Monomial ideals of minimal depth

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Abstract

Let S be a polynomial algebra over a field. We study classes of monomial ideals (as for example lexsegment ideals) of S having minimal depth. In particular, Stanley's conjecture holds for these ideals. Also we show that if I is a monomial ideal with $\text{Ass}(S/I) = \{P_1, P_2, \dots, P_s\}$ and $P_i \not\subset \sum_{j=1, j \neq i}^s P_j$ for all $i \in [s]$, then Stanley's conjecture holds for S/I .

Introduction

Let K be a field and $S = K[x_1, \dots, x_n]$ be a polynomial ring in n variables over K . Let $I \subset S$ be a monomial ideal and $I = \cap_{i=1}^s Q_i$ an irredundant primary decomposition of I , where the Q_i are monomial ideals. Let Q_i be P_i -primary. Then each P_i is a monomial prime ideal and $\text{Ass}(S/I) = \{P_1, \dots, P_s\}$.

According to Lyubeznik [9] the size of I , denoted $\text{size}(I)$, is the number $a + (n - b) - 1$, where a is the minimum number t such that there exist $j_1 < \dots < j_t$ with

$$\sqrt{\sum_{l=1}^t Q_{j_l}} = \sqrt{\sum_{j=1}^s Q_j},$$

and where $b = \text{ht}(\sum_{j=1}^s Q_j)$. It is clear from the definition that $\text{size}(I)$ depends only on the associated prime ideals of S/I . In the above definition if we replace “there exists $j_1 < \dots < j_t$ ” by “for all $j_1 < \dots < j_t$ ”, we obtain the definition of $\text{bigsize}(I)$, introduced by Popescu [11]. Clearly $\text{bigsize}(I) \geq \text{size}(I)$.

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Theorem 0.1. (*Lyubeznik [9]*) Let $I \subset S$ be a monomial ideal then $\text{depth}(I) \geq 1 + \text{size}(I)$.

Herzog, Popescu and Vladioiu say in [5] that a monomial ideal I has *minimal depth*, if $\text{depth}(I) = \text{size}(I) + 1$. Suppose above that $P_i \not\subset \sum_{j=1, j \neq i}^s P_j$ for all $i \in [s]$. Then I has minimal depth as shows our Corollary 1.3 which extends [11, Theorem 2.3]. It is easy to see that if I has bigsize 1 then it must have minimal depth (see our Corollary 1.5).

Next we consider the lexicographical order on the monomials of S induced by $x_1 > x_2 > \dots > x_n$. Let $d \geq 2$ be an integer and $\mathcal{M}_d$ the set of monomials of degree d of S . For two monomials $u, v \in \mathcal{M}_d$, with $u \geq_{\text{lex}} v$, the set

$$\mathcal{L}(u, v) = \{w \in \mathcal{M}_d \mid u \geq_{\text{lex}} w \geq_{\text{lex}} v\}$$

is called a lexsegment set. A lexsegment ideal in S is a monomial ideal of S which is generated by a lexsegment set. We show that a lexsegment ideal has minimal depth (see our Theorem 1.6).

Now, let M be a finitely generated multigraded S -module, $z \in M$ be a homogeneous element in M and $zK[Z]$, $Z \subseteq \{x_1, \dots, x_n\}$ the linear K -subspace of M of all elements zf , $f \in K[Z]$. Such a linear K -subspace $zK[Z]$ is called a Stanley space of dimension $|Z|$ if it is a free $K[Z]$ -module, where $|Z|$ denotes the number of indeterminates in Z . A presentation of M as a finite direct sum of spaces $\mathcal{D} : M = \bigoplus_{i=1}^r z_i K[Z_i]$ is called a Stanley decomposition. Stanley depth of a decomposition $\mathcal{D}$ is the number

$$\text{sdepth } \mathcal{D} = \min\{|Z_i| : i = 1, \dots, r\}.$$

The number

$$\text{sdepth}(M) := \max\{\text{sdepth}(\mathcal{D}) : \text{Stanley decomposition of } M\}$$

is called Stanley depth of M . In [14] R. P. Stanley conjectured that

$$\text{sdepth}(M) \geq \text{depth}(M).$$

Theorem 0.2 ([5]). Let $I \subset S$ be a monomial ideal then $\text{sdepth}(I) \geq 1 + \text{size}(I)$. In particular, Stanley's conjecture holds for the monomial ideals of minimal depth.

As a consequence, Stanley's depth conjecture holds for all ideals considered above since they have minimal depth. It is still not known a relation between $\text{sdepth}(I)$ and $\text{sdepth}(S/I)$, but our Theorem 2.3 shows that Stanley's conjecture holds also for S/I if $P_i \not\subset \sum_{j=1, j \neq i}^s P_j$ for all $i \in [s]$. Some of the recent development about the Stanley's conjecture is given in [12].

1 Minimal depth

We start this section extending some results of Popescu in [11]. Lemma 1.1, Proposition 1.2, Lemma 1.4 and Corollary 1.5 were proved by Popescu when I is a squarefree monomial ideal. We show that with some small changes the same proofs work even in the non-squarefree case.

Lemma 1.1. *Let $I = \bigcap_{i=1}^s Q_i$ be the irredundant presentation of I as an intersection of primary monomial ideals. Let $P_i := \sqrt{Q_i}$. If $P_s \not\subset \sum_{i=1}^{s-1} P_i$, then*

$$\text{depth}(S/I) = \min\{\text{depth}(S/\cap_{i=1}^{s-1} Q_i), \text{depth}(S/Q_s), 1 + \text{depth}(S/\cap_{i=1}^{s-1} (Q_i + Q_s))\}.$$

Proof. We have the following exact sequence

$$0 \longrightarrow S/I \longrightarrow S/\cap_{i=1}^{s-1} Q_i \oplus S/Q_s \longrightarrow S/\cap_{i=1}^{s-1} (Q_i + Q_s) \longrightarrow 0.$$

Clearly $\text{depth}(S/I) \leq \text{depth}(S/Q_s)$ by [1, Proposition 1.2.13]. Choosing x_j^a where $x_j \in P_s \not\subset \sum_{i=1}^{s-1} P_i$ and a is minimum such that $x_j^a \in Q_s$ we see that $I : x_j^a = \cap_{i=1}^{s-1} Q_i$ and by [13, Corollary 1.3] we have

$$\text{depth}(S/I) \leq \text{depth}(S/(I : x_j^a)) = \text{depth}(S/(\cap_{i=1}^{s-1} Q_i)).$$

Now by using Depth Lemma (see [15, Lemma 1.3.9]) we have

$$\text{depth}(S/I) = \min\{\text{depth}(S/\cap_{i=1}^{s-1} Q_i), \text{depth}(S/Q_s), 1 + \text{depth}(S/\cap_{i=1}^{s-1} (Q_i + Q_s))\},$$

which is enough. $\square$

Proposition 1.2. *Let $I = \bigcap_{i=1}^s Q_i$ be the irredundant presentation of I as an intersection of primary monomial ideals. Let $P_i := \sqrt{Q_i}$. If $P_i \not\subset \sum_{1 \leq j \neq i}^{s-1} P_j$ for all $i \in [s]$. Then $\text{depth}(S/I) = s - 1$.*

Proof. It is enough to consider the case when $\sum_{j=1}^s P_j = \mathfrak{m}$. We use induction on s . If $s = 1$ the result is trivial. Suppose that $s > 1$. By Lemma 1.1 we get

$$\text{depth}(S/I) = \min\{\text{depth}(S/\cap_{i=1}^{s-1} Q_i), \text{depth}(S/Q_s), 1 + \text{depth}(S/\cap_{i=1}^{s-1} (Q_i + Q_s))\}.$$

Then by induction hypothesis we have

$$\text{depth}(S/\cap_{i=1}^{s-1} Q_i) = s - 2 + \dim(S/(\sum_{i=1}^{s-1} Q_i)) \geq s - 1.$$

We see that $\cap_{i=1}^{s-1} (Q_i + Q_s)$ satisfies also our assumption, the induction hypothesis gives $\text{depth}(S/\cap_{i=1}^{s-1} (Q_i + Q_s)) = s - 2$. Since $Q_i \not\subset Q_s$, $i < s$ by our assumption we get $\text{depth}(S/Q_s) > \text{depth}(S/(Q_i + Q_s))$ for all $i < s$. It follows $\text{depth}(S/Q_s) \geq 1 + \text{depth}(S/\cap_{i=1}^{s-1} (Q_i + Q_s))$ which is enough. $\square$

Corollary 1.3. *Let $I \subset S$ be a monomial ideal such that $\text{Ass}(S/I) = \{P_1, \dots, P_s\}$ where $P_i \not\subset \sum_{j=1, j \neq i}^s P_j$ for all $i \in [s]$. Then I has minimal depth.*

Proof. Clearly $\text{size}(I) = s - 1$ and by Proposition 1.2 we have $\text{depth}(I) = s$, thus we have $\text{depth}(I) = \text{size}(I) + 1$, i.e. I has minimal depth. $\square$

Lemma 1.4. *Let $I = \cap_{i=1}^s Q_i$ be the irredundant primary decomposition of I and $\sqrt{Q_i} \neq \mathfrak{m}$ for all i . Suppose that there exists $1 \leq r < s$ such that $\sqrt{Q_i + Q_j} = \mathfrak{m}$ for each $r < j \leq s$ and $1 \leq i \leq r$. Then $\text{depth}(I) = 2$.*

Proof. The proof follows by using Depth Lemma on the following exact sequence.

$$0 \longrightarrow S/I \longrightarrow S/\cap_{i=1}^r Q_i \oplus S/\cap_{j>r}^s Q_j \longrightarrow S/\cap_{i=1}^r \cap_{j>r}^s (Q_i + Q_j) \longrightarrow 0.$$

$\square$

Corollary 1.5. *Let $I \subset S$ be a monomial ideal. If bigsize of I is one then I has minimal depth.*

Proof. We know that $\text{size}(I) \leq \text{bigsize}(I)$. If $\text{size}(I) = 0$ the $\text{depth}(I) = 1$ and the result follows in this case. Now let us suppose that $\text{size}(I) = 1$. By Lemma 1.4 we have $\text{depth}(I) = 2$. Hence the result follows. $\square$

Let $d \geq 2$ be an integer and $\mathcal{M}_d$ the set of monomials of degree d of S . For two monomials $u, v \in \mathcal{M}_d$, with $u \geq_{\text{lex}} v$, we consider the lexsegment set

$$\mathcal{L}(u, v) = \{w \in \mathcal{M}_d \mid u \geq_{\text{lex}} w \geq_{\text{lex}} v\}.$$

Theorem 1.6. *Let $I = (\mathcal{L}(u, v)) \subset S$ be a lexsegment ideal. Then $\text{depth}(I) = \text{size}(I) + 1$, that is I has minimal depth.*

Proof. For the trivial cases $u = v$ the result is obvious. Suppose that $u = x_1^{a_1} \cdots x_n^{a_n}$, $v = x_1^{b_1} \cdots x_n^{b_n} \in S$. First assume that $b_1 = 0$. If there exist r such that $a_1 = \cdots = a_r = 0$ and $a_{r+1} \neq 0$, then I is a lexsegment ideal in $S' := K[x_{r+1}, \dots, x_n]$. We get $\text{depth}(IS) = \text{depth}(IS') + r$ and by definition of size we have $\text{size}(IS) = \text{size}(IS') + r$. This means that without loss of generality we can assume that $a_1 > 0$. If $x_n u / x_1 \geq_{\text{lex}} v$, then by [3, Proposition 3.2] $\text{depth}(I) = 1$ which implies that $\mathfrak{m} \in \text{Ass}(S/I)$, thus $\text{size}(I) = 0$ and the result

follows in this case. Now consider the complementary case $x_n u / x_1 <_{lex} v$, then u is of the form $u = x_1 x_l^{a_l} \cdots x_n^{a_n}$ where $l \geq 2$. Let $I = \cap_{i=1}^s Q_i$ be an irredundant primary decomposition of I , where Q_i 's are monomial primary ideals. If $l \geq 4$ and $v = x_2^d$ then by [3, Proposition 3.4] we have $\text{depth}(I) = l - 1$. After [6, Proposition 2.5(ii)] we know that

$$\sqrt{\sum_{i=1}^s Q_i} = (x_1, x_2, x_l, \dots, x_n) \notin \text{Ass}(S/I),$$

but $(x_1, x_2), (x_2, x_l, \dots, x_n) \in \text{Ass}(S/I)$. Therefore, $\text{size}(I) = l - 2$ and we have $\text{depth}(I) = \text{size}(I) + 1$, so we are done in this case. Now consider the case $v = x_2^{d-1} x_j$ for some $3 \leq j \leq n - 2$ and $l \geq j + 2$, then again by [3, Proposition 3.4] we have $\text{depth}(I) = l - j + 1$ and by [6, Proposition 2.5(ii)] we have

$$\sqrt{\sum_{i=1}^s Q_i} = (x_1, \dots, x_j, x_l, \dots, x_n) \notin \text{Ass}(S/I)$$

and $(x_1, \dots, x_j), (x_2, \dots, x_j, x_l, \dots, x_n) \in \text{Ass}(S/I)$. Therefore, $\text{size}(I) = l - j$ and again we have $\text{depth}(I) = \text{size}(I) + 1$. Now for all the remaining cases by [3, Proposition 3.4] we have $\text{depth}(I) = 2$, and by [6, Proposition 2.5(i)]

$$\sqrt{\sum_{i=1}^s Q_i} = (x_1, \dots, x_n) \notin \text{Ass}(S/I),$$

but $(x_1, \dots, x_j), (x_2, \dots, x_n) \in \text{Ass}(S/I)$, for some $j \geq 2$. Therefore $\text{size}(I) = 1$. Thus the equality $\text{depth}(I) = \text{size}(I) + 1$ follows in all cases when $b_1 = 0$.

Now let us consider that $b_1 > 0$, then $I = x_1^{b_1} I'$ where $I' = (I : x_1^{b_1})$. Clearly I' is a lexsegment ideal generated by the lexsegment set $\mathcal{L}(u', v')$ where $u' = u/x_1^{b_1}$ and $v' = v/x_1^{b_1}$. The ideals I, I' are isomorphic, therefore $\text{depth}(I') = \text{depth}(I)$. It is enough to show that $\text{size}(I') = \text{size}(I)$. We have the exact sequence

$$0 \rightarrow S/I' \xrightarrow{x_1^{b_1}} S/I \rightarrow S/(I, x_1^{b_1}) = S/(x_1^{b_1}) \rightarrow 0,$$

and therefore

$$\text{Ass}(S/I') \subset \text{Ass}(S/I) \subset \text{Ass}(S/I') \cup \{(x_1)\}.$$

As $\{(x_1)\} \in \text{Ass}(S/I)$ since it is a minimal prime over I , we get $\text{Ass}(S/I) = \text{Ass}(S/I') \cup \{(x_1)\}$. Let s' be the minimum number such that there exist $P_1, \dots, P_{s'} \in \text{Ass } S/I'$ such that $\sum_{i=1}^{s'} P_i = a := \sum_{P \in \text{Ass}(S/I')} P$. Then

$\text{size}(I') = s' + \dim(S/a) - 1$. Let s be the minimum number t such that there exist t prime ideals in $\text{Ass}(S/I)$ whose sum is (a, x_1) . By [6, Lemma 2.1] we have that atleast one prime ideal from $\text{Ass}(S/I')$ contains necessarily x_1 , we have $x_1 \in a$. It follows $s \leq s'$ because anyway $\sum_{i=1}^{s'} P_i = a = \sum_{P \in \text{Ass}(S/I)} P$. If we have $P'_1, \dots, P'_{s-1} \in \text{Ass}(S/I')$ such that $\sum_{i=1}^{s-1} P'_i + (x_1) = a$ then we have also $\sum_{i=1}^{s-1} P'_i + P_1 = a$ for some $P_1 \in \text{Ass}(S/I')$ which contains x_1 . Thus $s = s'$ and so $\text{size}(I) = \text{size}(I')$. $\square$

2 Stanley depth of cyclic modules defined by ideals of minimal depth

Using Corollaries 1.3, 1.5 and Theorems 1.6, 0.2 we get the following theorem.

Theorem 2.1. *Stanley's conjecture holds for I , if it satisfies one of the following statements:*

1. $P_i \not\subseteq \sum_{1=j \neq i}^s P_j$ for all $i \in [s]$,
2. the bigsize of I is one,
3. I is a lexsegment ideal.

Remark 2.2. Usually, if Stanley's conjecture holds for an ideal I then we may show that it holds for the module S/I too. There exist no general explanation for this fact. If I is a monomial ideal of bigsize one then Stanley's conjecture holds for S/I . Indeed, case $\text{depth}(S/I) = 0$ is trivial. Suppose $\text{depth}(S/I) \neq 0$, then by Lemma 1.4 $\text{depth}(S/I) = 1$, therefore by [2, Proposition 2.13] $\text{sdepth}(S/I) \geq 1$. If I is a lexsegment ideal then Stanley's conjecture holds for S/I [6]. Below we show this fact in the first case of the above theorem.

Theorem 2.3. *Let $I = \bigcap_{i=1}^s Q_i$ be the irredundant presentation of I as an intersection of primary monomial ideals. Let $P_i := \sqrt{Q_i}$. If $P_i \not\subseteq \sum_{1=i \neq j}^s P_j$ for all $i \in [s]$ then $\text{sdepth}(S/I) \geq \text{depth}(S/I)$, that is the Stanley's conjecture holds for S/I .*

Proof. Using [4, Lemma 3.6] it is enough to consider the case $\sum_{i=1}^s P_i = \mathfrak{m}$. By Proposition 1.2 we have $\text{depth}(S/I) = s - 1$. We show that $\text{sdepth}(S/I) \geq s - 1$. Apply induction on s , case $s = 1$ being clear. Fix $s > 1$ and apply induction on n . If $n \leq 5$ then the result follows by [10]. Let $A := \cup_{i=1}^s (G(P_i) \setminus \sum_{1=j \neq i}^s G(P_j))$. If $(A) = \mathfrak{m}$ then note that $G(P_i) \cap G(P_j) = \emptyset$ for all $i \neq j$. By [7, Theorem 2.1] and [8, Theorem 3.1] we have $\text{sdepth}(S/I) \geq s - 1$. Now suppose that $(A) \neq \mathfrak{m}$. By renumbering the primes and variables we can

assume that $x_n \notin A$. There exists a number r , $2 \leq r \leq s$ such that $x_n \in G(P_j)$, $1 \leq j \leq r$ and $x_n \notin G(P_j)$, $r+1 \leq j \leq s$. Let $S' := K[x_1, \dots, x_{n-1}]$. First assume that $r < s$. Let $Q'_j = Q_j \cap S'$, $P'_j = P_j \cap S'$ and $J = \bigcap_{i=r+1}^s Q'_i \subset S'$, $L = \bigcap_{i=1}^r Q'_i \subset S'$. We have $(I, x_n) = ((J \cap L), x_n)$ because $(Q_j, x_n) = (Q'_j, x_n)$ using the structure of monomial primary ideals given in [15]. In the exact sequence

$$0 \longrightarrow S/(I : x_n) \longrightarrow S/I \longrightarrow S/(I, x_n) \longrightarrow 0,$$

the sdepth of the right end is $\geq s-1$ by induction hypothesis on n for $J \cap L \subset S'$ (note that we have $P'_i \not\subset \sum_{1=i \neq j}^s P'_j$ for all $i \in [s]$ since $x_n \notin A$). Let e_I be the maximum degree in x_n of a monomial from $G(I)$. Apply induction on e_I . If $e_I = 1$ then $(I : x_n) = JS$ and the sdepth of the left end in the above exact sequence is equal with $\text{sdepth}(S/JS) \geq (s-r-1)+r = s-1$ since there are at least r variables which do not divide the minimal monomial generators of ideal $(I : x_n)$ and we may apply induction hypothesis on s for J . By [13, Theorem 3.1] we have $\text{sdepth}(S/I) \geq \min\{\text{sdepth}(S/(I : x_n)), \text{sdepth}(S/(I, x_n))\} \geq s-1$. If $e_I > 1$ then note that $e_{(I:x_n)} < e_I$ and by induction hypothesis on e_I or s we get $\text{sdepth}(S/(I : x_n)) \geq s-1$. As above we obtain by [13, Theorem 3.1] $\text{sdepth}(S/I) \geq s-1$.

Now let $r = s$. If $e_I = 1$ then $I = (L, x_n)$ and by induction on n we have $\text{sdepth}(S/I) = \text{sdepth}(S'/L) \geq s-1$. If $e_I > 1$ then by induction hypothesis on e_I and s we get $\text{sdepth}(S/(I : x_n)) \geq s-1$. As above we are done using [13, Theorem 3.1]. $\square$

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