Pseudo-contractibility Of Weighted L^p –Algebras

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Abstract

Let G be a locally compact group, $1 < p < \infty$ and let ω be a weight function on G . Recently, we introduced the Lebesgue weighted L^p –algebra $\mathcal{L}_{\omega}^{1,p}(G)$. Here, we establish necessary and sufficient conditions for $\mathcal{L}_{\omega}^{1,p}(G)$ to be ϕ –contractible, pseudo-contractible or contractible. Moreover we give some similar results about $L^p(G, \omega)$.

1 Introduction

Let $\mathcal{A}$ be a Banach algebra and X be a Banach $\mathcal{A}$ –bimodule. A derivation is a linear map $D : \mathcal{A} \rightarrow X$ such that

$$D(ab) = aD(b) + D(a)b \quad (a, b \in \mathcal{A}).$$

A derivation D from $\mathcal{A}$ into X is inner if there is $\xi \in X$ such that

$$D(a) = a\xi - \xi a \quad (a \in \mathcal{A}).$$

A Banach algebra $\mathcal{A}$ is called contractible if every continuous derivation from $\mathcal{A}$ into any Banach $\mathcal{A}$ –bimodule is inner [15]. Accordingly, $\mathcal{A}$ is contractible if and only if it has a diagonal, i.e. there is an element $m \in \mathcal{A} \hat{\otimes} \mathcal{A}$ for which

Key Words: Contractible, Lebesgue weighted L^p –algebra, ϕ –contractible, pseudo-contractible, weight function.

2010 Mathematics Subject Classification: Primary 43A15, Secondary 43A20.

Received: April, 2012.

Revised: May, 2012.

Accepted: February, 2013.

$am = ma$ and $\pi(m)a = a$, for all $a \in \mathcal{A}$ [15]. Here and in the sequel, π always denotes the product morphism from $\mathcal{A} \hat{\otimes} \mathcal{A}$ into $\mathcal{A}$, specified by

$$\pi(a \otimes b) = ab.$$

In fact every contractible Banach algebra is unital. The structure of contractible Banach algebras has been studied by many authors; see for example [15], [24] and also [26]. Moreover, the concept of pseudo-contractibility was introduced and investigated by Ghahramani and Zhang [14], according to the existence of a central approximate diagonal; i.e., a net (m_α) in $\mathcal{A} \hat{\otimes} \mathcal{A}$ such that

$$\|\pi(m_\alpha)a - a\| \rightarrow_\alpha 0$$

and also $am_\alpha = m_\alpha a$, for all $a \in \mathcal{A}$ and all α . Although there are many pseudo-contractible Banach algebras which are not contractible, but it has been proved that pseudo-contractibility of the unitization algebra of any Banach algebra $\mathcal{A}$ is equivalent to contractibility of $\mathcal{A}$ [14, Theorem 2.4].

Moreover, let $\phi \in \Omega(\mathcal{A})$, the spectrum space of $\mathcal{A}$ consisting of all non-zero characters on $\mathcal{A}$. Then $\mathcal{A}$ is called ϕ -contractible if there is a ϕ -diagonal, i.e. an element $m \in \mathcal{A} \hat{\otimes} \mathcal{A}$ such that

$$\phi(\pi(m)) = 1 \quad \text{and} \quad am = \phi(a)m,$$

for each $a \in \mathcal{A}$. The notion of ϕ -contractibility of $\mathcal{A}$ was laid by Hu, Monfared, and Traynor [17]. Furthermore it has been investigated ϕ -contractibility of some classes of Banach algebras; see for example [2].

In the present paper, we present some characterizations for contractibility, pseudo-contractibility and also ϕ -contractibility of the Lebesgue weighted L^p -algebra $\mathcal{L}_\omega^{1,p}(G)$ and also weighted L^p -algebra $L^p(G, \omega)$, endowed with convolution product. In fact all of the obtained results about $\mathcal{L}_\omega^{1,p}(G)$, can be considered as the consequences for Segal algebra $L^1(G) \cap L^p(G)$; see [20] for basic definitions and information concerning Segal algebras.

2 Preliminaries

Let G be a locally compact group with a fixed left Haar measure λ , and let ω be a weight function on G ; that is, a positive Borel measurable function on G . The weight function ω is of left moderate growth on G if for each $x \in G$,

$$\operatorname{ess\,sup}_y \frac{\omega(xy)}{\omega(y)} < \infty.$$

For $1 < p < \infty$, the weighted L^p -space $L^p(G, \omega)$ with respect to λ is the set of all complex valued measurable functions f on G such that $f\omega \in L^p(G)$, the

usual Lebesgue space as defined in [16]. We denote this space by $\ell^p(G, \omega)$ when G is discrete. Then $L^p(G, \omega)$ is a Banach space with the norm $\|\cdot\|_{p, \omega}$ defined by $\|f\|_{p, \omega} = \|f\omega\|_p$, for all $f \in L^p(G, \omega)$. Recently, in [1] we introduced the Lebesgue weighted L^p -space $\mathcal{L}_\omega^{1,p}(G) := L^1(G) \cap L^p(G, \omega)$ with norm $\|f\|_{\mathcal{L}_\omega^{1,p}(G)} = \|f\|_1 + \|f\|_{p, \omega}$, and also the space $L^\infty(G) + L^q(G, 1/\omega)$ as its dual, with the action

$$\langle \phi + \psi, f \rangle = \int_G \phi f d\lambda + \int_G \psi f d\lambda \quad (f \in \mathcal{L}_\omega^{1,p}(G))$$

and the norm of $F = \phi + \psi$ being

$$\|F\|_{\mathcal{L}_\omega^{1,p}(G)^*} = \inf\{\max\{\|\phi\|_\infty, \|\psi\|_{q, 1/\omega}\}\},$$

where the infimum is taken over all $\phi \in L^\infty(G)$ and $\psi \in L^q(G, 1/\omega)$ such that $F = \phi + \psi$. Let $\mathcal{A}$ be one of the spaces $L^p(G, \omega)$ or $\mathcal{L}_\omega^{1,p}(G)$ and let $\mathcal{A}^*$ be its dual space. We will denote by $\tilde{G}_\mathcal{A}$ the subset of $\mathcal{A}^*$, consisting of all continuous homomorphisms $\rho : G \rightarrow \mathbb{C} \setminus \{0\}$.

It should be noted that the spaces $L^p(G, \omega)$ and $\mathcal{L}_\omega^{1,p}(G)$ and also their algebraic properties under the convolution product, have been studied very more completely in the decade of 1970. We found a lot of interesting results related to these algebras in many earlier publications. We just refer to some of them such as [7], [8], [9], [11], [12], [13] and also [25].

Note that $\mathcal{L}_\omega^{1,p}(G)$ is a Banach algebra whenever $L^p(G, \omega)$ is a Banach algebra. We provide two examples in this field. The following examples have been essentially introduced in [9, page 454].

Example 2.1. Take G to be either of $\mathbb{R}^m$ or $\mathbb{Z}^m$.

1. Define the weight function ω_a on G by

$$\omega_a(x) = (1 + |x|)^a \quad (x \in G).$$

Then $L^2(G, \omega_a)$ is a Banach algebra whenever $a > m/2$. It follows that $\mathcal{L}_\omega^{1,2}(G)$ is a Banach algebra, as well.

2. Define the weight function ω on G by

$$\omega(x) = 1 + |x_1|^{a_1} + \cdots + |x_m|^{a_m} \quad (x = (x_1, \dots, x_m) \in G),$$

whenever $0 \leq a_i < \infty$. Then $\mathcal{L}_\omega^{1,2}(G)$ is a Banach algebra.

It is noticeable to know that the space $\mathcal{L}_\omega^{1,p}(G)$ is in fact a weighted case of Segal algebra $L^1(G) \cap L^p(G)$. However there are important differences in their structures as Banach algebras. For instance, consider $\mathbb{Z}$, the additive

group of the integer numbers, and take a weight function ω on $\mathbb{Z}$ such that it is not submultiplicative and also $1/\omega \in \ell^q(\mathbb{Z})$. Then $\ell^1(\mathbb{Z}) \cap \ell^p(\mathbb{Z}) = \ell^1(\mathbb{Z})$ is always a Banach algebra whereas $\mathcal{L}_\omega^{1,p}(\mathbb{Z}) = \ell^p(\mathbb{Z}, \omega)$ is not necessarily a Banach algebra. In fact on a discrete group, the weight of any Banach algebra for all $p \geq 1$ is submultiplicative; see [19, page 573].

From now on, we assume that G is a locally compact group and ω is a weight function on G such that $L^p(G, \omega)$ or $\mathcal{L}_\omega^{1,p}(G)$ are Banach algebras under convolution, in their own situations. Moreover, we take as assumption that there are some $\rho \in \tilde{G}_A$. Note that the trivial character $\rho = 1$ always belongs to $\tilde{G}_A$, whenever $A = \mathcal{L}_\omega^{1,p}(G)$. If $A = L^p(G, \omega)$, then $1 \in \tilde{G}_A$ if and only if $1/\omega \in L^q(G)$. For example this condition is satisfied whenever G is abelian [18].

If $\mathcal{L}_\omega^{1,p}(G)$ or $L^p(G, \omega)$ is a Banach algebra, then [19, Lemma 2.1] implies that ω^p is locally summable; that is $\omega \in L^p(K)$, for each compact subset K of G .

For completeness, we also turn the attention to the spectrum space of A , where A is one of the Banach algebras $\mathcal{L}_\omega^{1,p}(G)$ or $L^p(G, \omega)$. It is not hard to see that each of them contains characters ϕ_ρ , defined by

$$\phi_\rho(f) = \int_G f(x) \rho(x) d\lambda(x),$$

where $\rho \in \tilde{G}_A$. See [18] for the proof of this statement, whenever $A = L^p(G, \omega)$ and G is abelian.

3 Main Results

We commence this section with the following proposition.

Proposition 3.1. *Let G be a locally compact group, $1 < p < \infty$ and let ω be a weight function on G and $\rho \in \tilde{G}_A$. Then the following assertions are equivalent.*

- (i) $\mathcal{L}_\omega^{1,p}(G)$ is ϕ_ρ -contractible.
- (ii) $L^p(G, \omega)$ is ϕ_ρ -contractible.
- (iii) G is compact.

Proof. (i) $\Rightarrow$ (iii). Let $\mathcal{L}_\omega^{1,p}(G)$ be ϕ_ρ -contractible. By assumption, there exists $m \in \mathcal{L}_\omega^{1,p}(G) \hat{\otimes} \mathcal{L}_\omega^{1,p}(G)$ such that for all $h \in \mathcal{L}_\omega^{1,p}(G)$

$$\phi_\rho(\pi(m)) = 1 \quad \text{and} \quad h * \pi(m) = \phi_\rho(h) \pi(m).$$

Set $g = \rho\pi(m)$. So $g \in L^1(G)$ and also $\phi_1(g) = \phi_\rho(\pi(m)) = 1$. Thus $f * g = \phi_1(f)g$, for all $f \in C_{00}(G)$; indeed,

$$\begin{aligned} f * g &= f * (\rho\pi(m)) \\ &= \rho((\tilde{\rho}f) * \pi(m)) \\ &= \rho\phi_1(f)\pi(m) \\ &= \phi_1(f)g. \end{aligned}$$

Since $C_{00}(G)$ is dense in $L^1(G)$, then for each $f \in L^1(G)$

$$f * g = \left(\int_G f(x) d\lambda(x) \right) g,$$

almost every where on G . It follows that G is compact; see for example [21, Exercise 1.1.7].

(iii) $\Rightarrow$ (i). Let G be compact. Since ω^p is locally summable, it follows that $\rho \in \mathcal{L}_\omega^{1,p}(G)$. Set $m = \tilde{\rho} \otimes \rho$, where $\tilde{\rho}(x) = \rho(x^{-1})$, for each $x \in G$. Then

$$\phi_\rho(\pi(m)) = \phi_\rho(\tilde{\rho})\phi_\rho(\rho) = 1.$$

For $f \in \mathcal{L}_\omega^{1,p}(G)$ and all $g, h \in \mathcal{L}_\omega^{1,p}(G)^*$, we have

$$\begin{aligned} ((f * \tilde{\rho}) \otimes \rho)(g, h) &= g(f * \tilde{\rho})h(\rho) \\ &= h(\rho) \int_G \int_G g(x) f(y) \tilde{\rho}(y^{-1}x) d\lambda(y) d\lambda(x) \\ &= h(\rho) \int_G g(x) \tilde{\rho}(x) d\lambda(x) \int_G f(y) \rho(y) d\lambda(y) \\ &= h(\rho) \langle g, \tilde{\rho} \rangle \phi_\rho(f) \\ &= \phi_\rho(f) (\tilde{\rho} \otimes \rho)(g, h). \end{aligned}$$

Thus for each $f \in \mathcal{L}_\omega^{1,p}(G)$,

$$(f * \tilde{\rho}) \otimes \rho = \phi_\rho(f) (\tilde{\rho} \otimes \rho).$$

It follows that $\mathcal{L}_\omega^{1,p}(G)$ is ϕ_ρ -contractible and so (i) is equivalent to (iii). One can easily follow the same steps to prove the equivalence between (ii) and (iii). $\square$

We state here the following theorem, as an immediate result of Proposition 3.1.

Theorem 3.2. *Let G be a locally compact group, $1 < p < \infty$ and let ω be a weight function on G . Then the following assertions are equivalent.*

- (i) $\mathcal{L}_\omega^{1,p}(G)$ is ϕ_ρ -contractible, for all $\rho \in \tilde{G}_A$.
- (ii) $\mathcal{L}_\omega^{1,p}(G)$ is ϕ_ρ -contractible, for some $\rho \in \tilde{G}_A$.
- (iii) $L^p(G, \omega)$ is ϕ_ρ -contractible, for all $\rho \in \tilde{G}_A$.
- (iv) $L^p(G, \omega)$ is ϕ_ρ -contractible, for some $\rho \in \tilde{G}_A$.
- (v) G is compact.

By [2], every pseudo-contractible Banach algebra $\mathcal{A}$ is ϕ -contractible, for each $\phi \in \Omega(\mathcal{A})$. Thus the following result is obtained from Proposition 3.1.

Corollary 3.3. *Let G be a locally compact group, $1 < p < \infty$ and let ω be a weight function on G . If $\mathcal{L}_\omega^{1,p}(G)$ or $L^p(G, \omega)$ is pseudo-contractible then G is compact.*

It would be interesting to know whether if the converse of Corollary 3.3 also holds. An affirmative answer for a particular case is given bellow. As a basic property of Segal algebras, $L^1(G) \cap L^p(G)$ is always a Banach left $L^1(G)$ -module. Also [22, Theorem 3.5] implies that $L^1(G) \cap L^p(G)$ is pseudo-contractible if and only if G is compact. Since $\mathcal{L}_\omega^{1,p}(G) \subseteq L^1(G)$, it also comes to mind to verify this result for $\mathcal{L}_\omega^{1,p}(G)$, in the case where it is a Banach left $L^1(G)$ -module. In the next result, we take this property as an assumption for $\mathcal{L}_\omega^{1,p}(G)$ and $L^p(G, \omega)$.

Theorem 3.4. *Let G be a locally compact group, $1 < p < \infty$ and let ω be a weight function on G . Then the following assertions are equivalent.*

- (i) $\mathcal{L}_\omega^{1,p}(G)$ is pseudo-contractible and it is a Banach left $L^1(G)$ -module.
- (ii) $L^p(G, \omega)$ is pseudo-contractible and it is a Banach left $L^1(G)$ -module.
- (iii) G is compact and ω is of left moderate growth.

Proof. (i) $\Rightarrow$ (iii). Corollary 3.3 implies that G is compact. Now we show that ω is of left moderate growth. We follow an argument similar to [19, Theorem 3.1]. Since $\mathcal{L}_\omega^{1,p}(G)$ is a Banach left $L^1(G)$ -module, it follows that with some constant $K > 0$

$$\|f * g\|_{\mathcal{L}_\omega^{1,p}(G)} \leq K \|f\|_1 \|g\|_{\mathcal{L}_\omega^{1,p}(G)},$$

for all $f \in L^1(G)$ and $g \in \mathcal{L}_\omega^{1,p}(G)$. Since ω^p is locally summable, then $\mathcal{L}_\omega^{1,p}(G)$ contains characteristic functions χ_A , for each subset A of G of finite measure. We also have the following inequality, pointwise

$$\lambda(A) \chi_{xyB} \leq \chi_{xA} * \chi_{A^{-1}yB}, \quad (1)$$

for all $x, y \in G$ and relatively compact subsets A and B of G with positive measure. Hence inequality (1) implies that

$$\lambda(A) \|\chi_{xyB}\|_{\mathcal{L}_{\omega}^{1,p}(G)} \leq \|\chi_{xA} * \chi_{A^{-1}yB}\|_{\mathcal{L}_{\omega}^{1,p}(G)} \leq K \|\chi_{xA}\|_1 \|\chi_{A^{-1}yB}\|_{\mathcal{L}_{\omega}^{1,p}(G)}.$$

Thus

$$\|\chi_{xyB}\|_{\mathcal{L}_{\omega}^{1,p}(G)} \leq K \|\chi_{A^{-1}yB}\|_{\mathcal{L}_{\omega}^{1,p}(G)}.$$

It follows that

$$\|\chi_{xyB}\|_1 + \|\chi_{xyB}\|_{p,\omega} \leq K (\|\chi_{A^{-1}yB}\|_1 + \|\chi_{A^{-1}yB}\|_{p,\omega}). \quad (2)$$

Let $x \in G$ be fixed. Since ω^p is locally summable, it follows that there exists a family $\mathcal{V}$ of sets of positive measure such that every $V \in \mathcal{V}$ contains the identity and every neighborhood of identity contains eventually all $V \in \mathcal{V}$ and also the following equations hold:

$$\lim_{V \in \mathcal{V}} \frac{1}{\lambda(V)} \int_{yV} \omega^p(r) dr = \omega^p(y)$$

and

$$\lim_{V \in \mathcal{V}} \frac{1}{\lambda(V)} \int_{yV} \omega^p(xr) dr = \omega^p(xy),$$

for locally almost all $y \in G$; see [3]. For such y and any $\varepsilon > 0$ for sufficiently small $V \in \mathcal{V}$ we have

$$\|\chi_{yV}\|_{p,\omega}^p = \int_{yV} \omega^p(r) dr < \lambda(V) \omega^p(y) (\varepsilon + 1) \quad (3)$$

and

$$\|\chi_{xyV}\|_{p,\omega}^p = \int_{yV} \omega^p(xr) dr > \frac{\lambda(V) \omega^p(xy)}{\varepsilon + 1}. \quad (4)$$

Moreover, there exists a relatively compact neighborhood U of identity such that

$$\|\chi_{U^{-1}yV}\|_{p,\omega} < (1 + \varepsilon) \|\chi_{yV}\|_{p,\omega} \quad (5)$$

and also

$$\|\chi_{U^{-1}yV}\|_1 < \frac{(1 + \varepsilon)}{K} \|\chi_{yV}\|_1. \quad (6)$$

By (2) we have

$$\|\chi_{xyV}\|_1 + \|\chi_{xyV}\|_{p,\omega} \leq K (\|\chi_{U^{-1}yV}\|_1 + \|\chi_{U^{-1}yV}\|_{p,\omega}). \quad (7)$$

As a consequence of the observations (3), (4), (5) and (6) together with (7), we have the following inequality,

$$\lambda(V) + \frac{\lambda(V)^{1/p} \omega(xy)}{(\varepsilon + 1)^{1/p}} \leq K \left(\frac{1 + \varepsilon}{K} \lambda(V) + (1 + \varepsilon)^{1+1/p} \omega(y) \lambda(V)^{1/p} \right)$$

and hence

$$(1 + \varepsilon)^{1/p} \lambda(V)^{1-1/p} + \omega(xy) \leq \left((1 + \varepsilon)^{1+1/p} \lambda(V)^{1-1/p} + K(1 + \varepsilon)^{1+2/p} \omega(y) \right).$$

Since G is compact, it follows that the net $(\lambda(V))_{V \in \mathcal{V}}$ is bounded and in the limit as $\varepsilon \rightarrow 0$, we conclude that

$$\frac{\omega(xy)}{\omega(y)} < K,$$

locally almost every where $y \in G$. Consequently ω is of left moderate growth.

(iii) $\Rightarrow$ (i). Let G be compact and let ω be of left moderate growth. Since ω is locally summable, it follows that ω is equivalent to a continuous function [10, Theorem 2.7]. So the compactness of G implies that $L^p(G, \omega) = L^p(G)$. Consequently $\mathcal{L}_\omega^{1,p}(G) = L^p(G)$. Now the identity map is a topological isomorphism of $L^p(G)$ onto $\mathcal{L}_\omega^{1,p}(G)$. Since $L^p(G)$ is pseudo-contractible [14, Theorem 4.5], it follows that $\mathcal{L}_\omega^{1,p}(G)$ is also pseudo-contractible.

A similar argument shows that (i) $\Leftrightarrow$ (ii). $\square$

The following known result is immediately obtained from Theorem 3.4, which is in fact [22, Theorem 3.5].

Corollary 3.5. *Let G be a locally compact group and $1 < p < \infty$. Then $L^1(G) \cap L^p(G)$ is pseudo-contractible if and only if G is compact.*

Remark 3.6. Let G be a locally compact group, $1 < p < \infty$ and let ω be a weight function on G . Also let $\mathcal{A}$ denote one of the Banach space $\mathcal{L}_\omega^{1,p}(G)$ or $L^p(G, \omega)$ (not necessarily an algebra), such that $\mathcal{A}$ is a Banach left $L^1(G)$ -module.

- (i) Whenever $\mathcal{A}$ is an essential Banach left $L^1(G)$ -module, a stronger result may be obtained. Indeed, we show that if $\mathcal{A}$ is an essential Banach left $L^1(G)$ -module then $\mathcal{A}$ is left translation invariant. Since $L^1(G)$ has a bounded approximate identity, one can conclude that $L^1(G) * \mathcal{A} = \mathcal{A}$, by the factorization theorem [16]. Then for every $f \in \mathcal{A}$ one can choose such $g \in L^1(G)$, $h \in \mathcal{A}$ that $f = g * h$. Now take an approximate identity ξ_ν in $L^1(G)$ and any $x \in G$. Denote by ϕ^x the left translation of a function ϕ by x . We have then $f^x = g^x * h$ and $\xi_\nu^x * g \rightarrow g^x$ in $L^1(G)$. So $\xi_\nu^x * f = (\xi_\nu^x * g) * h$ is a Cauchy net in $\mathcal{A}$. Its limit is obviously f^x . Thus $f^x \in \mathcal{A}$ and consequently $\mathcal{A}$ is translation-invariant.

- (ii) If ω is locally summable then $C_{00}(G)$, the set of all compactly supported continuous functions on G , is contained in $\mathcal{A}$. It follows that $L^1(G) * \mathcal{A}$ is dense in $\mathcal{A}$, whenever $\mathcal{A} = L^p(G, \omega)$ and G is compact. By a similar proof to part (i) we obtain that $\mathcal{A}$ is translation invariant.
- (iii) By [19, Lemma 2.1], if $L^p(G, \omega)$ is a Banach algebra then ω is locally summable. Thus part (ii) implies that if $\mathcal{A} = L^p(G, \omega)$ is a Banach algebra and G is compact then $\mathcal{A}$ is essential and again left translation invariance of $\mathcal{A}$ is obtained by part (i).
- (iv) Let G be compact and let $\mathcal{A} = L^p(G, \omega)$ be a Banach algebra. Then one gets not only $\mathcal{A}$ is translation invariant, but also the translation is a bounded operator on $\mathcal{A}$; see for example [6]. Now one can conclude that ω is of left moderate growth by [7]. Consequently the implication (ii) $\Rightarrow$ (iii) of Theorem 3.4 can be proved more easier.

We finish the paper with the presentation of an equivalent condition to contractibility of $\mathcal{L}_\omega^{1,p}(G)$ and also $L^p(G, \omega)$. Let us first recall the concept of approximation property (AP).

A Banach space $\mathcal{X}$ is said to have the approximation property (AP), if every compact operator on $\mathcal{X}$ is a limit of finite-rank operators.

Proposition 3.7. *Let G be a locally compact group, $1 < p < \infty$ and let ω be a weight function on G . Then the following assertions are equivalent.*

- (i) $L^p(G, \omega)$ is contractible.
- (ii) $\mathcal{L}_\omega^{1,p}(G)$ is contractible.
- (iii) G is finite.

Proof. (i) $\Rightarrow$ (iii). It follows from previously known results. First, one can show that $L^p(G, \omega)$ has the approximation property (AP); see for example [5, Example 11, p. 245]. From a result of Selivanov [23], it follows then that if $L^p(G, \omega)$ is contractible then G is finite.

(iii) $\Rightarrow$ (i) is obvious.

(ii) $\Rightarrow$ (iii). Let $\mathcal{L}_\omega^{1,p}(G)$ be contractible. Then it is unital and since $\mathcal{L}_\omega^{1,p}(G)$ is dense in $L^1(G)$, it follows that $L^1(G)$ is unital. Consequently G is discrete [4]. Since contractibility implies pseudo contractibility, it follows by Corollary 3.3 that G is compact. Therefore G is finite.

The statement (iii) $\Rightarrow$ (ii) is clear. □

Acknowledgments. The author expresses its sincere gratitude to Professor Hans Georg Feichtinger for his invaluable advice in continuously pointing out a number of earlier publications in the field. This research was partially supported by the Banach algebra Center of Excellence for Mathematics, University of Isfahan

References

- [1] F. Abtahi, *Lebesgue weighted L^p -algebra on locally compact groups*, Acta. Math. Hungar., **133/4**, (2011), 324-331.
- [2] M. Alaghmandan, R. Nasr Isfahani and M. Nemati, *On ϕ -contractibility of the Lebesgue-Fourier algebra of a locally compact group*, Arch. Math. (Basel), **95**, (2010), 373-379.
- [3] A. M. Bruckner, *Differentiation of integrals*, Amer. Math. Monthly, **78** No. 9 pt. 2, (1971).
- [4] A. Deitmar and S. Echterhoff, *Principles of harmonic analysis*, Springer, New York, 2009.
- [5] J. Diestel and J. J. Uhl, Jr., *Vector measures*, AMS, 1977.
- [6] D. H. Dunford, *Segal algebras and left normed ideals*, J. London Math. Soc. (2), **8**, (1974), 514-516.
- [7] R. E. Edwards, *The stability of weighted Lebesgue spaces*, Trans. Amer. Math. Soc., **93**, (1959), 369-394.
- [8] H. G. Feichtinger, *On a class of convolution algebras of functions*, Ann. Inst. Fourier (Grenoble), **27**, no. 3, (1977), 135-162.
- [9] H. G. Feichtinger, *Some remarks on Banach convolution algebras of functions (*)*, Istituto Nazionale di Alta Matematica Symposia Mathematica, Volume XXII, (1977), 453-455.
- [10] H. G. Feichtinger, *Gewichtsfunktionen auf lokalkompakten Gruppen*, Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II, **188/8-10**, (1979), 451-471.
- [11] H. G. Feichtinger, *Weighted L^p -spaces and the canonical mapping $T_H : L^1(G) \rightarrow L^1(G/H)$* , Bollettino U. M. I. (5) **16-B**, (1979), 989-999.
- [12] H. G. Feichtinger and A. T. Gürkanli, *On a family of weighted convolution algebras*, Internat. J. Math. Math. Sci., **13/3**, (1990), 517-525.
- [13] G. Fendler, K. Grchenig, M. Leinert, J. Ludwig and C. Molitor-Braun, *Weighted group algebras on groups of polynomial growth*, Math. Z., **245/4**, (2003), 791-821.
- [14] F. Ghahramani and Y. Zhang, *Pseudo-amenable and pseudo-contractible Banach algebras*, Math. Proc. Cambridge Philos. Soc., **142**, (2007), no. 1, 111-123.
- [15] A. Ya. Helemskii, A. Ya. Helemskii, *Banach and Locally Convex Algebras*, The Clarendon Press Oxford University Press, New York, 1993.
- [16] E. Hewitt and K. Ross, *Abstract harmonic analysis I,II*, Springer-Verlag, New York, 1970.
- [17] Z. Hu, M. S. Monfared and T. Traynor, *On character amenable Banach algebras*, Studia Math., **193**, (2009), no. 1, 53-78.
- [18] Yu. N. Kuznetsova, *Weighted L_p -Algebras on Groups*, Funct. Anal. Appl., **40/3**, (2006), 234-236.
- [19] Yu. N. Kuznetsova, *Invariant weighted algebras $L^p(G, \omega)$* , Mat. Zametki., **84/4**, (2008), 567-576.
- [20] H. Reiter, *L^1 -algebras and segal algebras*, Lecture Notes in Mathematics, **231**, Springer-Verlag, Berlin, 1971.
- [21] V. Runde, *Lectures on amenability*, Lecture Notes in Mathematics, **1774**, Springer-Verlag, Berlin, 2002.

- [22] E. Samei, N. Spronk and R. Stokke, *Biflatness and pseudo-amenability of Segal algebras*, Canad. J. Math., **62**, (2010), no. 4, 845-869.
- [23] Ju. V. Selivanov, *Banach Algebras of Small Global Dimension Zero*, Uspehi Mat. Nauk., **31**, (1976), 227-228.
- [24] J. L. Taylor, *Homology and cohomology for topological algebras*, Advances in Math., **9**, (1972), 137-182.
- [25] J. Wermer, *On a class of normed rings*, Ark. Mat., **2**, (1954), 537-551.
- [26] Y. Zhang, *Maximal ideals and the structure of contractible and amenable Banach algebras*, Bull. Austral. Math. Soc., **62**, (2000), 221-226.

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