A viscosity projection method for class $\mathcal{T}$ mappings

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Abstract

In this paper, we firstly introduce a viscosity projection method for the class $\mathcal{T}$ mappings

$$x_{n+1} = \alpha_n P_H(x_n, S_n x_n) f(x_n) + (1 - \alpha_n) S_n x_n,$$

where $S_n = (1 - w) I + w T_n$, $w \in (0, 1), T_n \in \mathcal{T}$ and prove strong convergence theorems of the proposed method. It is verified that the viscosity projection method converges locally faster than the viscosity method. Furthermore, we present a viscosity projection method for a quasi-nonexpansive and nonexpansive mappings in Hilbert spaces. A numerical test provided in the paper shows that the viscosity projection method converges faster than the viscosity method.

1 Introduction and preliminaries

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Recall that a mapping $T : H \to H$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$. The set of fixed points of $T$ is $\text{Fix}(T) := \{ x \in H : Tx = x \}$. A mapping $T : H \to H$ is said to be quasi-nonexpansive if $\text{Fix}(T)$ is nonempty and $\|Tx - p\| \leq \|x - p\|$ for all $x \in H$ and $p \in \text{Fix}(T)$. A mapping $f : H \to H$ is said to be a contraction with constant $\rho \in [0, 1)$ if

$$\|f(x) - f(y)\| \leq \rho \|x - y\| \quad \forall x, y \in H.$$
Given \( x, y \in H \), let
\[
H(x, y) := \{ z \in H : \langle z - y, x - y \rangle \leq 0 \},
\]
be the half-space generated by \((x, y)\). The boundary \( \partial H \) of \( H \) is
\[
\partial H(x, y) = \{ z \in H : \langle z - y, x - y \rangle = 0 \}.
\]
It is clear that \( \partial H(x, y) \) is a closed and convex subset of \( H \).

**Remark 1.1.** The class \( \mathcal{T} \) is fundamental because it contains several types of operators commonly found in various areas of applied mathematics and in particular in approximation and optimization theory (see [1, 2] for details).

Let \( C \) be a nonempty closed convex subset of a Hilbert space \( H \). For a mapping \( T : C \to C \), Moudafi [10] and many other researchers (e.g., [7, 8, 11, 12, 13, 14]) studied the viscosity approximation method as follow: for given \( x_0 \in C \), the sequence \( \{x_n\} \) is generated by
\[
x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n,
\]
where \( \alpha_n \subset (0, 1) \) and \( f : C \to C \) is a contraction. It was proved in [10] (also see Xu [13]) that the sequence \( \{x_n\} \) generated by (1) converges strongly to the unique solution of the variational inequality problem \( VI(I - f, \text{Fix}(T)) : \) find \( x^* \) in \( \text{Fix}(T) \) such that
\[
\forall v \in \text{Fix}(T), \quad \langle (I - f)x^*, v - x^* \rangle \geq 0.
\]

A special case of (1) was considered by Halpern [5] who introduced following iterative process:
\[
x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n,
\]
where \( u, x_0 \in C \) are arbitrary (but fixed) and \( \{\alpha_n\} \subset (0, 1) \).

Recently, Maingé [9] studied following algorithm for a quasi-nonexpansive mapping \( T \):
\[
x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T_wx_n,
\]
where \( \alpha_n \subset (0, 1) \), \( T_w = (1 - w)I + wT \), \( w \in (0, 1) \). He proposed a new analysis of the viscosity approximation method to prove the convergence of the algorithm (2).

Inspired by Maingé [9] and others (e.g., [1, 2, 3, 6]), in this paper we firstly discuss the following viscosity projection method for a sequence of class \( \mathcal{T} \) mappings \( T_n : H \to H \) as follow:
\[
x_{n+1} = \alpha_n P_{H(x_n, S_nx_n)}f(x_n) + (1 - \alpha_n)S_nx_n,
\]
where \{\alpha_n\} \subset (0,1), S_n = (1 - w)I + wT_n, w \in (0,1), I is the identity mapping on H and \(P_K\) denotes the metric projection from H onto a closed convex subset \(K\) of H (see below Lemma 1.3 for the definition). We prove that the sequence \(\{x_n\}\) generated by (3) converges strongly to the unique solution of the variational inequality problem \(VI(I - f, \bigcap_{n=1}^{\infty} \text{Fix}(T_n)) : \text{find } x^* \text{ in } \bigcap_{n=1}^{\infty} \text{Fix}(T_n) \text{ such that} \)

\[ \forall v \in \bigcap_{n=1}^{\infty} \text{Fix}(T_n), \quad \langle (I - f)x^*, v - x^* \rangle \geq 0. \]  

(4)

We will use the following notations:
1. \(\rightharpoonup\) for weak convergence and \(\rightarrow\) for strong convergence.
2. \(\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}\) denotes the weak \(\omega\)-limit of \(\{x_n\}\).

We need some facts and tools in a real Hilbert space \(H\) which are listed below.

**Definition 1.1.** Suppose that \(\{x_n\}_{n=1}^{\infty}\) and \(\{y_n\}_{n=1}^{\infty}\) are two iterations which converge to a point \(p\). Then \(\{x_n\}_{n=1}^{\infty}\) is said to converge locally faster than \(\{y_n\}_{n=1}^{\infty}\) if \(x_n = y_n\) implies

\[ \|x_{n+1} - p\| \leq \|y_{n+1} - p\| \]

for any \(n \in \mathbb{N}\).

**Lemma 1.1.** Let \(H\) be a Hilbert space and \(I\) be the identity operator of \(H\).

(i) If \(\text{dom } T = H\), then \(2T - I\) is quasi-nonexpansive if and only if \(T \in \mathfrak{T}\),

(ii) If \(T \in \mathfrak{T}\), then \(\lambda I + (1 - \lambda)T \in \mathfrak{T}, \forall \lambda \in [0,1]\).

(iii) If \(T \in \mathfrak{T}\), then \(T\) is quasi-nonexpansive.

(iv) If \(T \in \mathfrak{T}\), then \(\|x - Tx\|^2 \leq \langle x - Tx, x - u \rangle \) for all \(x \in H\) and \(u \in \text{Fix}(T)\).

(v) If \(T \in \mathfrak{T}\) and \(S = wI + (1 - w)T, w \in (0,1),\) then \(H(x, Tx) \subset H(x, Sx), \forall x \in H\).

**Proof.** The proof of (i)-(iv) can be found in [1]. Here we just prove (v). For any \(y \in H(x, Tx)\), we have

\[ \langle y - Tx, x - Tx \rangle \leq 0. \]

So, we get

\[ \langle y - Sx, x - Sx \rangle = (1 - w)\langle y - Tx, x - Tx \rangle - (1 - w)w\|x - Tx\|^2 \leq 0, \]

which implies \(y \in H(x, Sx)\).
Remark 1.2. Let $T \in \mathcal{S}$ with $\text{Fix}(T) \neq \emptyset$ and set $T_w := (1-w)I + wT$ for $w \in (0,1)$. Then the following statements are reached:

(a1) $\text{Fix}(T) = \text{Fix}(T_w)$ if $w \neq 0$;
(a2) $\text{Fix}(T)$ is a closed convex subset of $H$.
(a3) $\langle x - T_w x, x - q \rangle \geq w \|x - Tx\|^2$ for all $x \in H$, $q \in \text{Fix}(T)$.

From Lemma 1.1 (i) and (ii), it is an easy matter to show (a1)-(a3) by using Remarks 1.2 and 2.1 in [9].

Definition 1.2. A sequence of mappings $\{T_n\}$ having common fixed points is said to satisfy the condition (Z) if every bounded sequence $\{x_n\}$ with $\|x_n - T_n x_n\| \to 0$ satisfies $\omega_w(x_n) \subset \bigcap_{n=1}^{\infty} \text{Fix}(T_n)$.

Definition 1.3. A mapping $T$ is called demiclosed at $y \in H$ if $Tx = y$ whenever $\{x_n\} \subset H$, $x_n \rightharpoonup x$ and $Tx_n \to y$.

Next Lemma shows that nonexpansive mappings are demiclosed at 0.

Lemma 1.2. [4] Let $C$ be a closed convex subset of a real Hilbert space $H$ and let $T : C \to C$ be a nonexpansive mapping such that $\text{Fix}(T) \neq \emptyset$. If a sequence $\{x_n\}$ in $C$ is such that $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$, then $z = Tz$.

Lemma 1.3. [4] Let $K$ be a closed convex subset of real Hilbert space $H$ and let $P_K$ be the (metric or nearest point) projection from $H$ onto $K$ (i.e., for $x \in H$, $P_K x$ is the only point in $K$ such that $\|x - P_K x\| = \inf \{\|x - z\| : z \in K\}$). Given $x \in H$ and $z \in K$. Then $z = P_K x$ if and only if there holds the relation:

$\langle x - z, y - z \rangle \leq 0$, for all $y \in K$.

Lemma 1.4. [6] Let $C = \{z \in H : \langle x - u, z - u \rangle \leq 0\}$. Assume $x \neq u$ and $x_0 \notin C$. Then

$P_C x_0 = x_0 - \frac{\langle x - u, x_0 - u \rangle}{\|x - u\|^2}(x - u)$.

Lemma 1.5. Let $F := I - P_{H(x,Tx)}f$, where $x \in H$ and $f$ is the contraction with constant $\rho$. Then the operator $F$ is $(1 - \rho)$-strongly monotone, i.e.,

$\langle Fy - Fz, y - z \rangle \geq (1 - \rho)\|y - z\|^2$ for all $x, y \in H$.

Proof. Note that $P_{H(x,Tx)}$ is a metric projection, so it is firmly nonexpansive and thus is nonexpansive. It is easy to see that, for all $y, z \in H$,

$\|P_{H(x,Tx)} f(y) - P_{H(x,Tx)} f(z)\| \leq \|f(y) - f(z)\| \leq \rho \|y - z\|$.
From (6), we have

\[(Fy - Fz, y - z) = \|y - z\|^2 - \langle P_{H(x, T0)}f(y) - P_{H(x, T0)}f(z), y - z \rangle\]
\[\geq \|y - z\|^2 - \|P_{H(x, T0)}f(y) - P_{H(x, T0)}f(z)\| \|y - z\|\]
\[\geq (1 - \rho)\|y - z\|^2.\]

**Lemma 1.6.** ([9] (Lemma 2.1)). Let \(\{\Gamma_n\}\) be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence \(\{\Gamma_{n_j}\}_{j \geq 0}\) of \(\{\Gamma_n\}\) which satisfies \(\Gamma_{n_j} < \Gamma_{n_j+1}\) for all \(j \geq 0\). Also consider the sequence of integers \(\{\tau(n)\}_{n \geq n_0}\) defined by

\[\tau(n) = \max\{k \leq n | \Gamma_k < \Gamma_{k+1}\}.\]

Then \(\{\tau(n)\}_{n \geq n_0}\) is a nondecreasing sequence verifying \(\lim_{n \to \infty} \tau(n) = \infty\) and, for all \(n \geq n_0\), it holds that \(\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}\) and we have

\[\Gamma_n \leq \Gamma_{\tau(n)+1}.\]

2 Main results

**Lemma 2.1.** Let \(T_n \in \mathcal{T}\) with \(\mathcal{T} := \bigcap_{n=1}^{\infty} \text{Fix}(T_n) \neq \emptyset\), \(\{\alpha_n\} \subset (0,1)\) and \(w \in (0,1)\). Let \(f\) be a contraction with constant \(\rho\). The sequence \(\{x_n\}\) generated by (3) is bounded.

**Proof.** By \(T_n \in \mathcal{T}\) and Lemma 1.1 (v), \(\text{Fix}(T_n) \subset H(x, S_n x)\), for all \(x \in H\), therefore, we have \(P_{H(x, S_n x)} p = p\), for all \(p \in \mathcal{T}\). So, using Lemma 1.1 (ii)-(iii) and (6), we have

\[\|x_{n+1} - p\| = \|\alpha_n P_{H(x_n, S_n x_n)} f(x_n) + (1 - \alpha_n) S_n x_n - p\|
\leq \alpha_n \|P_{H(x_n, S_n x_n)} f(x_n) - p\| + (1 - \alpha_n) \|S_n x_n - p\|
\leq \alpha_n \|P_{H(x_n, S_n x_n)} f(x_n) - P_{H(x_n, S_n x_n)} f(p)\|
+ \alpha_n \|P_{H(x_n, S_n x_n)} f(p) - P_{H(x_n, S_n x_n)} p\| + (1 - \alpha_n) \|x_n - p\|
\leq \alpha_n \|f(p) - p\| + (1 - \alpha_n (1 - \rho)) \|x_n - p\|
= \alpha_n (1 - \rho) \frac{\|f(p) - p\|}{1 - \rho} + (1 - \alpha_n (1 - \rho)) \|x_n - p\|.
\]

Thus, by induction on \(n\),

\[\|x_n - p\| \leq \max \left\{ \frac{\|f(p) - p\|}{1 - \rho}, \|x_0 - p\| \right\},\]

for every \(n \in \mathbb{N}\). This shows that \(\{x_n\}\) is bounded, and hence, \(\{P_{H(x_n, S_n x_n)} f(x_n)\}\) is also bounded.
Lemma 2.2. Assume a sequence of mappings \( T_n \in \mathcal{F} : H \to H \) satisfies the condition (Z). If \( x^* \) is the solution of (4) and \( \{x_n\} \) is a bounded sequence such that \( \|T_n x_n - x_n\| \to 0 \), then

\[
\liminf_{n \to \infty} \langle (I - P_{H(x_n, T_n x_n)} f)x^*, x_n - x^* \rangle \geq 0. \tag{7}
\]

Proof. Since the sequence \( \{T_n\} \) satisfies the condition (Z) and \( \{x_n\} \) is a bounded sequence, \( \omega_n(x_n) \subset \mathcal{F} \). It is also a simple matter to see that there exists \( \bar{x} \) and a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( x_{n_k} \to \bar{x} \) as \( k \to \infty \) (hence \( \bar{x} \in \mathcal{F} \)) and such that

\[
\liminf_{n \to \infty} \langle (I - f)x^*, x_n - x^* \rangle = \lim_{k \to \infty} \langle (I - f)x^*, x_{n_k} - x^* \rangle,
\]

which by (4) obviously leads to

\[
\liminf_{n \to \infty} \langle (I - f)x^*, x_n - x^* \rangle = \langle (I - f)x^*, \bar{x} - x^* \rangle \geq 0.
\]

So,

\[
\liminf_{n \to \infty} \langle (I - f)x^*, x_n - x^* \rangle \geq 0. \tag{8}
\]

If \( f(x^*) \in H(x_n, T_n x_n) \), then \( P_{H(x_n, T_n x_n)} f(x^*) = f(x^*) \) and (8) implies (7). Otherwise, assume \( f(x^*) \notin H(x_n, T_n x_n) \). Then, by definition of \( H(x_n, T_n x_n) \), we have

\[
\langle x_n - T_n x_n, f(x^*) - T_n x_n \rangle > 0. \tag{9}
\]

By \( x^* \in \mathcal{F} \subset H(x_n, T_n x_n) \), we get

\[
\langle x_n - T_n x_n, x_n - x^* \rangle = \|x_n - T_n x_n\|^2 + \langle x_n - T_n x_n, T_n x_n - x^* \rangle > 0. \tag{10}
\]

From (5), it follows

\[
P_{H(x_n, T_n x_n)} f(x^*) = f(x^*) - \frac{\langle x_n - T_n x_n, f(x^*) - T_n x_n \rangle}{\|x_n - T_n x_n\|^2} (x_n - T_n x_n). \tag{11}
\]

Combining (9), (10) and (11), we obtain

\[
\langle (I - P_{H(x_n, T_n x_n)} f)x^*, x_n - x^* \rangle = \langle (I - f)x^*, x_n - x^* \rangle + \frac{\langle x_n - T_n x_n, f(x^*) - T_n x_n \rangle}{\|x_n - T_n x_n\|^2} \langle x_n - T_n x_n, x_n - x^* \rangle
\]

\[
> \langle (I - f)x^*, x_n - x^* \rangle,
\]

which together with (8) implies

\[
\liminf_{n \to \infty} \langle (I - P_{H(x_n, T_n x_n)} f)x^*, x_n - x^* \rangle \geq \liminf_{n \to \infty} \langle (I - f)x^*, x_n - x^* \rangle \geq 0.
\]

Therefore, we obtain the desired result.
Theorem 2.1. Suppose that a sequence \( \{T_n\} \subset \mathcal{T} \) satisfies
\[ \mathcal{T} := \bigcap_{n=1}^{\infty} \text{Fix}(T_n) \neq \emptyset \] and the condition (Z). Let \( f \) be a contraction with constant \( \rho \in [0, 1) \). Assume \( \omega \in (0, 1) \), and \( \{\alpha_n\} \subset (0, 1) \) such that \( \lim_{n \to \infty} \alpha_n = 0 \), \( \sum_{n=1}^{\infty} \alpha_n = \infty \). Then, \( \{x_n\} \) generated by (3) converges strongly to \( x^* \in \mathcal{T} \) verifying
\[ x^* = (P_\mathcal{T} \circ f)x^*, \]
which equivalently solves the following variational inequality problem:
\[ x^* \in \mathcal{T}, \quad \forall v \in \mathcal{T}, \quad \langle (I-f)x^*, v-x^* \rangle \geq 0. \] (13)

Proof. Let \( x^* \) be the solution of (13). From (3) we obviously have
\[ x_{n+1} - x_n + \alpha_n(x_n - PH(x_n,S_n) f(x_n)) = (1 - \alpha_n)(S_n x_n - x_n), \] (14)
hence
\[ \langle x_{n+1} - x_n + \alpha_n(x_n - PH(x_n,S_n) f(x_n)), x_n - x^* \rangle = -(1 - \alpha_n)\langle x_n - S_n x_n, x_n - x^* \rangle. \] (15)
Moreover, by \( x^* \in \mathcal{T} \), and using Remark 1.2 (a3), we have
\[ \langle x_n - S_n x_n, x_n - x^* \rangle \geq \omega \|x_n - T_n x_n\|^2, \]
which together with (15) entails
\[ \langle x_{n+1} - x_n + \alpha_n(x_n - PH(x_n,S_n) f(x_n)), x_n - x^* \rangle \leq -\omega(1 - \alpha_n)\|x_n - T_n x_n\|^2, \]
or equivalently
\[ -\langle x_n - x_{n+1}, x_n - x^* \rangle \leq -\alpha_n\langle x_n - PH(x_n,S_n) f(x_n), x_n - x^* \rangle - \omega(1 - \alpha_n)\|x_n - T_n x_n\|^2. \] (16)
Setting \( \Gamma_n := \frac{1}{2}\|x_n - x^*\|^2 \), we have
\[ \langle x_n - x_{n+1}, x_n - x^* \rangle = -\Gamma_{n+1} + \Gamma_n + \frac{1}{2}\|x_n - x_{n+1}\|^2. \]
So that (16) can be equivalently rewritten as
\[ \Gamma_{n+1} - \Gamma_n - \frac{1}{2}\|x_n - x_{n+1}\|^2 \leq -\alpha_n\langle x_n - PH(x_n,S_n) f(x_n), x_n - x^* \rangle - \omega(1 - \alpha_n)\|x_n - T_n x_n\|^2. \] (17)
Now using (14) again, we have
\[ \|x_{n+1} - x_n\|^2 = \|\alpha_n(Ph(x_n,S_n) f(x_n) - x_n) + (1 - \alpha_n)(S_n x_n - x_n)\|^2. \]
Hence it is a classical matter to see that

\[ ||x_{n+1} - x_n||^2 \leq 2\alpha_n^2 ||P_{H(x_n,S_n)}f(x_n) - x_n||^2 + 2(1 - \alpha_n)^2 ||S_nx_n - x_n||^2, \]

which by \( ||S_nx_n - x_n|| = w||T_nx_n - x_n|| \) and \( (1 - \alpha_n)^2 \leq (1 - \alpha_n) \) yields

\[ \frac{1}{2}||x_{n+1} - x_n||^2 \leq \alpha_n^2 ||P_{H(x_n,S_n)}f(x_n) - x_n||^2 + w^2(1 - \alpha_n) ||T_nx_n - x_n||^2. \] (18)

Then from (17) and (18) we obtain

\[ \Gamma_{n+1} - \Gamma_n + (1 - w)w(1 - \alpha_n)||x_n - T_nx_n||^2 \leq \alpha_n(\alpha_n||P_{H(x_n,S_n)}f(x_n) - x_n||^2 - (x_n - P_{H(x_n,S_n)}f(x_n)), x_n - x^*) \]. (19)

The rest of the proof will be divided into two parts:

Case 1. Suppose that there exists \( n_0 \) such that \( \{\Gamma_n\}_{n \geq n_0} \) is nonincreasing. In this situation, \( \{\Gamma_n\} \) is then convergent because it is also nonnegative (hence it is bounded from below), so that \( \lim_{n \to \infty}(\Gamma_{n+1} - \Gamma_n) = 0 \), hence, in light of (19) together with \( \alpha_n \to 0 \), and the boundedness of \( \{x_n\} \) (hence, thanks Lemma 2.1, \( \{P_{H(x_n,S_n)}f(x_n)\} \) is also bounded), we obtain

\[ \lim_{n \to \infty} ||x_n - T_nx_n|| = 0, \]

which together with \( S_n = (1 - w)I + wT_n \), \( w \in (0,1) \), implies

\[ \lim_{n \to \infty} ||x_n - S_nx_n|| = 0. \] (20)

From (19) again, we have

\[ \alpha_n(-\alpha_n||P_{H(x_n,S_n)}f(x_n) - x_n||^2 + (x_n - P_{H(x_n,S_n)}f(x_n)), x_n - x^*) \leq \Gamma_n - \Gamma_{n+1}. \]

Then, by \( \sum_n \alpha_n = \infty \), we obviously deduce that

\[ \liminf_{n \to \infty}(-\alpha_n||P_{H(x_n,S_n)}f(x_n) - x_n||^2 + (x_n - P_{H(x_n,S_n)}f(x_n)), x_n - x^*) \leq 0, \]

or equivalently (as \( \alpha_n||P_{H(x_n,S_n)}f(x_n)||^2 \to 0 \))

\[ \liminf_{n \to \infty}(x_n - P_{H(x_n,S_n)}f(x_n)), x_n - x^* \leq 0. \] (21)

Moreover, by Lemma 1.5, we have

\[ 2(1 - \rho)\Gamma_n + (x^* - P_{H(x_n,S_n)}f(x^*)), x_n - x^* \leq (x_n - P_{H(x_n,S_n)}f(x_n)), x_n - x^*, \] (22)
which by (21) entails
\[
\liminf_{n \to \infty} (2(1 - \rho) \Gamma_n + \langle x^* - P_H(x_n, S_n x_n) f(x^*), x_n - x^* \rangle) \leq 0.
\]

Hence, recalling that \(\lim_{n \to \infty} \Gamma_n\) exists, we equivalently obtain
\[
2(1 - \rho) \lim_{n \to \infty} \Gamma_n + \liminf_{n \to \infty} \langle x^* - P_H(x_n, S_n x_n) f(x^*), x_n - x^* \rangle \leq 0,
\]

namely,
\[
2(1 - \rho) \lim_{n \to \infty} \Gamma_n \leq - \liminf_{n \to \infty} \langle x^* - P_H(x_n, S_n x_n) f(x^*), x_n - x^* \rangle. \tag{23}
\]

From (20) and invoking Lemma 2.2, we have
\[
\liminf_{n \to \infty} \langle x^* - P_H(x_n, S_n x_n) f(x^*), x_n - x^* \rangle \geq 0,
\]

which by (23) yields \(\lim_{n \to \infty} \Gamma_n = 0\), so that \(\{x_n\}\) converges strongly to \(x^*\).

Case 2. Suppose there exists a subsequence \(\{\Gamma_{n_k}\}_{k \geq 0}\) of \(\{\Gamma_n\}_{n \geq 0}\) such that \(\Gamma_{n_k} < \Gamma_{n_k+1}\) for all \(k \geq 0\). In this situation, we consider the sequence of indices \(\{\tau(n)\}\) as defined in Lemma 1.6. It follows that \(\Gamma_{\tau(n)+1} - \Gamma_{\tau(n)} > 0\), which by (19) amounts to
\[
(1 - w)w(1 - \alpha_{\tau(n)}) \|x_{\tau(n)} - T_{\tau(n)} x_{\tau(n)}\|^2 \\
\leq \alpha_{\tau(n)} \left( (1 - \alpha_{\tau(n)}) \|P_H(x_{\tau(n)}, S_{\tau(n)} x_{\tau(n)}) f(x_{\tau(n)}) - x_{\tau(n)}\|^2 \\
\quad - \langle x_{\tau(n)} - P_H(x_{\tau(n)}, S_{\tau(n)} x_{\tau(n)}) f(x_{\tau(n)}), x_{\tau(n)} - x^* \rangle \right).
\tag{24}
\]

Hence, by the boundedness of \(\{x_n\}\) and \(\{P_H(x_n, S_n x_n) f(x_n)\}\), and \(\alpha_n \to 0\), we immediately obtain
\[
\lim_{n \to \infty} \|x_{\tau(n)} - T_{\tau(n)} x_{\tau(n)}\| = 0, \tag{25}
\]

which together with \(S_{\tau(n)} = (1 - w)I + w T_{\tau(n)}, w \in (0, 1)\), implies
\[
\lim_{n \to \infty} \|x_{\tau(n)} - S_{\tau(n)} x_{\tau(n)}\| = 0. \tag{26}
\]

Using (3), we have
\[
\|x_{\tau(n)+1} - x_{\tau(n)}\| \leq \alpha_{\tau(n)} \|P_H(x_{\tau(n)}, S_{\tau(n)} x_{\tau(n)}) f(x_{\tau(n)}) - x_{\tau(n)}\| \\
\quad + (1 - \alpha_{\tau(n)}) \|x_{\tau(n)} - S_{\tau(n)} x_{\tau(n)}\|,
\]

which together with (26) and \(\alpha_n \to 0\) yields
\[
\lim_{n \to \infty} \|x_{\tau(n)+1} - x_{\tau(n)}\| = 0. \tag{27}
\]
Now by (24), we clearly have
\[
\langle x_{\tau(n)} - P_{H(x_{\tau(n)}, S_{\tau(n)}x_{\tau(n)})}(x_{\tau(n)}), x_{\tau(n)} - x^* \rangle \\
\leq \alpha_{\tau(n)}\|P_{H(x_{\tau(n)}, S_{\tau(n)}x_{\tau(n)})}(x_{\tau(n)}) - x_{\tau(n)}\|^2,
\]
which in the light of (22) yields
\[
2(1 - \rho)\Gamma_{\tau(n)} + \langle x^* - P_{H(x_{\tau(n)}, S_{\tau(n)}x_{\tau(n)})}(x_{\tau(n)}), x_{\tau(n)} - x^* \rangle \\
\leq \alpha_{\tau(n)}\|P_{H(x_{\tau(n)}, S_{\tau(n)}x_{\tau(n)})}(x_{\tau(n)}) - x_{\tau(n)}\|^2.
\]
Hence (as \(\alpha_{\tau(n)}\|P_{H(x_{\tau(n)}, S_{\tau(n)}x_{\tau(n)})}(x_{\tau(n)})) \to 0\) it follows that
\[
2(1 - \rho) \limsup_{n \to \infty} \Gamma_{\tau(n)} \leq - \liminf_{n \to \infty} \langle x^* - P_{H(x_{\tau(n)}, S_{\tau(n)}x_{\tau(n)})}(x_{\tau(n)}), x_{\tau(n)} - x^* \rangle.
\]
From (26) and invoking Lemma 2.2, we have
\[
\liminf_{n \to \infty} \langle x^* - P_{H(x_{\tau(n)}, S_{\tau(n)}x_{\tau(n)})}(x_{\tau(n)}), x_{\tau(n)} - x^* \rangle \geq 0,
\]
which by (28) yields \(\limsup_{n \to \infty} \Gamma_{\tau(n)} = 0\), so that \(\lim_{n \to \infty} \Gamma_{\tau(n)} = 0\). Applying (27), we have \(\lim_{n \to \infty} \Gamma_{\tau(n)+1} = 0\). Then, recalling that \(\Gamma_n \leq \Gamma_{\tau(n)+1}\) (by Lemma 1.6), we get \(\lim_{n \to \infty} \Gamma_n = 0\), so that \(x_n \to x^*\) strongly.

Remark 2.1. Assume that \(f(x_n) \notin H(x_n, S_n x_n)\). From Lemma 1.4, we have
\[
P_{H(x_n, S_n x_n)}(x_n) = f(x_n) - \langle x_n - S_n x_n, f(x_n) - S_n x_n \rangle \|x_n - S_n x_n\|^2 (x_n - S_n x_n).
\]
So, the algorithm (3) can be rewritten as the form:
\[
x_{n+1} = \begin{cases} 
\alpha_n f(x_n) + (1 - \alpha_n)S_n x_n, & \text{if } f(x_n) \in H(x_n, S_n x_n) \\
\alpha_n P_{H(x_n, S_n x_n)}(x_n) + (1 - \alpha_n)S_n x_n, & \text{if } f(x_n) \notin H(x_n, S_n x_n) 
\end{cases}
\]
where \(P_{H(x_n, S_n x_n)}(x_n)\) is given by (29). From (30), we know the algorithm (3) can be easily realized although there is a metric projection.

From (2), the classical viscosity method for class \(\mathcal{F}\) mappings \(\{T_n\}\) is
\[
y_{n+1} = \alpha_n f(y_n) + (1 - \alpha_n)S_n y_n,
\]
where \(S_n = (1 - w)I + wT_n\).

Next, we will compare the convergence rate of the viscosity projection method with the viscosity method.
**Theorem 2.2.** Suppose that a sequence \( \{T_n\} \subset \mathcal{F} \) satisfies \( \mathcal{F} := \bigcap_{n=1}^{\infty} \text{Fix}(T_n) \neq \emptyset \). Take the same parameters \( \{\alpha_n\} \) and \( w \) in (3) and (31). Let \( y_n = x_n \) and \( p \in \mathcal{F} \). Then it holds

\[
\|x_{n+1} - p\| \leq \|y_{n+1} - p\|. \tag{32}
\]

**Proof.** From \( T_n \in \mathcal{F} \) and Lemma 1.1 (v), it follows \( \mathcal{F} \in H(x_n, S_n x_n) \). If \( f(x_n) \in H(x_n, S_n x_n) \) and then \( P_{H(x_n, S_n x_n)} f(x_n) = f(x_n) \), then, it is obvious that \( y_{n+1} = x_{n+1} \) and (32) follows.

Next, assume \( f(x_n) \notin H(x_n, S_n x_n) \), then it is easy to verify \( P_{H(x_n, S_n x_n)} f(x_n) \in \partial H(x_n, S_n x_n) \). Actually, from (29), it follows

\[
\langle f(x_n) - S_n x_n, f(x_n) - S_n x_n \rangle - \langle x_n - S_n x_n, f(x_n) - S_n x_n \rangle \frac{\langle x_n - S_n x_n, x_n - S_n x_n \rangle}{\|x_n - S_n x_n\|^2} = 0,
\]

which yields

\[
\langle P_{H(x_n, S_n x_n)} f(x_n) - f(x_n), S_n x_n - P_{H(x_n, S_n x_n)} f(x_n) \rangle
= \frac{\langle x_n - S_n x_n, f(x_n) - S_n x_n \rangle}{\|x_n - S_n x_n\|^2} (x_n - S_n x_n, P_{H(x_n, S_n x_n)} f(x_n) - S_n x_n) \tag{33}
\]

\[
= 0.
\]

On the other hand, since \( p \in \mathcal{F} \subset H(x_n, S_n x_n) \), using Lemma 1.3, we get

\[
\langle P_{H(x_n, S_n x_n)} f(x_n) - f(x_n), P_{H(x_n, S_n x_n)} f(x_n) - p \rangle \leq 0. \tag{34}
\]

Applying (33), (34) and \( x_n = y_n \), we obtain

\[
\|x_{n+1} - p\|^2 = \|\alpha_n P_{H(x_n, S_n x_n)} f(x_n) + (1 - \alpha_n) S_n x_n - p\|^2
\]

\[
= \|\alpha_n P_{H(x_n, S_n x_n)} f(x_n) - f(y_n) + (y_{n+1} - p)\|^2
\]

\[
\leq \|y_{n+1} - p\|^2 + 2\alpha_n \langle P_{H(x_n, S_n x_n)} f(x_n) - f(x_n), x_{n+1} - p \rangle
\]

\[
= \|y_{n+1} - p\|^2 + 2\alpha_n \langle P_{H(x_n, S_n x_n)} f(x_n) - f(x_n), P_{H(x_n, S_n x_n)} f(x_n) - p \rangle
\]

\[
+ 2\alpha_n (1 - \alpha_n) \langle P_{H(x_n, S_n x_n)} f(x_n) - f(x_n), S_n x_n - P_{H(x_n, S_n x_n)} f(x_n) \rangle
\]

\[
\leq \|y_{n+1} - p\|^2,
\]

which implies \( \|x_{n+1} - p\| \leq \|y_{n+1} - p\|. \)
Remark 2.2. From the Definition 1.1 and Theorem 2.2, it follows that the viscosity projection method converges locally faster than viscosity method.

Remark 2.3. In [3], Dong et al proved the strong convergence theorem of the shrinking projection methods under the assumption that a sequence of class $T$ mappings $\{T_n\}$ is coherent (see definition 1.1 in [3]). In Theorem 2.1, the condition (Z) is needed for a sequence of class $T$ mappings $\{T_n\}$. Comparing the definition of coherent and condition (Z), it is obvious that a sequence $\{T_n\}$ satisfies condition (Z) if it is coherent. So, in order to obtain strong convergence results, in this paper we just need a weaker condition than that in [3].

3 Deduced results

In this section, using Theorem 2.1, we obtain some strong convergence results for a class $T$ mapping, a quasi-nonexpansive mapping and a nonexpansive mapping in a Hilbert space.

Theorem 3.1. Assume $T \in T$ with $\text{Fix}(T) \neq \emptyset$ satisfies that $I - T$ is demiclosed at 0. Let $f$ be a contraction with constant $\rho \in [0, 1)$. Define a sequence $\{x_n\}$ as follow:

$$x_{n+1} = \alpha_n P_{H(x_n, Sx_n)} f(x_n) + (1 - \alpha_n) Sx_n,$$

(35)

where $S = (1 - w)I + wT$, $w \in (0, 1)$, and $\{\alpha_n\} \subset (0, 1)$ satisfies $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to $x^* \in \text{Fix}(T)$ verifying

$$x^* = (P_{\text{Fix}(T)} \circ f)x^*,$$

which equivalently solves the following variational inequality problem:

$$x^* \in \text{Fix}(T), \quad \text{and} \quad (\forall v \in \text{Fix}(T)), \quad \langle (I - f)x^*, v - x^* \rangle \geq 0.$$

Proof. Let $T_n = T$ in (3) for all $n \in \mathbb{N}$. From Lemma 2.1, it follows that $\{x_n\}$ is bounded. Using the definition of demiclosed, we get that $T$ satisfies condition (Z). From Theorem 2.1, the desired result follows.

Theorem 3.2. Assume $U : H \to H$ is a quasi-nonexpansive mapping with $\text{Fix}(U) \neq \emptyset$ and satisfies that $I - U$ is demiclosed at 0. Let $f$ be a contraction with constant $\rho \in [0, 1)$. Define a sequence $\{x_n\}$ as follow:

$$x_{n+1} = \alpha_n P_{H(x_n, Vx_n)} f(x_n) + (1 - \alpha_n)Vx_n,$$
where $V = (1 - \gamma)I + \gamma U$, $\gamma \in (0, \frac{1}{2})$, and $\{\alpha_n\} \subset (0, 1)$ satisfies $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to $x^* \in \text{Fix}(U)$ verifying

$$x^* = (P_{\text{Fix}(U)} \circ f)x^*,$$

which equivalently solves the following variational inequality problem:

$$x^* \in \text{Fix}(U), \quad \text{and} \quad (\forall v \in \text{Fix}(U)), \quad \langle (I - f)x^*, v - x^* \rangle \geq 0.$$

**Proof.** By Lemma 1.1 (i), $U + \frac{I}{2} \in \mathcal{T}_m$. Substitute $T$ in (35) by $U + \frac{I}{2}$. Then,

$$S = (1 - w)I + wT = (1 - w)I + w\frac{U + I}{2} = (1 - \frac{w}{2})I + \frac{w}{2}U.$$

Set $\gamma = \frac{w}{2} \in (0, \frac{1}{2})$ and $V = S = (1 - \gamma)I + \gamma U$. Since $I - U$ is demiclosed at 0, $I - \frac{U + I}{2} = \frac{I - U}{2}$ is demiclosed at 0. So we can obtain the result by using Theorem 3.1.

Since a nonexpansive mapping is quasi-nonexpansive and demiclosed (see Lemma 1.2), using Theorem 3.2, we have following theorem.

**Theorem 3.3.** Let $U : H \to H$ be a nonexpansive mapping with $\text{Fix}(U) \neq \emptyset$ and $f$ be a contraction with constant $\rho \in [0, 1)$. Define a sequence $\{x_n\}$ as follow:

$$x_{n+1} = \alpha_n P_{H(x_n, V x_n)}f(x_n) + (1 - \alpha_n)V x_n,$$

where $V = (1 - \gamma)I + \gamma U$, $\gamma \in (0, \frac{1}{2})$, and $\{\alpha_n\} \subset (0, 1)$ satisfies $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to $x^* \in \text{Fix}(U)$ verifying

$$x^* = (P_{\text{Fix}(U)} \circ f)x^*,$$

which equivalently solves the following variational inequality problem:

$$x^* \in \text{Fix}(U), \quad \text{and} \quad (\forall v \in \text{Fix}(U)), \quad \langle (I - f)x^*, v - x^* \rangle \geq 0.$$

### 4 Numerical tests

For comparing the convergent rate of viscosity projection with viscosity method, we compute two simple examples. Let $w = \frac{1}{3}$, $\alpha_n = \frac{1}{n}$, and $x_0 = y_0 = -0.3$. Consider two cases:

**Case 1.** $T_1(x) = \sin(x)$ and $f_1(x) = \cos(\frac{x}{2})$ with constant $\frac{1}{2}$;

**Case 2.** $T_2(x) = \cos(x)$ and $f_2(x) = \sin(\frac{x}{2})$ with constant $\frac{1}{2}$.

It is obvious $T_1$ and $T_2$ are two nonexpansive mappings on $\mathbb{R}$. From Figure 1, It illustrates that viscosity projection methods converges faster than viscosity methods for the given examples.
Figure 1: (a) Case 1 \( \|x_n - Tx_n\| \); (b) Case 2 \( \|x_n - Tx_n\| \).

Remark 4.1. We just prove that viscosity projection method converges locally faster than viscosity in Theorem 2.2, and don’t know if viscosity projection method converges faster than viscosity. It is an open problem.

Acknowledgements The authors would like to thank Paul-Emile Maingé for helpful correspondences and the referees for their pertinent comments and suggestions. This work is supported by National Natural Science Foundation of China (No. 11201476) and Fundamental Research Funds for the Central Universities (No. ZXH2012K001), in part by the Foundation of Tianjin Key Lab for Advanced Signal Processing.

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