



Composition hyperrings

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Abstract

In this paper we introduce the notion of composition hyperring. We show that the composition structure of a composition hyperring is determined by a class of its strong multiendomorphisms. Finally, the three isomorphism theorems of ring theory are derived in the context of composition hyperrings.

1 Introduction

The hyperrings have appeared as a new class of algebraic hyperstructures more general than that of hyperfields, introduced by Krasner [9] in the theory of valued fields. A Krasner hyperring is a nonempty set R endowed with a hyperoperation (the addition) and a binary operation (the multiplication) such that $(R, +)$ is a canonical hypergroup, (R, \cdot) is a semigroup and the multiplication is distributive with respect to the addition. The theory of these hyperrings has been developing since the beginning of seventies, thanks to the contributions of Mittas [14, 15], Krasner [10], Stratigopoulos [20], till nowadays [2, 3, 5, 8, 13, 17].

Several types of hyperrings have been proposed (for more details see [6, 11, 12, 16, 22] and their references), but the most general one is that introduced by Spartalis [18], used also in the context of P -hyperrings or (H, R) -hyperrings [19]. A comprehensive review of hyperrings theory is covered in Nakassis [16], Vougiouklis [22] and in the book [6] written by Davvaz, Leoreanu-Fotea. New applications of the theory of hyperrings in number theory and algebraic geometry can be found in [4, 21].

Key Words: Hyperring; Hyperideal; Multiendomorphism; Composition hyperoperation.
2010 Mathematics Subject Classification: Primary 20N20; Secondary 06E20.
Received: May 2012.
Accepted: June 2012.

Based on the notion of composition ring introduced by Adler [1], we define here the concept of composition hyperring, as a quadruple $(R, +, \cdot, \circ)$ such that $(R, +, \cdot)$ is a commutative hyperring in the general sense of Spartalis, and the composition hyperoperation \circ is an associative hyperoperation, distributive to the right side with respect to the addition and multiplication. Many of the familiar rings of functions are composition rings, where the composition operation is defined just as the composition between the functions. The idea to study a similar hyperstructure comes from the properties of the operations between the polynomials with coefficients in a commutative hyperring, the set of them forming a hyperring as shown in [7] by Jančić-Rašović.

The rest of the paper is organized as follows. After a short presentation of the main results from hyperring theory covered in Preliminaries, in Section 3, we define the notion of composition hyperring, proving that the composition structure of it is determined by a certain class $(\phi_y)_{y \in R}$ of strong multiendomorphisms of the considered hyperring R . We determine conditions under which an arbitrary family Ω of multiendomorphisms of R generates the class $(\phi_y)_{y \in R}$ of strong multiendomorphisms of R , and, consequently, the composition hyperoperation on R . In Section 4, using the notion of composition hyperideal of a composition hyperring, the three isomorphism theorems of ring theory are derived and discussed in the context of composition hyperrings. We end this note with some concluding remarks and some open problems.

2 Preliminaries

We recall some definitions concerning hyperrings theory and we fix the notations used in this paper.

A *canonical hypergroup* is a nonempty set H endowed with an additive hyperoperation $+: H \times H \rightarrow \mathcal{P}^*(H)$, satisfying the following properties:

1. for any $x, y, z \in H$, $x + (y + z) = (x + y) + z$
2. for any $x, y \in H$, $x + y = y + x$
3. there exists $0 \in H$ such that $0 + x = x + 0 = x$, for any $x \in H$
4. for every $x \in H$, there exists a unique element $x' \in H$, such that $0 \in x + x'$ (we write $-x$ instead of x' and we call it the *opposite* of x .)
5. $z \in x + y$ implies that $y \in -x + z$ and $x \in z - y$, that is $(H, +)$ is *reversible*.

A multivalued system $(R, +, \cdot)$ is a *hyperring* (in the general case of Spartalis [18]), if:

1. $(R, +)$ is a hypergroup
2. (R, \cdot) is a semihypergroup
3. the multiplication is distributive with respect to the addition, i.e. for all $x, y, z \in R$, $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$.

If R is commutative with respect to both addition and multiplication, then it is called a *commutative hyperring*.

A particular case of hyperring is that called *Krasner hyperring*, where $(R, +)$ is a canonical hypergroup, (R, \cdot) is a semigroup having 0 as a bilaterally absorbing element, and the multiplication is distributive with respect to the addition.

A nonempty subset S of a hyperring R is called a *subhyperring* of R if, $(S, +)$ is a subhypergroup of $(R, +)$ and $S \cdot S \subseteq S$. Moreover, a subhyperring S of a hyperring R is a *hyperideal* of R , if $r \cdot x \subseteq S$ and $x \cdot r \subseteq S$, for all $r \in R$ and $x \in S$.

Suppose now that $(R, +, \cdot)$ and $(T, +', \cdot')$ are two hyperrings. A map $\phi : R \rightarrow T$ is called a *multihomomorphism* from R to T if, for all $x, y \in R$, the following relations hold:

1. $\bigcup_{u \in x+y} \phi(u) \subseteq \phi(x) +' \phi(y)$
2. $\bigcup_{u \in x \cdot y} \phi(u) \subseteq \phi(x) \cdot' \phi(y)$

If, in the previous conditions, the equality is valid, then ϕ is called a *strong multihomomorphism*. A multihomomorphism from R to R is called *multiendomorphism* of R . If ϕ_1 and ϕ_2 are multiendomorphisms on a hyperring R , then their composition $\phi_1 \circ \phi_2$ defined by $(\phi_1 \circ \phi_2)(x) = \bigcup_{a \in \phi_2(x)} \phi_1(a)$ is also a multiendomorphism on R .

3 Composition hyperrings

In this section we introduce the notion of composition hyperring, giving several examples that illustrate the significance of this new hyperstructure. The composition rings constructed by Adler [1] represents a special case of composition hyperrings.

A *composition ring* is a commutative ring R with an additional binary operation \circ (called *composition*), satisfying the following properties:

1. $(x + y) \circ z = x \circ z + y \circ z$
2. $(xy) \circ z = (x \circ z)(y \circ z)$
3. $x \circ (y \circ z) = (x \circ y) \circ z$,

for any $x, y, z \in R$. The most significant and natural example of a such ring is represented by the ring of functions, where the composition operation is just the composition between two functions: $(f \circ g)(x) = f(g(x))$. Extending this construction to the case of hyperstructures, we obtain the following concept.

Definition 3.1. A *composition hyperring* is an algebraic structure $(R, +, \cdot, \circ)$, where $(R, +, \cdot)$ is a commutative hyperring and the hyperoperation \circ satisfies the following properties, for any $x, y, z \in R$:

1. $(x + y) \circ z = x \circ z + y \circ z$
2. $(x \cdot y) \circ z = (x \circ z) \cdot (y \circ z)$
3. $x \circ (y \circ z) = (x \circ y) \circ z$.

The binary hyperoperation \circ having the previous properties is called the *composition hyperoperation* of the hyperring $(R, +, \cdot)$.

Definition 3.2. Let $(R, +, \cdot, \circ)$ be a composition hyperring. An element $c \in R$ is called a *constant*, if $c \circ x = c$, for all $x \in R$. If A is an arbitrary subset of R , the set of all constants in A is called a *foundation* of A , denoted by $FoundA$.

Example 3.3. Let $(R, +, \cdot)$ be a commutative hyperring. A formal power series with coefficients in R is an infinite sequence $(a_0, a_1, \dots, a_n, \dots)$ in which all a_i belong to R . The set of all formal power series with coefficients in R will be denoted by $R[[x]]$. Defining the following hyperoperations \oplus and \odot by taking:

$$(a_0, a_1, \dots, a_n, \dots) \oplus (b_0, b_1, \dots, b_n, \dots) = \{(c_0, c_1, \dots, c_n, \dots) \mid c_k \in a_k + b_k\}$$

and

$$(a_0, a_1, \dots, a_n, \dots) \odot (b_0, b_1, \dots, b_n, \dots) = \{(c_0, c_1, \dots, c_n, \dots) \mid c_k \in \sum_{i+j=k} a_i b_j\},$$

then the obtained hyperstructure $(R[[x]], \oplus, \odot)$ is a hyperring.

Suppose now that the hypergroup $(R, +)$ has one identity, the zero element 0. Let $R[x]$ denote the set of all polynomials $(a_0, a_1, \dots, a_n, \dots)$ of $R[[x]]$ having $a_i = 0$ except a finite number of indices i . If $0 + 0 = 0$ and $a \cdot 0 = 0$, for all $a \in R$, then according to Theorem 3.2. [7], it follows that $(R[x], \oplus, \odot)$ is a subhyperring of $(R[[x]], \oplus, \odot)$.

Take $f = (a_0, a_1, \dots, a_n, \dots) \in R[x]$ such that $a_k = 0$, for all $k \geq n + 1$, and take $g \in R[x]$. Define a new hyperoperation by putting:

$$f \circ g = a_0 \oplus (a_1 \odot g) \oplus \dots \oplus (a_n \odot g^n).$$

It can be easily verified that $(R[x], \oplus, \odot, \circ)$ is a composition hyperring with $Found(R) = R$.

Example 3.4. If $(R, +, \cdot)$ is an arbitrary commutative hyperring and \circ is defined by $r \circ s = r$, for all $r, s \in R$, then $(R, +, \cdot, \circ)$ is a composition hyperring with $\text{Found}(R) = R$.

Example 3.5. Let $(R, +, \cdot)$ be a commutative ring and R^R be the ring of all functions from R into R . If we define the binary operation \circ as the composition of functions, then $(R^R, +, \cdot, \circ)$ becomes a composition ring with $\text{Found}(R) = R$.

In the following we propose a method to define the composition structure of a composition hyperring by a certain class of its strong multiendomorphisms.

Theorem 3.6. *Let $(R, +, \cdot, \circ)$ be a composition hyperring. For any element $y \in R$, the function $\Phi_y : R \rightarrow \mathcal{P}^*(R)$ defined by $\Phi_y(x) = x \circ y$, for all $x \in R$, is a strong multiendomorphism of the hyperring R . Moreover, if M is a nonempty subset of R , denote by*

$$\Phi_M(x) = \bigcup_{m \in M} \Phi_m(x), \forall x \in R.$$

Then, for all $x, y, z \in R$, it holds:

$$\Phi_{\Phi_x(y)}(z) = \bigcup_{t \in \Phi_y(z)} \Phi_x(t). \quad (1)$$

Conversely, if $(R, +, \cdot)$ is a commutative hyperring and $(\Phi_y)_{y \in R}$ is a family of its strong multiendomorphisms satisfying equation (1), then, defining the hyperoperation \circ by $x \circ y = \Phi_y(x)$, we obtain that $(R, +, \cdot, \circ)$ is a composition hyperring.

Proof. Let $(R, +, \cdot, \circ)$ be a composition hyperring and let $y \in R$. By the definition of the function Φ_y , for all $a, b \in R$, it holds:

$$\bigcup_{u \in a+b} \Phi_y(u) = \bigcup_{u \in a+b} u \circ y = (a+b) \circ y = a \circ y + b \circ y = \Phi_y(a) + \Phi_y(b)$$

and

$$\bigcup_{u \in a \cdot b} \Phi_y(u) = \bigcup_{u \in a \cdot b} u \circ y = (a \cdot b) \circ y = (a \circ y) \cdot (b \circ y) = \Phi_y(a) \cdot \Phi_y(b).$$

Thus, Φ_y is a strong multiendomorphism of the hyperring $(R, +, \cdot)$. Moreover, for all $x, y, z \in R$, it holds:

$$\begin{aligned} \Phi_{\Phi_x(y)}(z) &= \bigcup_{s \in y \circ x} \Phi_s(z) = \bigcup_{s \in y \circ x} z \circ s = z \circ (y \circ x) = (z \circ y) \circ x = \\ &= \bigcup_{t \in z \circ y} t \circ x = \bigcup_{t \in \Phi_y(z)} \Phi_x(t). \end{aligned}$$

Suppose now that $(\Phi_y)_{y \in R}$ is a family of strong multiendomorphisms of a hyperring $(R, +, \cdot)$ satisfying condition (1). If a hyperoperation \circ is defined taking $x \circ y = \Phi_y(x)$, then it can be easily verified that, for all $a, b, x \in R$, it holds $(a + b) \circ x = a \circ x + b \circ x$ and $(a \cdot b) \circ x = (a \circ x) \cdot (b \circ x)$.

Besides, using equation (1), we obtain

$$\begin{aligned} (a \circ b) \circ c &= \bigcup_{s \in \Phi_b(a)} s \circ c = \bigcup_{s \in \Phi_b(a)} \Phi_c(s) = \Phi_{\Phi_c(b)}(a) \\ &= \bigcup_{u \in \Phi_c(b)} \Phi_u(a) = \bigcup_{u \in b \circ c} a \circ u = a \circ (b \circ c). \end{aligned}$$

Thus, $(R, +, \cdot, \circ)$ is a composition hyperring and the proof is now complete. \square

It arises the following question: **Can every commutative hyperring give a composition structure?** In this proposal we determine conditions under which a family of multiendomorphisms of a hyperring $(R, +, \cdot)$ generates the class $(\Phi_y)_{y \in R}$ satisfying the conditions of the previous theorem.

Let Ω be a family of multiendomorphisms of a hyperring $(R, +, \cdot)$. For any $y \in R$, denote

$$P_y = \bigcup_{\Phi \in \Omega} \Phi(y).$$

The set P_y is called the *orbit* of y . An orbit P is said to be *principal* if, for all $x \in P$ and $\Phi_1, \Phi_2 \in \Omega$, it holds:

$$\Phi_1(x) \cap \Phi_2(x) \neq \emptyset \implies \Phi_1 = \Phi_2.$$

Let $(R, +, \cdot)$ be a commutative hyperring and 0 be an identity element of the hypergroup $(R, +, \cdot)$.

Lemma 3.7. *Let Ω be a family of strong multiendomorphisms of a hyperring $(R, +, \cdot)$, such that:*

1. $\Phi_1 \circ \Phi_2 \in \Omega$, for all $\Phi_1, \Phi_2 \in \Omega$, where $\Phi_1 \circ \Phi_2$ is defined by: $(\Phi_1 \circ \Phi_2)(x) = \bigcup_{v \in \Phi_2(x)} \Phi_1(v)$.
2. $\Phi(0) = 0$
3. For all $x, y \in R$ it holds:

$$\Phi \in \Omega \text{ and } x \in \Phi(y) \implies \exists \Phi_1 \in \Omega \text{ such that } y \in \Phi_1(x).$$

Then, Ω induces a partition of the set $\Omega(R) = \bigcup_{\Phi \in \Omega, r \in R} \Phi(r)$ into orbits.

Proof. It is clear that $\Omega(R) = \bigcup_{y \in R} P_y$.

First we prove that $x \in P_y$ implies that $P_x = P_y$. Indeed, if $x \in P_y$, then $x \in \Phi(y)$, for some $\Phi \in \Omega$. By the third condition of the hypothesis, it follows that $y \in \Phi_1(x)$, for some $\Phi_1 \in \Omega$. Thus, if $z \in P_y$, then there exists $\Phi_2 \in \Omega$ such that $z \in \Phi_2(y)$ and so $z \in (\Phi_2 \circ \Phi_1)(x)$. Since $\Phi_2 \circ \Phi_1 \in \Omega$, by the first condition of the hypothesis, we obtain $z \in P_x$. Therefore, $x \in P_y$ implies $P_y \subseteq P_x$. Moreover, if $x \in P_y$, then $y \in P_x$ and consequently $P_x \subseteq P_y$, that is $P_x = P_y$.

Thus if $P_x \cap P_y \neq \emptyset$, then there exists $z \in P_x \cap P_y$ which implies that $P_x = P_y = P_z$ as we have proved before. Thereby, $\Omega(R)$ can be partitioned into the orbits $P_y, y \in R$. \square

Notice that, if the family Ω satisfies the three conditions of the previous lemma and if Ω has at least two elements, then, for any principal orbit P , it holds $0 \notin P$.

Theorem 3.8. *Let Ω be a family of strong multiendomorphisms of a hyper-ring $(R, +, \cdot)$ satisfying conditions of Lemma 3.7. Let \mathcal{S} be a nonempty set of principal orbits with $0 \notin \mathcal{S}$ and for each $P \in \mathcal{S}$, let a_p be an element of P .*

For each $y \in R$ define the multiendomorphism $\Phi_y : R \rightarrow \mathcal{P}^(R)$ as follows. If y is an element of an orbit $P \in \mathcal{S}$, then $\Phi_y = \Phi$, where Φ is an element of Ω such that $y \in \Phi(a_p)$. If $y \notin \bigcup_{P \in \mathcal{S}} P$, then $\Phi_y = 0$.*

Then, the family $(\Phi_y)_{y \in R}$ satisfies equation (1), thus it generates a composition hyperoperation on R .

Proof. If $P_1, P_2 \in \mathcal{S}$ and $y \in P_1 \cap P_2$, then, by Lemma 3.7, it follows that $P_1 = P_2$. Besides, if $y \in P$ and $y \in \Phi_1(a_p) \cap \Phi_2(a_p)$, then $\Phi_1 = \Phi_2$, because P is a principal orbit. So the mapping $y \rightarrow \Phi_y$ is well defined.

Obviously, $(\Phi_y)_{y \in R}$ is a family of strong multiendomorphisms of R .

Let $x, y, a \in R$. We will prove the following relation:

$$\Phi_{\Phi_x(y)}(a) = \bigcup_{v \in \Phi_y(a)} \Phi_x(v). \quad (2)$$

We have to consider the following situations.

1) If $x \in \bigcup_{P \in \mathcal{S}} P$, then $\Phi_x = \Phi$, where Φ is an element of Ω such that $x \in \Phi(a_p)$, with $x \in P$. We have two possibilities:

- a) If $y \in P'$, for some $P' \in \mathcal{S}$, then $\Phi_y = \Phi'$, where Φ' is an element of Ω such that $y \in \Phi'(a_{p'})$. Thus, $\Phi_x(y) = \Phi(y) \subseteq (\Phi \circ \Phi')(a_{p'})$. Therefore, if $z \in \Phi_x(y)$, then $z \in (\Phi \circ \Phi')(a_{p'})$. Since $\Phi \circ \Phi' \in \Omega$, it follows that $z \in P'$ and $\Phi_z = \Phi \circ \Phi'$. Thus, $\bigcup_{z \in \Phi_x(y)} \Phi_z(a) = (\Phi \circ \Phi')(a) = (\Phi_x \circ \Phi_y)(a)$. So, the equation (2) holds.

b) Suppose $y \notin \bigcup_{P \in \mathcal{S}} P$. Then $\Phi_x(y) \cap \bigcup_{P \in \mathcal{S}} P = \emptyset$. Indeed, if $z \in \Phi_x(y) \cap P'$, for some $P' \in \mathcal{S}$, then $z \in \Phi_x(y) = \Phi(y)$ and so there exists $\Phi_1 \in \Omega$ such that $y \in \Phi_1(z)$. Since $z \in P'$, there exists $\Phi_2 \in \Omega$ such that $z \in \Phi_2(a_{P'})$. Therefore, $y \in (\Phi_1 \circ \Phi_2)(a_{P'})$, and because $\Phi_1 \circ \Phi_2 \in \Omega$, we obtain $y \in P'$, which contradicts the assumption.

Therefore, if $y \notin \bigcup_{P \in \mathcal{S}} P$, then, for all $z \in \Phi_x(y)$, it holds $\Phi_z = 0$ and so $\Phi_{\Phi_x(y)}(a) = 0$. Also, $\bigcup_{v \in \Phi_y(a)} \Phi_x(v) = \Phi_x(0) = 0$.

2) Suppose now $x \notin \bigcup_{P \in \mathcal{S}} P$. Then $\Phi_x = 0$ and so $\Phi_x(y) = 0$. Thus, $\Phi_{\Phi_x(y)}(a) = \Phi_0(a)$.

Notice that $0 \notin \bigcup_{P \in \mathcal{S}} P$. Indeed, if $0 \in P$, for some $P \in \mathcal{S}$, then, by Lemma 3.7, it follows that $P = P_0 = \bigcup_{\Phi \in \Omega} \Phi(0) = 0$, i.e. we obtain $0 \in \mathcal{S}$, contrary to the hypothesis. Thus, $0 \notin \bigcup_{P \in \mathcal{S}} P$ and so $\Phi_0 = 0$, i.e. $\Phi_0(a) = 0$. So, $\Phi_{\Phi_x(y)}(a) = 0$. Also $\bigcup_{v \in \Phi_y(a)} \Phi_x(v) = 0$, since $\Phi_x = 0$. Thus, the family $(\Phi_y)_{y \in R}$ satisfies conditions of Theorem 3.6, generating a composition hyperoperation on R . \square

Remark 3.9. Let $(R, +, \cdot)$ be a Krasner hyperring and $AutR$ be the group of its ordinary automorphisms. If Ω is a subgroup of $AutR$, then Ω satisfies conditions of previous theorem. The composition hyperoperation \circ associated with the corresponding family $(\Phi_y)_{y \in R}$ is an ordinary operation, since, for all $x, y \in R$, $|x \circ y| = |\Phi_y(x)| = 1$.

Example 3.10. Let $(R, +, \cdot)$ be the field \mathbb{R} of real numbers and $A = 2^{\mathbb{Q}} = \{2^q \mid q \in \mathbb{Q}\}$. Define hyperoperations \oplus_A and \odot_A on R by: $x \oplus_A y = xA + yA$ and $x \odot_A y = xAy$. It can be easily verified that (R, \oplus_A) is a commutative hypergroup and (R, \odot_A) is a commutative semihypergroup. Moreover, since, for all $a \in A$, it holds $aA = A$, it follows that:

$$\begin{aligned} (x \oplus_A y) \odot_A z &= (xA + yA)Az = (xAz + yAz)A = (xzA + yzA)A = \\ &= \bigcup_{a \in A} (xzAa + yzAa) = xzA + yzA = xzAA + yzAA = \\ &= (x \odot_A z) \oplus_A (y \odot_A z), \end{aligned}$$

for all $x, y, z \in R$. Thus, (R, \oplus_A, \odot_A) is a commutative hyperring.

Let us define now two functions $f : R \rightarrow \mathcal{P}^*(R)$ and $g : R \rightarrow \mathcal{P}^*(R)$ by $f(x) = A \cdot x = \{2^q \cdot x \mid q \in \mathbb{Q}\}$ and $g(x) = -A \cdot x = \{-2^q \cdot x \mid q \in \mathbb{Q}\}$. Obviously, f and g are strong multiendomorphisms of (R, \oplus_A, \odot_A) . Also, $f \circ f = g \circ g = f$ and $f \circ g = g \circ f = g$. If $x \in f(y)$, then $x = 2^q y$, for some $q \in \mathbb{Q}$, and so $y = 2^{-q} x \in Ax = f(x)$. Similarly, $x \in g(y)$ implies that $y \in g(x)$. Obviously $f(0) = 0$ and $g(0) = 0$. Let $\Omega = \{f, g\}$. It is easy to verify that Ω satisfies conditions of Lemma 3.7.

Besides, for any $y \in R$, its orbit has the form $P_y = f(y) \cup g(y) = \{\pm 2^q \cdot y \mid q \in \mathbb{Q}\}$.

If $y \neq 0$, then P_y is a principal orbit, since, for any $x \in P_y$, it holds $f(x) \cap g(x) = \emptyset$, because $2^{\mathbb{Q}}x \cap (-2^{\mathbb{Q}}x) = \emptyset$. Thus, by the previous theorem, each family \mathcal{S} of principal orbits generates corresponding composition hyperoperation on R .

For instance, if $\mathcal{S} = \{P_n \mid n \in \mathbb{N}\}$, then, for $y \in \bigcup_{n \in \mathbb{N}} P_n$ and $y > 0$, we put $\Phi_y = f$ and, for $y < 0$, we put $\Phi_y = g$. If $y \notin \bigcup_{n \in \mathbb{N}} P_n$, then $\Phi_y = 0$. Thus, the corresponding hyperoperation is defined by:

$$x \circ y = \begin{cases} 2^{\mathbb{Q}}x & \text{if } y \in 2^{\mathbb{Q}} \cdot \mathbb{N}, \\ -2^{\mathbb{Q}}x & \text{if } y \in -2^{\mathbb{Q}} \cdot \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

4 Isomorphism theorems of composition hyperrings

This section deals with the isomorphism theorems for the composition hyperrings. In order to state them, first we introduce the notion of *composition hyperideal* and then we construct the *quotient composition hyperring*.

Throughout this section, $(R, +, \cdot, \circ)$ is a composition hyperring such that $(R, +)$ is a canonical hypergroup and $x \cdot 0 = 0$, for all $x \in R$. Obviously, in a Krasner hyperring these conditions are satisfied.

Definition 4.1. Let $(R, +, \cdot, \circ)$ be a composition hyperring and N be a subset of R . N is called a *composition hyperideal* of R if the following three conditions are satisfied:

1. N is a hyperideal of the hyperring R
2. $n \circ r \subseteq N$, for all $n \in N$ and $r \in R$
3. If $r, s, t \in R$ and $r - s \cap N \neq \emptyset$, then $t \circ r - t \circ s \subseteq N$.

Let N be a composition hyperideal of R . Consider on R the following relation:

$$x\rho y \iff x + N = y + N.$$

Obviously, ρ is an equivalence on R and the equivalence class represented by x is $[x]_{\rho} = x + N$. Denote by R/N the set of all equivalence classes of the elements of R with respect to the equivalence relation ρ .

Lemma 4.2. Let R be a composition hyperring and N be a composition hyperideal of R . Defining on the quotient R/N the hyperoperations \oplus, \odot, \otimes as it

follows:

$$\begin{aligned}(x + N) \oplus (y + N) &= \{z + N \mid z \in x + y\} \\ (x + N) \odot (y + N) &= \{z + N \mid z \in x \cdot y\} \\ (x + N) \circledast (y + N) &= \{z + N \mid z \in x \circ y\},\end{aligned}$$

we obtain that $(R/N, \oplus, \odot, \circledast)$ is a composition hyperring, called the quotient composition hyperring related to the equivalence ρ .

Proof. First we prove that the hyperoperations $\oplus, \odot, \circledast$ are well-defined.

Let $x + N = x_1 + N$ and $y + N = y_1 + N$, for $x, x_1, y, y_1 \in R$. Set $L = (x + N) \oplus (y + N) = \{z + N \mid z \in x + y\}$ and $D = (x_1 + N) \oplus (y_1 + N) = \{z + N \mid z \in x_1 + y_1\}$. If $z \in x + y$, then $z + N \subseteq x + y + N = (x + N) + (y + N) = (x_1 + N) + (y_1 + N) = x_1 + y_1 + N$. Since $z \in z + N \subseteq x_1 + y_1 + N$, there exists $z_1 \in x_1 + y_1$ and $n_1 \in N$, such that $z \in z_1 + n_1$. It follows that $z + N \subseteq z_1 + n_1 + N = z_1 + N$. But $z \in z_1 + n_1$, so $z_1 \in z - n_1$ and then $z_1 + N \subseteq z - n_1 + N = z + N$. Thus, $z + N = z_1 + N$, while $z_1 \in x_1 + y_1$. Therefore, $L \subseteq D$. Similarly one proves that $D \subseteq L$.

Now set $L = (x + N) \odot (y + N) = \{z + N \mid z \in x \cdot y\}$ and $D = (x_1 + N) \odot (y_1 + N) = \{z + N \mid z \in x_1 \cdot y_1\}$. Let $z \in x \cdot y$. Since $x \in x_1 + N$ and $y \in y_1 + N$, there exist $n_1, n_2 \in N$ such that $z \in (x_1 + n_1) \cdot (y_1 + n_2) = x_1 \cdot y_1 + n_1 \cdot y_1 + x_1 \cdot n_2 + n_1 \cdot n_2 \subseteq x_1 \cdot y_1 + N$. Thereby, there exist $z_1 \in x_1 \cdot y_1$ and $n_1 \in N$ such that $z \in z_1 + n_1$. It implies that $z + N = z_1 + N$, i.e. $z + N \in D$. So, $L \subseteq D$. The converse inclusion can be similarly proved.

Suppose now $L = (x + N) \circledast (y + N) = \{z + N \mid z \in x \circ y\}$ and $D = (x_1 + N) \circledast (y_1 + N) = \{z + N \mid z \in x_1 \circ y_1\}$. Let $z \in x \circ y$. Because $y \in y_1 + N$, we can write $y \in y_1 + n_1$, for some $n_1 \in N$. It follows that $n_1 \in y - y_1$, that is $y - y_1 \cap N \neq \emptyset$. So $x \circ y - x \circ y_1 \subseteq N$, i.e. $x \circ y \subseteq x \circ y_1 + N$. Since $x \in x_1 + N$, there exists $n_2 \in N$ such that $x \in x_1 + n_2$ and then $x \circ y \subseteq (x_1 + n_2) \circ y_1 + N = x_1 \circ y_1 + n_2 \circ y_1 + N \subseteq x_1 \circ y_1 + N$. Thereby, if $z \in x \circ y$, then there exists $z_1 \in x_1 \circ y_1$ such that $z \in z_1 + N$ and thus $z + N = z_1 + N$, which means that $L \subseteq D$ and similarly $D \subseteq L$.

Finally, it is easy to verify that $(R/N, \oplus, \odot, \circledast)$ is a composition hyperring. We omit here the classical proof. \square

Definition 4.3. Let R_1 and R_2 be composition hyperrings. A mapping $f : R_1 \rightarrow R_2$ is called a *strong homomorphism* if the following conditions are satisfied, for all $x, y \in R_1$:

1. $f(x + y) = f(x) + f(y)$
2. $f(x \cdot y) = f(x) \cdot f(y)$

$$3. f(x \circ y) = f(x) \circ f(y)$$

$$4. f(0) = 0.$$

A strong homomorphism f is an *isomorphism* if f is one to one and onto. We write $R_1 \cong R_2$ if R_1 is isomorphic with R_2 .

Notice that, if f is a strong homomorphism from R_1 into R_2 , then, for all $x \in R_1$, it holds $f(-x) = -f(x)$. Indeed, since $0 \in x - x$, it follows that $0 = f(0) \in f(x) + f(-x)$, so $f(-x) = -f(x)$.

If f is a strong homomorphism from R_1 into R_2 , then the *kernel* of f is the set $\text{Ker}f = \{x \in R_1 \mid f(x) = 0\}$. Obviously, $\text{Ker}f$ is a hyperideal of $(R_1, +, \cdot)$, but generally it is not a composition hyperideal.

In the following, we will state and prove the isomorphism theorems for composition hyperrings. Note that, for the first theorem we need $\text{Ker}f$ to be a composition hyperideal.

Theorem 4.4. *Let R_1 and R_2 be composition hyperrings. If f is a strong homomorphism from R_1 into R_2 with the kernel K such that K is a composition hyperideal of R_1 , then $R_1/K \cong \text{Im}f$.*

Proof. Define $\phi : R_1/K \rightarrow \text{Im}f$ by $\phi(x + K) = f(x)$, for all $x \in R_1$. First we prove that ϕ is well-defined. Suppose that $x + K = y + K$. Then, there exists $z \in K$ such that $x \in y + z$. It follows that $z \in (-y + x) \cap K$, that is $0 = f(z) \in f(x) - f(y)$. Thus, $f(x) = f(y)$. Obviously, f is onto. It remains to show that ϕ is one to one. Suppose $\phi(x + K) = \phi(y + K)$. Then $f(x) = f(y)$, which means that $0 \in f(x - y)$. Thus, there exists $z \in x - y$ such that $z \in K = \text{Ker}f$ and so, $x \in z + y \subseteq K + y$ which implies that $x + K = y + K$. Thereby ϕ is a bijection.

Moreover, ϕ is a strong homomorphism, because

$$\begin{aligned} \phi((x + K) \oplus (y + K)) &= \phi(\{z + K \mid z \in x + y\}) = \{f(z) \mid z \in x + y\} = \\ &= f(x + y) = f(x) + f(y) = \phi(x + K) + \phi(y + K). \end{aligned}$$

Similarly,

$$\begin{aligned} \phi((x + K) \odot (y + K)) &= \phi(x + K) \cdot \phi(y + K), \\ \phi((x + K) \circ (y + K)) &= \phi(x + K) \circ \phi(y + K) \end{aligned}$$

and $\phi(K) = f(0) = 0$. □

Theorem 4.5. *If A and B are composition hyperideals of a composition hyperring R , then $A/(A \cap B) \cong (A + B)/B$.*

Proof. Clearly $A \cap B$ is a composition hyperideal of A (as the intersection between composition hyperideals) and $A + B$ is a subhyperring of $(R, +, \cdot)$. If $x, y \in A + B$, then there exist $a, a_1 \in A$ and $b, b_1 \in B$ such that $x \in a + b$ and $y \in a_1 + b_1$, and therefore $x \circ y \subseteq (a + b) \circ (a_1 + b_1) = \bigcup_{s \in a_1 + b_1} (a + b) \circ s = \bigcup_{s \in a_1 + b_1} (a \circ s + b \circ s) \subseteq \bigcup_{a' \in A, b' \in B} a' + b' = A + B$. Thus $A + B$ is a composition hyperring and, since B is a composition hyperideal of $A + B$, it follows that $(A + B)/B$ is well defined.

Let us take $f : A \rightarrow (A + B)/B$ by $f(a) = a + B$. It is easy to verify that f is a strong homomorphism.

We prove that f is onto. Let $y + B \in (A + B)/B$, with $y \in a + b$, for some $a \in A$ and $b \in B$. Then $a \in y - b$, that is $a \in y + B$. Thus $a + B = y + B$, so $f(a) = y + B$.

Besides, for any $a \in A$, it holds:

$$a \in \text{Ker } f \iff f(a) = B \iff a + B = B \iff a \in A \cap B.$$

Thereby $\text{Ker } f = A \cap B$, and by Theorem 4.4, we get the isomorphism $A/(A \cap B) \cong (A + B)/B$. \square

Theorem 4.6. *If A and B are composition hyperideals of a composition hyperring R such that $A \subseteq B$, then B/A is a composition hyperideal of R/A and $(R/A)/(B/A) \cong R/B$.*

Proof. As in the previous two theorems, one can verify that B/A is a composition hyperideal of R/A and that the application $f : R/A \rightarrow R/B$, defined by $f(x + A) = x + B$, is a strong homomorphism of R/A onto R/B with $\text{Ker } f = B/A$. \square

5 Conclusions and future work

The notion of hyperring is a natural generalization of that of ring, many properties of rings have been transferred to the case of hyperrings. This paper is a contribution to the development of the theoretical background of hyperrings starting from rings. The notion of composition ring introduced in 1962 [1] has been extended to that of composition hyperring, i.e. a hyperring $(R, +, \cdot, \circ)$ with a new hyperoperation \circ , called composition, which is associative and distributive with respect to the addition and multiplication of the hyperring. It is shown that the composition structure of R can be determined by a certain class of multiendomorphisms of R . The three isomorphism theorems have been proved for this class of hyperstructures.

This research could be continued further, for instance the theory of hyperideals (maximal or prime hyperideals) could be developed in this context, or to study the composition near-hyperrings, starting from near-rings.

Acknowledgements. The first author was partially supported by “Agencija za raziskovalno dejavnost Republike Slovenije, program $P1 - 0285$ ”.

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