On the Growth of Solutions of Some Second Order Linear Differential Equations With Entire Coefficients

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Abstract

In this paper, we investigate the order and the hyper-order of growth of solutions of the linear differential equation

$$f'' + Q(e^{-z}) f' + (A_1 e^{a_1 z} + A_2 e^{a_2 z})^n f = 0,$$

where $n \geq 2$ is an integer, $A_j(z) (\neq 0) (j = 1, 2)$ are entire functions with $\max\{\sigma(A_j) : j = 1, 2\} < 1$, $Q(z) = q_m z^m + \cdots + q_1 z + q_0$ is a nonconstant polynomial and $a_1, a_2$ are complex numbers. Under some conditions, we prove that every solution $f(z) \neq 0$ of the above equation is of infinite order and hyper-order 1.

1 Introduction and statement of results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna’s value distribution theory (see [8], [13]). Let $\sigma(f)$ denote the order of growth of an entire function $f$ and the hyper-order $\sigma_2(f)$ of $f$ is defined by (see [9], [13])

$$\sigma_2(f) = \limsup_{r \to +\infty} \frac{\log \log T(r, f)}{\log r} = \limsup_{r \to +\infty} \frac{\log \log \log M(r, f)}{\log r},$$

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where \( T(r,f) \) is the Nevanlinna characteristic function of \( f \) and \( M(r,f) = \max_{|z|=r} |f(z)| \).

In order to give some estimates of fixed points, we recall the following definition.

**Definition 1.1** ([3], [10]) Let \( f \) be a meromorphic function. Then the exponent of convergence of the sequence of distinct fixed points of \( f(z) \) is defined by

\[
\tau(f) = \lambda(f) = \limsup_{r \to +\infty} \frac{\log N(r, \frac{1}{f-z})}{\log r},
\]

where \( N(r, \frac{1}{f}) \) is the counting function of distinct zeros of \( f(z) \) in \( \{z : |z| < r\} \). We also define

\[
\lambda(f - \varphi) = \limsup_{r \to +\infty} \frac{\log N(r, \frac{1}{f-z})}{\log r}
\]

for any meromorphic function \( \varphi(z) \).

In [11], Peng and Chen have investigated the order and hyper-order of solutions of some second order linear differential equations and have proved the following result.

**Theorem A** ([11]) Let \( A_j(z) (\neq 0) (j = 1, 2) \) be entire functions with \( \sigma(A_j) < 1 \), \( a_1, a_2 \) be complex numbers such that \( a_1 a_2 \neq 0, a_1 \neq a_2 \) (suppose that \( |a_1| \leq |a_2| \)). If \( \arg a_1 \neq \pi \) or \( a_1 < -1 \), then every solution \( f \neq 0 \) of the equation

\[
f'' + e^{-z} f' + (A_1 e^{a_1 z} + A_2 e^{a_2 z}) f = 0
\]

has infinite order and \( \sigma_2(f) = 1 \).

The main purpose of this paper is to extend and improve the results of Theorem A to some second order linear differential equations. In fact we will prove the following results.

**Theorem 1.1** Let \( n \geq 2 \) be an integer, \( A_j(z) (\neq 0) (j = 1, 2) \) be entire functions with \( \max \{\sigma(A_j) : j = 1, 2\} < 1 \), \( Q(z) = q_n z^n + \cdots + q_1 z + q_0 \) be nonconstant polynomial and \( a_1, a_2 \) be complex numbers such that \( a_1 a_2 \neq 0, a_1 \neq a_2 \). If (1) \( \arg a_1 \neq \pi \) and \( \arg a_1 \neq \arg a_2 \) or (2) \( \arg a_1 \neq \pi, \arg a_1 = \arg a_2 \) and
|a_2| > n|a_1| or (3) a_1 < 0 and arg a_1 ≠ arg a_2 or (4) -\frac{1}{n}(|a_2| - m) < a_1 < 0, |a_2| > m and arg a_1 = arg a_2, then every solution f ≠ 0 of the equation

\[ f'' + Q(e^{-z}) f' + (A_1 e^{a_1 z} + A_2 e^{a_2 z})^n f = 0 \quad (1.1) \]

satisfies \( \sigma(f) = +\infty \) and \( \sigma_2(f) = 1 \).

**Theorem 1.2** Let \( A_j(z) \) (j = 1, 2), \( Q(z) \), \( a_1 \), \( a_2 \), \( n \) satisfy the additional hypotheses of Theorem 1.1. If \( \varphi \) ≠ 0 is an entire function of order \( \sigma(\varphi) < +\infty \), then every solution \( f \neq 0 \) of equation (1.1) satisfies

\[ \lambda(f - \varphi) = \lambda(f - \varphi) = \sigma(f) = +\infty, \]

\[ \lambda_2(f - \varphi) = \lambda_2(f - \varphi) = \sigma_2(f) = 1. \]

**Theorem 1.3** Let \( A_j(z) \) (j = 1, 2), \( Q(z) \), \( a_1 \), \( a_2 \), \( n \) satisfy the additional hypotheses of Theorem 1.1. If \( \varphi \) ≠ 0 is an entire function of order \( \sigma(\varphi) < 1 \), then every solution \( f \neq 0 \) of equation (1.1) satisfies

\[ \lambda(f - \varphi) = \lambda(f - \varphi) = +\infty. \]

Furthermore, if (i) \((2n + 2) a_1 ≠ (2 - p) a_1 + pa_2 - k \) \( (p = 0, 1, 2; \ k = 0, 1, \cdots, 2m), (n + 2 - p) a_1 + pa_2 - k \) \( (p = 0, 1, \cdots, n + 2; \ k = 0, 1, \cdots, m) \) or (ii) \((2n + 2) a_2 ≠ (2 - p) a_1 + pa_2 - k \) \( (p = 0, 1, 2; \ k = 0, 1, \cdots, 2m), (n + 2 - p) a_1 + pa_2 - k \) \( (p = 0, 1, \cdots, n + 2; \ k = 0, 1, \cdots, m) \), then

\[ \lambda(f'' - \varphi) = +\infty. \]

**Corollary 1.1** Let \( A_j(z) \) (j = 1, 2), \( Q(z) \), \( a_1 \), \( a_2 \), \( n \) satisfy the additional hypotheses of Theorem 1.1. If \( f \neq 0 \) is any solution of equation (1.1), then \( f, f' \) all have infinitely many fixed points and satisfy

\[ \tau(f) = \tau(f') = \infty. \]

Furthermore, if (i) \((2n + 2) a_1 ≠ (2 - p) a_1 + pa_2 - k \) \( (p = 0, 1, 2; \ k = 0, 1, \cdots, 2m), (n + 2 - p) a_1 + pa_2 - k \) \( (p = 0, 1, \cdots, n + 2; \ k = 0, 1, \cdots, m) \) or (ii) \((2n + 2) a_2 ≠ (2 - p) a_1 + pa_2 - k \) \( (p = 0, 1, 2; \ k = 0, 1, \cdots, 2m), (n + 2 - p) a_1 + pa_2 - k \) \( (p = 0, 1, \cdots, n + 2; \ k = 0, 1, \cdots, m) \), then \( f'' \) has infinitely many fixed points and satisfies

\[ \tau(f'') = \infty. \]
2 Preliminary lemmas

To prove our theorems, we need the following lemmas.

Lemma 2.1 ([7]) Let $f$ be a transcendental meromorphic function with
\[ \sigma(f) = \sigma < +\infty, \]
$H = \{(k_1, j_1), (k_2, j_2), \ldots, (k_q, j_q)\}$ be a finite set of distinct
pairs of integers satisfying $k_i > j_i > 0$ ($i = 1, \ldots, q$) and let $\varepsilon > 0$
be a given constant. Then,

(i) there exists a set $E_1 \subset \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$ with linear measure zero, such that, if
$\psi \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right) \setminus E_1$, then there is a constant $R_0 = R_0(\psi) > 1$,
such that for all $z$ satisfying $\arg z = \psi$ and $|z| \geq R_0$, and for all $k, j \in H$, we have
\[ \frac{|f^{(k)}(z)|}{|f^{(j)}(z)|} \leq |z|^{(k-j)(\sigma-1+\varepsilon)}, \quad (2.1) \]

(ii) there exists a set $E_2 \subset (1, +\infty)$ with finite logarithmic measure, such that
for all $z$ satisfying $|z| \notin E_2 \cup [0, 1]$ and for all $(k, j) \in H$, we have
\[ \frac{|f^{(k)}(z)|}{|f^{(j)}(z)|} \leq |z|^{(k-j)(\sigma-1+\varepsilon)}, \quad (2.2) \]

(iii) there exists a set $E_3 \subset (0, +\infty)$ with finite linear measure, such that
for all $z$ satisfying $|z| \notin E_3$ and for all $(k, j) \in H$, we have
\[ \frac{|f^{(k)}(z)|}{|f^{(j)}(z)|} \leq |z|^{(k-j)(\sigma+\varepsilon)}. \quad (2.3) \]

Lemma 2.2 ([4]) Suppose that $P(z) = (\alpha + i\beta)z^n + \cdots$ ($\alpha, \beta$ are real
numbers, $|\alpha| + |\beta| \neq 0$) is a polynomial with degree $n \geq 1$, that $A(z) \neq 0$
is an entire function with $\sigma(A) < n$. Set $g(z) = A(z)e^{P(z)}$, $z = re^{i\theta}$,
$\delta(P, \theta) = \alpha \cos n\theta - \beta \sin n\theta$. Then for any given $\varepsilon > 0$, there is a set $E_4 \subset [0, 2\pi)$
that has linear measure zero, such that for any $\theta \in [0, 2\pi) \setminus (E_4 \cup E_5)$,
there is $R > 0$, such that for $|z| = r > R$, we have

(i) if $\delta(P, \theta) > 0$, then
\[ \exp\{(1-\varepsilon)\delta(P, \theta)r^n\} \leq |g(re^{i\theta})| \leq \exp\{(1+\varepsilon)\delta(P, \theta)r^n\}; \quad (2.4) \]

(ii) if $\delta(P, \theta) < 0$, then
\[ \exp\{(1+\varepsilon)\delta(P, \theta)r^n\} \leq |g(re^{i\theta})| \leq \exp\{(1-\varepsilon)\delta(P, \theta)r^n\}, \quad (2.5) \]
where $E_5 = \{\theta \in [0, 2\pi) : \delta(P, \theta) = 0\}$ is a finite set.
Lemma 2.3 ([11]) Suppose that \( n \geq 1 \) is a positive entire number. Let \( P_j (z) = a_{jn} z^n + \cdots \) \( (j = 1, 2) \) be nonconstant polynomials, where \( a_{jq} \) \( (q = 1, \cdots, n) \) are complex numbers and \( a_{1n} a_{2n} \neq 0 \). Set \( z = r e^{i\theta} \), \( a_{jn} = |a_{jn}| e^{i\theta} \), \( \theta_j \in [-\frac{\pi}{2n}, \frac{3\pi}{2n}] \). \( \delta (P_j, \theta) = |a_{jn}| \cos (\theta_j + n\theta) \), then there is a set \( E_6 \subset [-\frac{\pi}{2n}, \frac{3\pi}{2n}] \) that has linear measure zero. If \( \theta_1 \neq \theta_2 \), then there exists a ray \( z = \theta, \theta \in (\frac{\pi}{2n}, \frac{3\pi}{2n}) \setminus (E_6 \cup E_7) \), such that
\[
\delta (P_1, \theta) > 0, \quad \delta (P_2, \theta) < 0
\] or
\[
\delta (P_1, \theta) < 0, \quad \delta (P_2, \theta) > 0,
\]
where \( E_7 = \{ \theta \in [-\frac{\pi}{2n}, \frac{3\pi}{2n}]: \delta (P_j, \theta) = 0 \} \) is a finite set, which has linear measure zero.

Remark 2.1 ([11]) In Lemma 2.3, if \( \theta \in (\frac{\pi}{2n}, \frac{3\pi}{2n}) \setminus (E_6 \cup E_7) \), then we obtain the same result.

Lemma 2.4 ([5]) Suppose that \( k \geq 2 \) and \( B_0, B_1, \cdots, B_{k-1} \) are entire functions of finite order and let \( \sigma = \max \{ \sigma (B_j) : j = 0, \cdots, k-1 \} \). Then every solution \( f \) of the equation
\[
f^{(k)} + B_{k-1} f^{(k-1)} + \cdots + B_1 f' + B_0 f = 0
\]
satisfies \( \sigma_2 (f) \leq \sigma \).

Lemma 2.5 ([7]) Let \( f(z) \) be a transcendental meromorphic function, and let \( \alpha > 1 \) be a given constant. Then there exist a set \( E_8 \subset (1, \infty) \) with finite logarithmic measure and a constant \( B > 0 \) that depends only on \( \alpha \) and \( i, j \) \( (0 \leq i < j \leq k) \), such that for all \( z \) satisfying \( |z| = r \notin [0, 1] \cup E_8 \), we have
\[
\left( \frac{f^{(j)}(z)}{f^{(i)}(z)} \right) \leq B \left\{ \frac{T(\alpha r, f)}{r} \log T(\alpha r, f) \right\}^{j-i}.
\] (2.9)

Lemma 2.6 ([2]) Let \( A_0, A_1, \cdots, A_{k-1} \), \( F \neq 0 \) be finite order meromorphic functions. If \( f \) is a meromorphic solution with \( \sigma (f) = +\infty \) of the equation
\[
f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_1 f' + A_0 f = F,
\]
then \( f \) satisfies
\[
\overline{\lambda} (f) = \lambda (f) = \sigma (f) = +\infty.
\] (2.10)

Lemma 2.7 ([1]) Let \( A_0, A_1, \cdots, A_{k-1}, F \neq 0 \) be finite order meromorphic functions. If \( f \) is a meromorphic solution of equation (2.10) with \( \sigma (f) = +\infty \) and \( \sigma_2 (f) = \sigma \), then \( f \) satisfies
\[
\overline{\lambda}_2 (f) = \lambda_2 (f) = \sigma_2 (f) = \sigma.
\] (2.11)
Lemma 2.8 ([6], [13]) Suppose that \( f_1(z), f_2(z), \ldots, f_n(z) (n \geq 2) \) are meromorphic functions and \( g_1(z), g_2(z), \ldots, g_n(z) \) are entire functions satisfying the following conditions:

(i) \( \sum_{j=1}^{n} f_j(z) e^{g_j(z)} = 0; \)

(ii) \( g_j(z) - g_k(z) \) are not constants for \( 1 \leq j \neq k \leq n; \)

(iii) For \( 1 \leq j \leq n, 1 \leq h < k \leq n, T(r, f_j) = o \left\{ T(r, e^{g_h(z)} - g_k(z)) \right\} \) \((r \to \infty, r \notin E_0)\), where \( E_0 \) is a set with finite linear measure.

Then \( f_j(z) = 0 \) \((j = 1, \ldots, n)\).

Lemma 2.9 ([12]) Suppose that \( f_1(z), f_2(z), \ldots, f_n(z) (n \geq 2) \) are meromorphic functions and \( g_1(z), g_2(z), \ldots, g_n(z) \) are entire functions satisfying the following conditions:

(i) \( \sum_{j=1}^{n} f_j(z) e^{g_j(z)} = f_{n+1}; \)

(ii) If \( 1 \leq j \leq n + 1, 1 \leq k \leq n, \) the order of \( f_j \) is less than the order of \( e^{g_k(z)}. \) If \( n \geq 2, 1 \leq j \leq n + 1, 1 \leq h \leq k \leq n, \) and the order of \( f_j \) is less than the order of \( e^{g_h(z)} - g_k. \) Then \( f_j(z) = 0 \) \((j = 1, 2, \ldots, n + 1)\).

3 Proof of Theorem 1.1

Assume that \( f \neq 0 \) is a solution of equation \((1.1).\)

**First step:** We prove that \( \sigma(f) = +\infty. \) Suppose that \( \sigma(f) = \sigma < +\infty. \) We rewrite \((1.1)\) as

\[
\frac{f''}{f} + Q(e^{-z}) \frac{f'}{f} + A_1^n e^{a_1 z} + A_2^n e^{a_2 z} + \sum_{p=1}^{n-1} C_p A_1^{n-p} e^{(n-p)a_1 z} A_2^p e^{pa_2 z} = 0.
\]

By Lemma 2.1, for any given \( \varepsilon, \)

\[
0 < \varepsilon < \min \left\{ \frac{|a_2| - n |a_1|}{2[(2n - 1) |a_2| + n |a_1|]}, \frac{1}{2(2n - 1)} \right\},
\]

there exists a set \( E_1 \subseteq \left[ -\frac{\pi}{2}, \frac{3\pi}{2} \right] \) of linear measure zero, such that if \( \theta \in \left[ -\frac{\pi}{2}, \frac{3\pi}{2} \right] \setminus E_1, \) then there is a constant \( R_0 = R_0(\theta) > 0, \) such that for all \( z \) satisfying \( \arg z = \theta \) and \( |z| = r \geq R_0, \) we have

\[
\left| \frac{f^{(j)}(z)}{f(z)} \right| \leq r^{j(\sigma - 1 + \varepsilon)} \quad (j = 1, 2).
\]

Let \( z = re^{i\theta}, a_1 = |a_1|e^{i\theta_1}, a_2 = |a_2|e^{i\theta_2}, \theta_1, \theta_2 \in \left[ -\frac{\pi}{2}, \frac{3\pi}{2} \right]. \) We know that \( \delta(pa_1 z, \theta) = p\delta(a_1 z, \theta) \) and \( \delta(pa_2 z, \theta) = p\delta(a_2 z, \theta), \) where \( p > 0. \)
Case 1: Assume that \( \arg a_1 \neq \pi \) and \( \arg a_1 \neq \arg a_2 \), which is \( \theta_1 \neq \pi \) and \( \theta_1 \neq \theta_2 \).

By Lemma 2.2 and Lemma 2.3, for the above \( \varepsilon \), there is a ray \( \arg z = \theta \) such that \( \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus (E_1 \cup E_6 \cup E_7) \) (where \( E_6 \) and \( E_7 \) are defined as in Lemma 2.3, \( E_1 \cup E_6 \cup E_7 \) is of the linear measure zero), and satisfying

\[
\delta (a_1 z, \theta) > 0, \delta (a_2 z, \theta) < 0
\]

or

\[
\delta (a_1 z, \theta) < 0, \delta (a_2 z, \theta) > 0.
\]

a) When \( \delta (a_1 z, \theta) > 0, \delta (a_2 z, \theta) < 0 \), for sufficiently large \( r \), we get by Lemma 2.2

\[
|A_1^ne^{na_1z}| \geq \exp \{(1-\varepsilon)n\delta (a_1 z, \theta) r\}, \quad (3.3)
\]
\[
|A_2^ne^{na_2z}| \leq \exp \{(1-\varepsilon)n\delta (a_2 z, \theta) r\} < 1, \quad (3.4)
\]
\[
|A_1^{n-p}e^{(n-p)a_1z}| \leq \exp \{(1+\varepsilon)(n-p)\delta (a_1 z, \theta) r\}
\]
\[
\leq \exp \{(1+\varepsilon)(n-1)\delta (a_1 z, \theta) r\}, \quad p = 1, \ldots, n-1, \quad (3.5)
\]
\[
|A_2^pe^{pa_2z}| \leq \exp \{(1-\varepsilon)p\delta (a_2 z, \theta) r\} < 1, \quad p = 1, \ldots, n-1. \quad (3.6)
\]

For \( \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \) we have

\[
|Q (e^{-z})| = |q_m e^{-mz} + \cdots + q_1 e^{-z} + q_0|
\]
\[
\leq |q_m| |e^{-mz}| + \cdots + |q_1| |e^{-z}| + |q_0|
\]
\[
\leq |q_m| e^{-mr \cos \theta} + \cdots + |q_1| e^{-r \cos \theta} + |q_0| \leq M, \quad (3.7)
\]

where \( M > 0 \) is a some constant. By (3.1) – (3.7), we get

\[
\exp \{(1-\varepsilon)n\delta (a_1 z, \theta) r\} \leq |A_1^ne^{na_1z}|
\]
\[
\leq \left| \frac{f'}{f} \right| + |Q (e^{-z})| \left| \frac{f'}{f} \right| + |A_2^ne^{na_2z}| + \sum_{p=1}^{n-1} C_n^p |A_1^{n-p}e^{(n-p)a_1z}| |A_2^pe^{pa_2z}|
\]
\[
\leq r^{2(\sigma-1+\varepsilon)} + Mr^{\sigma-1+\varepsilon} + 2^n \exp \{(1+\varepsilon)(n-1)\delta (a_1 z, \theta) r\}
\]
\[
\leq M_1 r^{M_2} \exp \{(1+\varepsilon)(n-1)\delta (a_1 z, \theta) r\}, \quad (3.8)
\]

where \( M_1 > 0 \) and \( M_2 > 0 \) are some constants. By \( 0 < \varepsilon < \frac{1}{2(2n-1)} \) and (3.8), we have

\[
\exp \left\{ \frac{1}{2} \delta (a_1 z, \theta) r \right\} \leq M_1 r^{M_2}. \quad (3.9)
\]
By $\delta ( a_1 z, \theta ) > 0$ we know that (3.9) is a contradiction.

b) When $\delta ( a_1 z, \theta ) < 0$, $\delta ( a_2 z, \theta ) > 0$, using a proof similar to the above, we can also get a contradiction.

**Case 2:** Assume that $\arg a_1 \neq \pi$, $\arg a_1 = \arg a_2$ and $|a_2| > n |a_1|$, which is $\theta_1 \neq \pi$ and $\theta_1 = \theta_2$ and $|a_2| > n |a_1|.$

By Lemma 2.3, for the above $\varepsilon$, there is a ray $z = \theta$ such that $\theta \in ( -\frac{\pi}{2}, \frac{\pi}{2} ) \setminus ( E_1 \cup E_2 \cup E_7 )$ and $\delta ( a_1 z, \theta ) > 0$. Since $|a_2| > n |a_1|$ and $n \geq 2$, then $|a_2| > |a_1|$, thus $\delta ( a_2 z, \theta ) > \delta ( a_1 z, \theta ) > 0$. For sufficiently large $r$, we have by using Lemma 2.2

\[ A_2^n e^{n a_2 z} \geq \exp \left\{ (1 - \varepsilon) n \delta ( a_2 z, \theta ) r \right\}, \tag{3.10} \]

\[ A_1^n e^{n a_1 z} \leq \exp \left\{ (1 + \varepsilon) n \delta ( a_1 z, \theta ) r \right\}, \tag{3.11} \]

\[ A_1^n e^{n a_1 z} \leq \exp \left\{ (1 + \varepsilon) (n - 1) \delta ( a_1 z, \theta ) r \right\}, \tag{3.12} \]

\[ A_2^n e^{n a_2 z} \leq \exp \left\{ (1 + \varepsilon) (n - 1) \delta ( a_2 z, \theta ) r \right\}, \tag{3.13} \]

By (3.1), (3.2), (3.7) and (3.10) – (3.13) we get

\[ \exp \left\{ (1 - \varepsilon) n \delta ( a_2 z, \theta ) r \right\} \leq | A_2^n e^{n a_2 z} | \]

\[ \leq \left| \frac{f'}{f} \right| + \left| Q ( e^{-z} ) \right| \left| \frac{f'}{f} \right| + \left| A_1^n e^{n a_1 z} \right| + \sum_{p=1}^{n-1} C_n^p \left| A_1^n e^{(n-p) a_1 z} \right| \left| A_2^n e^{p a_2 z} \right| \]

\[ \leq r^{2(n-1+\varepsilon)} + M r^{n-1+\varepsilon} + \exp \left\{ (1 + \varepsilon) n \delta ( a_1 z, \theta ) r \right\} \]

\[ + 2^n \exp \left\{ (1 + \varepsilon) (n - 1) \delta ( a_1 z, \theta ) r \right\} \exp \left\{ (1 + \varepsilon) (n - 1) \delta ( a_2 z, \theta ) r \right\} \exp \left\{ (1 + \varepsilon) (n - 1) \delta ( a_2 z, \theta ) r \right\}. \tag{3.14} \]

Therefore, by (3.14), we obtain

\[ \exp \{ \alpha r \} \leq M_1 r^{M_2} \tag{3.15} \]

where

\[ \alpha = [1 - \varepsilon (2n - 1)] \delta ( a_2 z, \theta ) - (1 + \varepsilon) n \delta ( a_1 z, \theta ). \]

Since $0 < \varepsilon < \frac{|a_2| - n |a_1|}{2n|a_2| + n|a_1|}$, $\theta_1 = \theta_2$ and $\cos (\theta_1 + \theta) > 0$, then

\[ \alpha = [1 - \varepsilon (2n - 1)] |a_2| \cos (\theta_2 + \theta) - (1 + \varepsilon) n |a_1| \cos (\theta_1 + \theta) \]

\[ = \{|a_2| - n |a_1| - \varepsilon [(2n - 1)|a_2| + n |a_1|]\} \cos (\theta_1 + \theta) \]
Lemma 2.2

which is \( \theta \). By Lemma 2.3, for the above \( \varepsilon \), there is a ray \( z = \theta \) such that \( \theta \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \setminus (E_1 \cup E_5 \cup E_7) \) and \( \delta (a_2z, \theta) > 0 \). Because \( \cos \theta > 0 \), we have \( \delta (a_1z, \theta) = |a_1| \cos (\theta_1 + \theta) = -|a_1| \cos \theta < 0 \). For sufficiently large \( r \), we obtain by Lemma 2.2

\[
|A_2^n e^{na_2z}| > \exp \{(1 - \varepsilon) n \delta (a_2z, \theta) r \}, \tag{3.16}
\]

\[
|A_1^n e^{na_1z}| \leq \exp \{(1 - \varepsilon) n \delta (a_1z, \theta) r \} < 1, \tag{3.17}
\]

\[
\left| A_1^n - P e^{(n-p)a_1z} \right| \leq \exp \{(1 - \varepsilon) (n-p) \delta (a_1z, \theta) r \} < 1, \quad p = 1, \cdots, n - 1, \tag{3.18}
\]

\[
|A_2^n e^{pa_2z}| \leq \exp \{(1 + \varepsilon) (n-1) \delta (a_2z, \theta) r \}, \quad p = 1, \cdots, n - 1. \tag{3.19}
\]

Using the same reasoning as in Case 1(a), we can get a contradiction.

**Case 3:** Assume that \( a_1 < 0 \) and \( a_1 \neq a_2 \), which is \( \theta_1 = \pi \) and \( \theta_2 \neq \pi \).

By Lemma 2.3, for the above \( \varepsilon \), there is a ray \( z = \theta \) such that \( \theta \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \setminus (E_1 \cup E_5 \cup E_7) \) and \( \delta (a_2z, \theta) > 0 \). Because \( \cos \theta > 0 \), we have \( \delta (a_1z, \theta) = |a_1| \cos (\theta_1 + \theta) = -|a_1| \cos \theta < 0 \). For sufficiently large \( r \), we obtain by Lemma 2.2

\[
|A_2^n e^{na_2z}| > \frac{|a_2| - n|a_1|}{2} \cos (\theta_1 + \theta) > 0.
\]

Hence (3.15) is a contradiction.

**Case 4:** Assume that \( -\frac{1}{n} (|a_2| - m) < a_1 < 0, \) \( |a_2| > m \) and \( a_1 = a_2 \), which is \( \theta_1 = \theta_2 = \pi \) and \( |a_1| < \frac{1}{n} (|a_2| - m) \), then \( |a_2| > n|a_1| + m \), hence \( |a_2| > n|a_1| \).

By Lemma 2.3, for the above \( \varepsilon \), there is a ray \( z = \theta \) such that \( \theta \in \left( \frac{\pi}{2}, \frac{3\pi}{2} \right) \setminus (E_1 \cup E_5 \cup E_7) \), then \( \cos \theta < 0 \), \( \delta (a_1z, \theta) = |a_1| \cos (\theta_1 + \theta) = -|a_1| \cos \theta > 0 \), \( \delta (a_2z, \theta) = |a_2| \cos (\theta_2 + \theta) = -|a_2| \cos \theta < 0 \). Since \( |a_2| > n|a_1| \) and \( n \geq 2 \), then \( |a_2| > |a_1| \), thus \( \delta (a_2z, \theta) > \delta (a_1z, \theta) > 0 \), for sufficiently large \( r \), we get (3.10) – (3.13) hold. For \( \theta \in \left( \frac{\pi}{2}, \frac{3\pi}{2} \right) \) we have

\[
|Q (e^{-z})| \leq Me^{-mr \cos \theta}. \tag{3.20}
\]

By (3.1), (3.2), (3.10) – (3.13) and (3.20), we get

\[
\exp \{(1 - \varepsilon) n \delta (a_2z, \theta) r \} \leq |A_2^n e^{na_2z}|
\]

\[
\leq \left| \frac{f'}{f} + |Q (e^{-z})| \frac{f'}{f} \right| + \left| A_1^n e^{na_1z} \right| + \sum_{p=1}^{n-1} C_p \left| A_1^{n-p} e^{(n-p)a_1z} \right| \left| A_2^p e^{pa_2z} \right|
\]

\[
\leq r^{2(\sigma - 1 + \varepsilon)} + M r^{\sigma - 1 + \varepsilon} e^{-mr \cos \theta} + \exp \{(1 + \varepsilon) n \delta (a_1z, \theta) r \}
\]

\[
+ 2^n \exp \{(1 + \varepsilon) (n-1) \delta (a_1z, \theta) r \} \exp \{(1 + \varepsilon) (n-1) \delta (a_2z, \theta) r \}
\]
\[ \leq M_1 r^{M_2} e^{-m r \cos \theta} \exp \{(1 + \varepsilon) n \delta (a_1 z, \theta) r\} \exp \{(1 + \varepsilon) (n - 1) \delta (a_2 z, \theta) r\}. \]  
(3.21)

Therefore, by (3.21), we obtain

\[ \exp \{\beta r\} \leq M_1 r^{M_2}, \]  
(3.22)

where

\[ \beta = \left[1 - \varepsilon (2n - 1)\right] \delta (a_2 z, \theta) - (1 + \varepsilon) n \delta (a_1 z, \theta) + m \cos \theta. \]

Since \(|a_2| - n |a_1| - m > 0\), then

\[ 2 \left[(2n - 1) |a_2| + n |a_1|\right] > |a_2| - n |a_1| - m > 0. \]

Therefore,

\[ \frac{|a_2| - n |a_1| - m}{2 \left[(2n - 1) |a_2| + n |a_1|\right]} < 1. \]

Then, we can take \(0 < \varepsilon < \frac{|a_2| - n |a_1| - m}{2 \left[(2n - 1) |a_2| + n |a_1|\right]}. \) Since \(0 < \varepsilon < \frac{|a_2| - n |a_1| - m}{2 \left[(2n - 1) |a_2| + n |a_1|\right]}\), \(\theta_1 = \theta_2 = \pi\) and \(\cos \theta < 0\), then

\[ \beta = - \cos \theta \left\{ |a_2| - n |a_1| - m - \varepsilon \left[(2n - 1) |a_2| + n |a_1|\right]\right\} \]
\[ > - \frac{1}{2} \left(|a_2| - n |a_1| - m\right) \cos \theta > 0. \]

Hence, (3.22) is a contradiction. Concluding the above proof, we obtain \(\sigma(f) = +\infty.\)

**Second step:** We prove that \(\sigma_2(f) = 1.\) By

\[ \max\{\sigma(Q(e^{-z})), \sigma((A_1 e^{a_1 z} + A_2 e^{a_2 z})^n)\} = 1 \]

and the Lemma 2.4, we get \(\sigma_2(f) \leq 1.\) By Lemma 2.5, we know that there exists a set \(E_8 \subset (1, +\infty)\) with finite logarithmic measure and a constant \(B > 0,\) such that for all \(z\) satisfying \(|z| = r \notin [0, 1] \cup E_8,\) we get

\[ \left| \frac{f^{(j)}(z)}{f(z)} \right| \leq B |T(2r, f)|^{j+1} \quad (j = 1, 2). \]  
(3.23)

**Case 1:** \(\theta_1 \neq \pi\) and \(\theta_1 \neq \theta_2.\) In first step, we have proved that there is a ray \(\arg z = \theta\) where \(\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus (E_1 \cup E_6 \cup E_7),\) satisfying

\[ \delta (a_1 z, \theta) > 0, \delta (a_2 z, \theta) < 0 \text{ or } \delta (a_1 z, \theta) < 0, \delta (a_2 z, \theta) > 0. \]
a) When $\delta (a_1 z, \theta) > 0$, $\delta (a_2 z, \theta) < 0$, for sufficiently large $r$, we get (3.3) – (3.7) holds. By (3.1), (3.3) – (3.7) and (3.23), we obtain

$$\exp \{ (1 - \varepsilon) n\delta (a_1 z, \theta) r \} \leq |A_1^n e^{na_1 z}|$$

$$\leq \left| \frac{f''}{f} \right| + |Q(e^{-z})| \left| \frac{f'}{f} \right| + |A_2^n e^{na_2 z}| + \sum_{p=1}^{n-1} C_p^n A_1^n r^{-p} e^{(n-p) a_1 z} |A_2^n e^{na_2 z}|$$

$$\leq B |T(2r,f)|^3 + MB |T(2r,f)|^2 + 2^n \exp \{ (1 + \varepsilon) (n - 1) \delta (a_1 z, \theta) r \}$$

$$\leq M_1 \exp \{ (1 + \varepsilon) (n - 1) \delta (a_1 z, \theta) r \} |T(2r,f)|^3.$$  (3.24)

By $0 < \varepsilon < \frac{1}{2(2n-1)}$ and (3.24), we have

$$\exp \left\{ \frac{1}{2} \delta (a_1 z, \theta) r \right\} \leq M_1 |T(2r,f)|^3.$$  (3.25)

By $\delta (a_1 z, \theta) > 0$ and (3.25), we have $\sigma_2(f) \geq 1$, then $\sigma_2(f) = 1$.

b) When $\delta (a_1 z, \theta) < 0$, $\delta (a_2 z, \theta) > 0$, using a proof similar to the above, we can also get $\sigma_2(f) = 1$.

**Case 2:** $\theta_1 \neq \pi, \theta_1 = \theta_2$ and $|a_2| > n |a_1|$. In first step, we have proved that there is a ray $\text{arg} \ z = \theta$ where $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus (E_1 \cup E_6 \cup E_7)$, satisfying

$$\delta (a_2 z, \theta) > \delta (a_1 z, \theta) > 0$$

and for sufficiently large $r$, we get (3.7) and (3.10) – (3.13) hold. By (3.1), (3.7), (3.10) – (3.13) and (3.23), we get

$$\exp \{ \alpha r \} \leq M_1 |T(2r,f)|^3,$$  (3.26)

where

$$\alpha = \left[ 1 - \varepsilon (2n-1) \right] \delta (a_2 z, \theta) - (1 + \varepsilon) n\delta (a_1 z, \theta) > 0.$$  

By $\alpha > 0$ and (3.26), we have $\sigma_2(f) \geq 1$, then $\sigma_2(f) = 1$.

**Case 3:** $a_1 < 0$ and $\theta_1 \neq \theta_2$. In first step, we have proved that there is a ray $\text{arg} \ z = \theta$ where $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus (E_1 \cup E_6 \cup E_7)$, satisfying

$$\delta (a_2 z, \theta) > 0$$

and $\delta (a_1 z, \theta) < 0$ and for sufficiently large $r$, we get (3.16) – (3.19) hold. Using the same reasoning as in second step ([Case 1 (a)]), we can get $\sigma_2(f) = 1$. 
Case 4: $-\frac{1}{n}(|a_2| - m) < a_1 < 0$, $|a_2| > m$ and $\theta_1 = \theta_2$. In first step, we have proved that there is a ray $\arg z = \theta$ where $\theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \setminus (E_1 \cup E_6 \cup E_7)$, satisfying

$$
\delta (a_2z, \theta) > \delta (a_1z, \theta) > 0
$$

and for sufficiently large $r$, we get (3.10)–(3.13) hold. By (3.1), (3.10)–(3.13), (3.20) and (3.23) we obtain

$$
\exp \{ \beta r \} \leq M_1 |T(2r, f)|^3,
$$

(3.27)

where

$$
\beta = [1 - \varepsilon (2n - 1)] \delta (a_2z, \theta) - (1 + \varepsilon) n\delta (a_1z, \theta) + m \cos \theta > 0.
$$

By $\beta > 0$ and (3.27), we have $\sigma_2 (f) \geq 1$, then $\sigma_2 (f) = 1$. Concluding the above proof, we obtain $\sigma_2 (f) = 1$. The proof of Theorem 1.1 is complete.

Example 1.1 Consider the differential equation

$$
f'' + (-4e^{-3z} - 4ie^{z} - 1) f' + (ie^z + 2e^{-z})f = 0,
$$

(3.28)

where $Q (z) = -4z^3 - 4iz - 1$, $a_1 = 1$, $a_2 = -1$, $A_1 (z) = i$ and $A_2 (z) = 2$. Obviously, the conditions of Theorem 1.1 (1) are satisfied. The entire function

$$
f (z) = e^{ez},
$$

with $\sigma (f) = +\infty$ and $\sigma_2 (f) = 1$, is a solution of (3.28).

Example 1.2 Consider the differential equation

$$
f'' + \left(-8e^{-2z} - 12e^{iz}e^{-z} - 1 - 6e^{iz}\right) f' + \left(e^{2z}e^{2z} + 2e^{iz}\right)^3 f = 0,
$$

(3.29)

where $Q (z) = -8z^2 - 12e^{iz}z - 1 - 6e^{iz}$, $a_1 = \frac{\pi}{2}$, $a_2 = -\frac{1}{2}$, $A_1 (z) = e^{iz}$ and $A_2 (z) = 2$. Obviously, the conditions of Theorem 1.1 (1) are satisfied. The entire function

$$
f (z) = e^{ez},
$$

with $\sigma (f) = +\infty$ and $\sigma_2 (f) = 1$, is a solution of (3.29).

Example 1.3 Consider the differential equation

$$
f'' + \left(-e^{-3z} - 4e^{iz}e^{-2z} - 6ie-e^{-z} - 1 - 4e^{iz}\right) f' + \left(e^{-\frac{z}{2}} + e^{iz}e^{\frac{iz}{2}}\right)^4 f = 0,
$$

(3.30)

where $Q (z) = -z^3 - 4e^{iz}z^2 - 6iz - 1 - 4e^{iz}$, $a_1 = -\frac{1}{2}$, $a_2 = \frac{1}{2}$, $A_1 (z) = 1$ and $A_2 (z) = e^{iz}$. Obviously, the conditions of Theorem 1.1 (3) are satisfied. The entire function

$$
f (z) = e^{ez},
$$

with $\sigma (f) = +\infty$ and $\sigma_2 (f) = 1$, is a solution of (3.30).
4 Proof of Theorem 1.2

We prove that $\lambda (f - \varphi) = \lambda (f - \varphi) = \sigma (f) = +\infty$ and $\lambda_2 (f - \varphi) = \lambda_2 (f - \varphi) = \sigma_2 (f) = 1$. First, setting $\omega = f - \varphi$. Since $\sigma (\varphi) < \infty$, then we have $\sigma (\omega) = \sigma (f) = +\infty$. From (1.1), we have

$$\omega'' + Q (e^{-z}) \omega' + (A_1 e^{a_{1z}} + A_2 e^{a_{2z}}) \omega = H,$$

where $H = -[\varphi'' + Q (e^{-z}) \varphi' + (A_1 e^{a_{1z}} + A_2 e^{a_{2z}}) \varphi]$. Now we prove that $H \neq 0$. In fact if $H \equiv 0$, then

$$\varphi'' + Q (e^{-z}) \varphi' + (A_1 e^{a_{1z}} + A_2 e^{a_{2z}}) \varphi = 0.$$  

Hence $\varphi$ is a solution of equation (1.1) with $\sigma (\varphi) = \infty$ and by Theorem 1.1, it is a contradiction. Since $\sigma (f) = \infty$, $\sigma (\varphi) < \infty$ and $\sigma_2 (f) = 1$, we get $\sigma_2 (\omega) = \sigma_2 (f - \varphi) = \sigma_2 (f) = 1$. By the Lemma 2.6 and Lemma 2.7, we have $\lambda (\omega) = \sigma (\omega) = \sigma (f) = +\infty$ and $\lambda_2 (\omega) = \lambda_2 (\omega) = \sigma_2 (\omega) = \sigma_2 (f) = 1$, i.e., $\lambda (f - \varphi) = \lambda (f - \varphi) = \sigma (f) = +\infty$ and $\lambda_2 (f - \varphi) = \lambda_2 (f - \varphi) = \sigma_2 (f) = 1$.

5 Proof of Theorem 1.3

Suppose that $f \neq 0$ is a solution of equation (1.1), then $\sigma (f) = +\infty$ by Theorem 1.1. Since $\sigma (\varphi) < 1$, then by Theorem 1.2, we have $\lambda (f - \varphi) = +\infty$. Now we prove that $\lambda (f' - \varphi) = \infty$. Set $g_1 (z) = f' (z) - \varphi (z)$, then $\sigma (g_1) = \sigma (f') = \sigma (f) = \infty$. Set $B (z) = Q (e^{-z})$ and $R (z) = A_1 e^{a_{1z}} + A_2 e^{a_{2z}}$, then $B' (z) = -e^{-z} Q' (e^{-z})$ and $R' = (A_1 + a_1 A_1) e^{a_{1z}} + (A_2 + a_2 A_2) e^{a_{2z}}$. Differentiating both sides of equation (1.1), we have

$$f''' + B f'' + (B' + R') f' + n R' R^{n-1} f = 0. \tag{5.1}$$

By (1.1), we have

$$f = -\frac{1}{R^n} (f'' + B f'). \tag{5.2}$$

Substituting (5.2) into (5.1), we have

$$f''' + \left( B - n \frac{R'}{R} \right) f'' + \left( B' + R' - n B \frac{R'}{R} \right) f = 0. \tag{5.3}$$

Substituting $f' = g_1 + \varphi$, $f'' = g_1' + \varphi'$, $f''' = g_1'' + \varphi''$ into (5.3), we get

$$g_1'' + E_1 g_1' + E_0 g_1 = E, \tag{5.4}$$
Now we prove that $E \not\equiv 0$. In fact, if $E \equiv 0$, then we get
\[
\frac{\varphi''}{\varphi} R + \frac{\varphi'}{\varphi} (BR - nR') + B'R - nBR' + R^{n+1} = 0.
\] (5.5)

Obviously $\varphi''$, $\varphi'$ are meromorphic functions with $\sigma \left(\frac{\varphi''}{\varphi}\right) < 1$, $\sigma \left(\frac{\varphi'}{\varphi}\right) < 1$.

We can rewrite (5.5) in the form
\[
\sum_{k=0}^{m} f_k e^{(a_1 - k)z} + \sum_{l=0}^{m} h_l e^{(a_2 - l)z} + \sum_{p=1}^{n} \alpha_{p+1} A_{1} e^{[(n+1-p)a_1 + pa_2]z} + A_{1}^{n+1} e^{(n+1)a_1z} + A_{2}^{n+1} e^{(n+1)a_2z} = 0,
\] (5.6)

where $f_k$ ($k = 0, 1, \cdots, m$) and $h_l$ ($l = 0, 1, \cdots, m$) are meromorphic functions with $\sigma (f_k) < 1$ and $\sigma (h_l) < 1$. Set $I = \{a_1 - k (k = 0, 1, \cdots, m), a_2 - l (l = 0, 1, \cdots, m), (n+1-p) a_1 + pa_2 (p = 1, 2, \cdots, n), (n+1) a_1, (n+1) a_2\}$. By the conditions of the Theorem 1.1, it is clear that $(n+1) a_1 \neq a_1, (n+1) a_2, (n+1-p) a_1 + pa_2 (p = 1, 2, \cdots, n)$.

(i) If $(n+1) a_1 \neq a_1 - k (k = 1, \cdots, m), a_2 - l (l = 0, 1, \cdots, m)$, then we write (5.6) in the form
\[
A_{1}^{n+1} e^{(n+1)a_1z} + \sum_{\beta \in \Gamma_1} \alpha_{\beta} e^{\beta z} = 0,
\]
where $\Gamma_1 \subseteq I \setminus \{(n+1) a_1\}$. By Lemma 2.8 and Lemma 2.9, we get $A_1 \equiv 0$, it is a contradiction.

(ii) If $(n+1) a_1 = \gamma$ such that $\gamma \in \{a_1 - k (k = 1, \cdots, m), a_2 - l (l = 0, 1, \cdots, m)\}$, then $(n+1) a_2 \neq \beta$ for all $\beta \in I \setminus \{(n+1) a_2\}$. Hence, we write (5.6) in the form
\[
A_{2}^{n+1} e^{(n+1)a_2z} + \sum_{\beta \in \Gamma_2} \alpha_{\beta} e^{\beta z} = 0,
\]
where $\Gamma_2 \subseteq I \setminus \{(n+1) a_2\}$. By Lemma 2.8 and Lemma 2.9, we get $A_2 \equiv 0$, it is a contradiction. Hence, $E \not\equiv 0$ is proved. We know that the functions $E_1, E_0$ and $E$ are of finite order. By Lemma 2.6 and (5.4), we have $\lambda (g_1) = \lambda (f' - \varphi) = \infty$. 

where
\[
E_1 = B - n \frac{R'}{R}, \quad E_0 = B' + R' - n B \frac{R'}{R},
\]
\[
E = \left\{ \varphi'' + \left( B - n \frac{R'}{R} \right) \varphi' + \left( B' + R' - n B \frac{R'}{R} \right) \varphi \right\}.
\]
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Now we prove that \( \lambda (f'' - \varphi) = \infty \). Set \( g_2(z) = f''(z) - \varphi(z) \), then \( \sigma(g_2) = \sigma(f'') = \sigma(f) = \infty \). Differentiating both sides of equation (1.1), we have

\[
f^{(4)} + B f''' + (2B' + R^n) f'' + (B'' + 2nR'R^{n-1}) f' + n \left[R''R^{n-1} + (n-1) R'^2R^{n-2}\right] f = 0. \tag{5.7}
\]

Combining (5.2) with (5.7), we get

\[
f^{(4)} + B f''' + \left(2B' + R^n - n \frac{R''}{R} - n(n-1) \frac{R'^2}{R^2}\right) f'' + \left(B'' + 2nR'R^{n-1} - nB \frac{R''}{R} - n(n-1) B \frac{R'^2}{R^2}\right) f' = 0. \tag{5.8}
\]

Now we prove that \( B' + R^n - nB \frac{R'}{R} \neq 0 \). Suppose that \( B' + R^n - nB \frac{R'}{R} \equiv 0 \), then we have

\[
B'R + R^{n+1} - nBR' = 0. \tag{5.9}
\]

We can write (5.9) in the form (5.6), then by the same reasoning as in the proof of \( \lambda (f' - \varphi) = \infty \) we get a contradiction. Hence \( B' + R^n - nB \frac{R'}{R} \neq 0 \) is proved. Set

\[
\psi(z) = B'R + R^{n+1} - nBR', \tag{5.10}
\]

\[
S_1 = 2B'R^2 + R^{n+2} - nR''R - n(n-1) R'^2, \tag{5.11}
\]

\[
S_2 = B''R^2 + 2nR'R^{n+1} - nBR''R - n(n-1) BR'^2, \tag{5.12}
\]

\[
S_3 = BR - nR'. \tag{5.13}
\]

By (5.3), (5.10) and (5.13), we get

\[
f' = -\frac{R}{\psi(z)} \left(f''' + \frac{S_3}{R} f''\right). \tag{5.14}
\]

By (5.14), (5.11), (5.12) and (5.8), we obtain

\[
f^{(4)} + \left(B - \frac{S_2}{R \psi(z)}\right) f''' + \left(S_1 \frac{1}{R^2} - \frac{S_2 S_3}{R^2 \psi(z)}\right) f'' = 0. \tag{5.15}
\]

Substituting \( f'' = g_2 \), \( f''' = g_2' \), \( f^{(4)} = g_2'' + \varphi'' \) into (5.15) we get

\[
g''_2 + H_1 g'_2 + H_0 g_2 = H, \tag{5.16}
\]

where

\[
H_1 = B - \frac{S_2}{R \psi(z)}, \quad H_0 = \frac{S_1}{R^2} - \frac{S_2 S_3}{R^2 \psi(z)},
\]
where

\[ f = J \]

meromorphic functions with \( \sigma \) obviously, \( \phi \)

\[ \phi = 50 \]

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where

\[ \phi = 1. \] By (5.18), (5.19) and (5.21), we can rewrite (5.22) in the form

\[ A_1^{2n+2} e^{(2n+2)\alpha_1 z} + A_2^{2n+2} e^{(2n+2)\alpha_2 z} + \sum_{p=1}^{2n+1} C_{2n+2}^p A_1^{2n+2-p} A_2^p e^{[(2n+2-p)\alpha_1 + \alpha_2]z} + \sum_{0 \leq p \leq 2, 0 \leq k \leq 2m} f_{p,k} e^{[(2-p)\alpha_1 + \alpha_2 - k]z} + \sum_{0 \leq p \leq n+2, 0 \leq k \leq m} h_{p,k} e^{[(n+2-p)\alpha_1 + \alpha_2 - k]z} = 0, \]

where \( f_{p,k} \) and \( h_{p,k} \) are meromorphic functions with \( \sigma(f_{p,k}) < 1 \) and \( \sigma(h_{p,k}) < 1 \). Set \( J = \{(2n+2)\alpha_1, (2n+2)\alpha_2, (2n+2-p)\alpha_1 + \alpha_2 (p = 1, 2, \cdots, 2n+1), (2-p)\alpha_1 + \alpha_2 - k (p = 0, 1, 2; k = 0, \cdots, m), (n+2-p)\alpha_1 + \alpha_2 - k (p = 0, 1, \cdots, n; k = 0, 1, \cdots, m)\} \). By the conditions of Theorem 1.3, it is clear that \( (2n+2)\alpha_1 \neq (2n+2)\alpha_2, (2n+2-p)\alpha_1 + \alpha_2 (p = 1, 2, \cdots, 2n+1), \)
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2a_1, (n + 2) a_1 and (2n + 2) a_2 \neq (2n + 2) a_1, (2n + 2 - p) a_1 + pa_2 (p = 1, 2, \cdots, 2n + 1), 2a_2, (n + 2) a_2.

(1) By the conditions of Theorem 1.3 (i), we have (2n + 2) a_1 \neq \beta for all \beta \in J \setminus \{(2n + 2) a_1\}, hence we write (5.23) in the form

A_1^{2n+2} e^{(2n+2)a_1 z} + \sum_{\beta \in \Gamma_1} \alpha_{\beta} e^{\beta z} = 0,

where \Gamma_1 \subseteq J \setminus \{(2n + 2) a_1\}. By Lemma 2.8 and Lemma 2.9, we get A_1 \equiv 0, it is a contradiction.

(2) By the conditions of Theorem 1.3 (ii), we have (2n + 2) a_2 \neq \beta for all \beta \in J \setminus \{(2n + 2) a_2\}, hence we write (5.23) in the form

A_2^{2n+2} e^{(2n+2)a_2 z} + \sum_{\beta \in \Gamma_2} \alpha_{\beta} e^{\beta z} = 0,

where \Gamma_2 \subseteq J \setminus \{(2n + 2) a_2\}. By Lemma 2.8 and Lemma 2.9, we get A_2 \equiv 0, it is a contradiction. Hence, H \not\equiv 0 is proved. We know that the functions H_1, H_0 and H are of finite order. By Lemma 2.6 and (5.16), we have \overline{\lambda}(g_2) = \overline{\lambda}(f'' - \varphi) = \infty. The proof of Theorem 1.3 is complete.

References


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