,1}(\{1\}) = \{1, 2\}, \dots, \Psi_{n,1}(\{n-1\}) = \{n-1, n\}$ and $\Psi_{n,1}(\{n\}) = \{1, n\}$. Now, assume $n \geq 3$ and $d \geq 2$. If $n = 2d + 1$, we consider the bijective map $\Phi_{n,d} : A_{n,d} \rightarrow A_{n,d}$ such that $\phi(F) \cap F = \emptyset$ for all $F \in A_{n,d}$ and we define $\Psi_{n,d}(F) := [n] \setminus \Phi_{n,d}(F)$. The map $\Psi_{n,d}$ satisfies the required condition.

If $n < 2d + 1$, we define $\Psi_{n,d}(F) := \Psi_{n-1,d}(F)$ if $F \subset [n-1]$ and $\Psi_{n,d}(F) := \Psi_{n-1,d-1}(F \setminus \{n\}) \cup \{n\}$ if $n \in F$. Note that both $\Psi_{n-1,d}$ and $\Psi_{n-1,d-1}$ are well defined and injective by induction hypothesis, since $n-1 \leq 2d+1$. It follows that $\Psi_{n,d}$ is well defined and injective, as required. $\square$

Corollary 1.5. *Let n, d be two positive integers such that $2d + 1 \leq n \leq 3d$. Then $\text{sdepth}(I_{n,d}) = d + 1$.*

Proof. As in the proof of 1.1, we denote

$$\Delta_{n,d} := \{\text{supp}(u) : u \in I_{n,d} \text{ monomial}\} = \{F \subset [n] : |F| \geq d\}.$$

We consider the following partition of $\Delta_{n,d}$:

$$\Delta_{n,d} = \bigcup_{|F|=d} [F, \Psi_{n,d}(F)] \cup \bigcup_{|F|>d+1} [F, F],$$

where $\Psi_{n,d}$ is given by the previous corollary. It follows that $\text{sdepth}(I_{n,d}) \geq d + 1$. On the other hand, by 1.1, $\text{sdepth}(I_{n,d}) \leq d + 1$ and thus $\text{sdepth}(I_{n,d}) = d + 1$, as required. $\square$

We conclude this paper with the following conjecture.

Conjecture 1.6. *For any positive integers $d \leq n$ such that $d \leq n/2$, we have*

$$\text{sdepth}(I_{n,d}) = \left\lfloor \frac{n-d}{d+1} \right\rfloor + d.$$

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