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Controlled G-Frames and Their G-Multipliers in Hilbert spaces

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Abstract

Multipliers have been recently introduced by P. Balazs as operators for Bessel sequences and frames in Hilbert spaces. These are operators that combine (frame-like) analysis, a multiplication with a fixed sequence (called the symbol) and synthesis. One of the last extensions of frames is weighted and controlled frames that introduced by P.Balazs, J-P. Antoine and A. Grybos to improve the numerical efficiency of iterative algorithms for inverting the frame operator. Also g-frames are the most popular generalization of frames that include almost all of the frame extensions. In this manuscript the concept of the controlled g-frames will be defined and we will show that controlled g-frames are equivalent to g-frames and so the controlled operators C and C' can be used as preconditions in applications. Also the multiplier operator for this family of operators will be introduced and some of its properties will be shown.

1 Introduction

In [30], R. Schatten provided a detailed study of ideals of compact operators by using their singular decomposition. He investigated the operators of the form $\sum_k \lambda_k \varphi_k \otimes \overline{\psi_k}$ where (ϕ_k) and (ψ_k) are orthonormal families. In [3], the orthonormal families were replaced with Bessel and frame sequences to define Bessel and frame multipliers.

Key Words: Frame; g-frame, g-Bessel, g-Riesz basis, g-orthonormal basis, multiplier, Schatten p-class, Hilbert-Schmidt, trace class, controlled frame, weighted frame, controlled g-frame, (C, C')-controlled g-frame, (C, C')-controlled g-multiplier operator.
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Definition 1.1. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, let $(\psi_k) \subseteq \mathcal{H}_1$ and $(\phi_k) \subseteq \mathcal{H}_2$ be Bessel sequences. Fix $m = (m_k) \in l^\infty$. The operator $\mathbf{M}_{m,(\phi_k),(\psi_k)} : \mathcal{H}_1 \to \mathcal{H}_2$ defined by*

$$\mathbf{M}_{m,(\phi_k),(\psi_k)}(f) = \sum_k m_k \langle f, \psi_k \rangle \phi_k$$

*is called the **Bessel multiplier** for the Bessel sequences (ψ_k) and (ϕ_k). The sequence m is called the symbol of $\mathbf{M}$.*

Several basic properties of these operators were investigated in [3]. Multipliers are not only interesting from a theoretical point of view, see e.g. [3, 11, 14], but they are also used in applications, in particular in the field of audio and acoustic. They have been investigated for fusion frames [2], for generalized frames [28], p-frames in Banach spaces [29] and for Banach frames [15, 17]. In signal processing they are used for Gabor frames under the name of Gabor filters [22], in computational auditory scene analysis they are known by the name of time-frequency masks [23]. In real-time implementation of filtering system they approximate time-invariant filters [5]. As a particular way to implement time-variant filters they are used for example for sound morphing [10] or psychoacoustical modeling [6].

G-frames, introduced by W. Sun in [31] and improved by the first author [27], are a natural generalization of frames which cover many other extensions of frames, e.g. bounded quasi-projectors [18, 19], pseudo-frames [21], frame of subspaces or fusion frames [8], outer frames [1], oblique frames [9, 13], and a class of time-frequency localization operators [12]. Also it was shown that g-frames are equivalent to stable spaces splitting studied in [26]. All of these concepts are proved to be useful in many applications. Multipliers for g-frames introduced in [28] and some of its properties investigated.

Weighted and controlled frames have been introduced recently to improve the numerical efficiency of iterative algorithms for inverting the frame operator on abstract Hilbert spaces [4], however they are used earlier in [7] for spherical wavelets. In this manuscript the concept of controlled g- frame will be defined and we will show that any controlled g-frame is equivalent a g-frame and the role of controller operators are like the role of preconditions matrices or operators in linear algebra. Furthermore the multiplier operator for these family will be investigated.

The paper is organized as follows. In Section 2 we fix the notations of this paper, summarize known and prove some new results needed for the rest of the paper. In Section 3 we will define the concept of controlled g-frames and we will show that a controlled g-frame is equivalent to a g-frame and so the controlling operators can be used as precondition matrices in the problems related to applications. In section 4 we will define multipliers of controlled g-frame operators and we will prove some of its properties.

2 Preliminaries

Now we state some notations and theorems which are used in the present paper. Through this paper, $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces and $\{\mathcal{H}_i : i \in I\}$ is a sequence of Hilbert spaces, where I is a subset of Z. $\mathcal{L}(\mathcal{H},\mathcal{K})$ and $\mathcal{L}(\mathcal{H})$ is the collection of all bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$ and $\mathcal{H}$ respectively.

A bounded operator T is called *positive* (respectively *non-negative*), if $\langle Tf, f\rangle > 0$ for all $f \neq 0$ (respectively $\langle Tf, f\rangle \geq 0$ for all f). Every non-negative operator is clearly self-adjoint. For $T_1, T_2 \in \mathcal{L}(\mathcal{H})$, we write $T_1 \leq T_2$ whenever

$$\langle T_1(f), f\rangle \leq \langle T_2(f), f\rangle, \quad \forall f \in \mathcal{H}.$$

If $U \in \mathcal{L}(\mathcal{H})$ is non-negative, then there exists a unique non-negative operator V such that $V^2 = U$. Furthermore V commutes with every operator that commute with U. This will be denoted by $V = U^{\frac{1}{2}}$. Let $\mathcal{GL}(\mathcal{H})$ be the set of all bounded operators with a bounded inverse and $\mathcal{GL}^+(\mathcal{H})$ be the set of positive operators in $\mathcal{GL}(\mathcal{H})$. For $U \in \mathcal{L}(\mathcal{H})$, $U \in \mathcal{GL}^+(\mathcal{H})$ if and only if there exists $0 < m \leq M < \infty$ such that

$$m \leq U \leq M.$$

For U^{-1} we have

$$M^{-1} \leq U^{-1} \leq m^{-1}.$$

The following theorem can be found in [20].

Theorem 2.1. *Let $T_1, T_2, T_3 \in \mathcal{L}(\mathcal{H})$ and $T_1 \leq T_2$. Suppose $T_3 > 0$ commutes with T_1 and T_2 then*

$$T_1T_3 \leq T_2T_3.$$

Recall that if T is a compact operator on a separable Hilbert space $\mathcal{H}$, then in [30] it is proved that there exist orthonormal sets $\{e_n\}$ and $\{\sigma_n\}$ in $\mathcal{H}$ such that

$$Tx = \sum_n \lambda_n \langle x, e_n\rangle \sigma_n,$$

for $x \in \mathcal{H}$, where λ_n is the n-th singular value of T. Given $0 < p < \infty$, the **Schatten p-class** of $\mathcal{H}$ [30], denoted $\mathcal{S}_p$, is the space of all compact operators T on $\mathcal{H}$ with the singular value sequence $\{\lambda_n\}$ belonging to ℓ^p. It was shown that [32] , $\mathcal{S}_p$ is a Banach space with the norm

$$\|T\|_p = [\sum_n |\lambda_n|^p]^{\frac{1}{p}}. \tag{1}$$

$\mathcal{S}_1$ is called the *trace class* of $\mathcal{H}$ and $\mathcal{S}_2$ is called the *Hilbert-Schmidt class.* $T \in \mathcal{S}_p$ if and only if $T^p \in \mathcal{S}_1$. Moreover $\|T\|_p^p = \|T^p\|_1$. Also, $T \in \mathcal{S}_p$ if and only if $|T|^p = (T^*T)^{\frac{p}{2}} \in \mathcal{S}_1$ if and only if $T^*T \in \mathcal{S}_{\frac{p}{2}}$. Moreover, $\|T\|_p^p = \|T^*\|_p^p = \||T|\|_p^p = \||T|^p\|_1 = \|T^*T\|_{\frac{p}{2}}$.

2.1 G-Frames

For any sequence $\{ \mathcal{H}_i : i \in I\}$, we can assume that there exits a Hilbert space $\mathcal{K}$ such that for all $i \in I, \mathcal{H}_i \subseteq \mathcal{K}$ (for example $\mathcal{K} = (\bigoplus_{i\in I} \mathcal{H}_i)_{\ell^2}$).

Definition 2.2. *A sequence $\Lambda = \{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ is called generalized frame, or simply a g-**frame**, for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in I\}$ if there exist constants $A > 0$ and $B < \infty$ such that*

$$A\|f\|^2 \leq \sum_{i\in I} \|\Lambda_i f\|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H}. \tag{2}$$

The numbers A and B are called g-frame bounds.

$\Lambda = \{\Lambda_i : i \in I\}$ is called *tight* g-frame if $A = B$ and *Parseval* g-frame if $A = B = 1$. If the second inequality in (2) holds, the sequence is called g-Bessel sequence.

$\Lambda = \{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ is called a g-frame sequence, if it is a g-frame for $\overline{span}\{\Lambda_i^*(\mathcal{H}_i)\}_{i\in I}$.

It is easy to see that, if $\{f_i\}_{i\in I}$ is a frame for $\mathcal{H}$ with bounds A and B, then by putting $\mathcal{H}_i = C$ and $\Lambda_i(\cdot) = \langle \cdot, f_i\rangle$, the family $\{\Lambda_i : i \in I\}$ is a g-frame for $\mathcal{H}$ with bounds A and B.

Let

$$\Big(\bigoplus_{i\in I} \mathcal{H}_i\Big)_{\ell_2} = \Big\{ \{f_i\}_{i\in I} \mid f_i \in \mathcal{H}_i, \forall i \in I \quad and \quad \sum_{i\in I} \|f_i\|^2 < +\infty\Big\}. \tag{3}$$

Proposition 2.3. *([25]) $\Lambda = \{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ is a g-Bessel sequence for $\mathcal{H}$ with bound B, if and only if the operator*

$$T_\Lambda : \Big(\bigoplus_{i\in I} \mathcal{H}_i\Big)_{\ell_2} \longrightarrow \mathcal{H}$$

defined by

$$T_\Lambda(\{f_i\}_{i\in I}) = \sum_{i\in I} \Lambda_i^*(f_i)$$

is a well-defined and bounded operator with $\|T_{\mathbf{\Lambda}}\| \leq \sqrt{B}$. Furthermore

$$T_\Lambda^* : \mathcal{H} \longrightarrow \Big(\bigoplus \mathcal{H}_i\Big)_{\ell_2}$$

$$T_\Lambda^*(f) = \{\Lambda_i f\}_{i\in I}.$$

If $\Lambda = \{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g-frame, the operators T_Λ and T^*_Λ in Proposition 2.3 are called **synthesis operator** and **analysis operator** of $\Lambda = \{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$, respectively.

Proposition 2.4. *([24]) $\Lambda = \{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g-frame for $\mathcal{H}$ if and only if the synthesis operator T_Λ is well-defined, bounded and onto.*

We use some results which are proved in the context of pair frames [15, 16, 17]. The ordinary version of the next theorem which is proved in [15], can be extended easily to the general case.

Theorem 2.5. *([15]) $\Lambda = \{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g-Bessel sequence for $\mathcal{H}$ if and only if the operator*

$$S_\Lambda : \mathcal{H} \longrightarrow \mathcal{H}, \quad S_\Lambda = \sum_{i \in I} \Lambda_i^* \Lambda_i f, \tag{4}$$

is a welldefined operator. In this case S_Λ is bounded.

Theorem 2.6. *([16]) $\Lambda = \{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g-frame for $\mathcal{H}$ if and only if the operator*

$$S_\Lambda : \mathcal{H} \longrightarrow \mathcal{H}, \quad S_\Lambda = \sum_{i \in I} \Lambda_i^* \Lambda_i f,$$

is a welldefined invertible operator. In this case S_Λ is bounded.

S_Λ is called the g**-frame operator** of $\Lambda = \{\Lambda_i : i \in I\}$ and it is known [25] that S_Λ is a positive and

$$AI \leq S_\Lambda \leq BI,$$

where A and B are the frame bounds. Every $f \in \mathcal{H}$ has an expansion $f = \sum_i \Lambda_i^* \Lambda_i S_\Lambda^{-1} f$. One of the most important advantages of g-frames is a resolution of identity $\sum_i \Lambda_i^* \Lambda_i S_\Lambda^{-1} = I$.

2.2 Multipliers of g-frames

The concept of multipliers for g-Bessel sequences introduced by the first author in [28] and some of their properties will be shown.

Definition 2.7. *Let $\Lambda = \{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\Theta = \{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be g-Bessel sequences. If for $m = \{m_i\} \subseteq C$, the operator*

$$\mathbf{M} = \mathbf{M}_{m,\Lambda,\Theta} : \mathcal{H} \to \mathcal{H}$$

$$\mathbf{M}(f) = \sum_i m_i \Lambda_i^* \Theta_i f \tag{5}$$

*is welldefined, then $\mathbf{M}$ is called the g-**multiplier** of Λ,Θ and m.*

If $m = (m_i) = (1,1,1,...)$ and $\mathbf{M} = I$, (Λ, Θ) is called a *pair dual* (i.e. $I = \sum_{i\in I} \Lambda_i^* \Theta_i$).

Let $\{\lambda_i\}$ and $\{\varphi_i\}$ be Bessel sequences and $m \in \ell^\infty$, consider the corresponding g-Bessel sequences $\Lambda_i \cdot = \langle \cdot, \lambda_i \rangle$ and $\Theta_i \cdot = \langle \cdot, \varphi_i \rangle$. For any $f \in \mathcal{H}$ we have:

$$\mathbf{M}_{m,\Lambda,\Theta}(f) = \mathbf{M}_{m,(\lambda_k),(\phi_k)}(f) = \sum_i m_i \langle f, \varphi_i \rangle \lambda_i.$$

It is easy to show that the adjoint of $\mathbf{M}_{m,\Lambda,\Theta}$ is $\mathbf{M}_{\overline{m},\Theta,\Lambda}$.

Lemma 2.8. *([28]) If $\Theta = \{ \Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ is a g-Bessel sequence with bound B_Θ and $m = (m_i) \in \ell^\infty$, then $\{m_i \Theta_i\}_{i\in I}$ is a g-Bessel sequence with bound $\|m\|_\infty B_\Theta$.*

Like weighted frames [4], $\{m_i \Theta_i\}_{i\in I}$ can be called *weighted g- frame (g-Bessel).* By using the synthesis and the analysis operators of Λ and $m\Theta$, respectively, we can write

$$\mathbf{M}_{m,\Lambda,\Theta} f = \sum_i m_i \Lambda_i^* \Theta_i f = \sum_i \Lambda_i^* (m_i \Theta_i) f = T_\Lambda \{m_i \Theta_i f\} = T_\Lambda T_{m\Theta}^* f.$$

So

$$\mathbf{M}_{m,\Lambda,\Theta} = T_\Lambda T_{m\Theta}^*. \tag{6}$$

If we define the diagonal operator

$$D_m : \Big(\bigoplus \mathcal{H}_i\Big)_{\ell_2} \to \Big(\bigoplus \mathcal{H}_i\Big)_{\ell_2},$$

$$D_m((\xi_i)) = (m_i \xi_i)_{i\in I} \tag{7}$$

then

$$\mathbf{M}_{m,\Lambda,\Theta} = T_\Lambda D_m T_\Theta^*. \tag{8}$$

The notations in (6), (7) and (8) were used for proving the following propositions in [28] .

Proposition 2.9. *([28]) Let $m \in \ell^\infty$, $\Lambda = \{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ be a g-Riesz base and $\Theta = \{ \Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ be a g-Bessel sequence. The map $m \to \mathbf{M}_{m,\Lambda,\Theta}$ is injective.*

Proposition 2.10. *([28]) Let $\Lambda = \{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ and $\Theta = \{ \Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ be g-Bessel sequences for $\mathcal{H}$. If $m = (m_i) \in c_0$ and $(rank\Theta_i) \in \ell^\infty$, then $\mathbf{M}_{m,\Lambda,\Theta}$ is compact.*

Proposition 2.11. *([28]) Let* $\Lambda = \{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ *and* $\Theta = \{ \Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ *be g-Bessel sequences for* $\mathcal{H}$. *If* $m = (m_i) \in \ell^p$ *and* $(dim\mathcal{H}_i)_{i\in I} \in \ell^\infty$, *then* $\mathbf{M}_{m,\Lambda,\Theta}$ *is a Schatten p-class operator.*

Corollary 2.12. *([28]) Let* $\Lambda = \{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ *and* $\Theta = \{ \Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ *be g-Bessel sequences for* $\mathcal{H}$.

1. *If* $m = (m_i) \in \ell^1$ *and* $(dim\mathcal{H}_i)_{i\in I} \in \ell^\infty$, *then* $\mathbf{M}_{m,\Lambda,\Theta}$ *is a trace- class operator.*
2. *If* $m = (m_i) \in \ell^2$ *and* $(dim\mathcal{H}_i)_{i\in I} \in \ell^\infty$, *then* $\mathbf{M}_{m,\Lambda,\Theta}$ *is a Hilbert-Schmit operator.*

3 Controlled g-frames

Weighted and controlled frames have been introduced recently to improve the numerical efficiency of iterative algorithms for inverting the frame operator. In [4], it was shown that the controlled frames are equivalent to standard frames and it was used in the sense of preconditioning.

In this section, the concepts of controlled frames and controlled Bessel sequences will be extended to g-frames and we will show that controlled g-frames are equivalent g-frames.

Definition 3.1. *Let* $C, C' \in \mathcal{GL}^+(\mathcal{H})$. *The family* $\Lambda = \{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ *will be called a* (C, C')*-controlled g-frame for* $\mathcal{H}$, *if* Λ *is a g-Bessel sequence and there exists constants* $A > 0$ *and* $B < \infty$ *such that*

$$A\|f\|^2 \leq \sum_{i\in I} \langle \Lambda_i C f, \Lambda_i C^{'} f\rangle \leq B\|f\|^2, \quad \forall f \in \mathcal{H}. \tag{9}$$

A and B will be called controlled frame bounds. *If* $C' = I$, *we call* $\Lambda = \{ \Lambda_i \}$ *a* C***-controlled*** g***-frame*** *for* $\mathcal{H}$ *with bounds* A *and* B. *If the second part of the above inequality holds, it will be called* (C, C')***-controlled*** g***-Bessel sequence*** *with bound* B.

The proof of the following lemmas is straightforward.

Lemma 3.2. *Let* $C \in \mathcal{GL}^+(\mathcal{H})$. *The g-Bessel sequence* $\Lambda = \{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ *is* (C, C)*-controlled Bessel sequence (or* (C, C)*-controlled g-frame) if and only if there exists constant* $B < \infty$ *(and* $A > 0$ *)such that*

$$\sum_{i\in I} \|\Lambda_i C f\|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H}$$

$$(\text{ or } A\|f\|^2 \leq \sum_{i\in I} \|\Lambda_i C f\|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H}).$$

We call the (C, C)-controlled Bessel sequence and (C, C)-controlled g-frame, C^2-controlled Bessel sequence and C^2-controlled g-frame with bounds A, B.

Lemma 3.3. *For $C, C' \in \mathcal{GL}^+(\mathcal{H})$, the family $\Lambda = \{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ is a (C, C')-controlled Bessel sequence for $\mathcal{H}$ if and only if the operator*

$$L_{C\Lambda C'} : \mathcal{H} \to \mathcal{H}, \quad L_{C\Lambda C'} f := \sum_{i \in I} C' \Lambda_i^* \Lambda_i C f,$$

is well defined and there exists constant $B < \infty$ such that

$$\sum_{i \in I} \langle \Lambda_i C f, \Lambda_i C^{'} f \rangle \leq B \|f\|^2, \quad \forall f \in \mathcal{H}.$$

The operator

$$L_{C\Lambda C'} : \mathcal{H} \to \mathcal{H}, \quad L_{C\Lambda C'} f := \sum_{i \in I} C' \Lambda_i^* \Lambda_i C f,$$

is called the (C, C')-controlled Bessel sequence operator, also $L_{C\Lambda C'} = C S_\Lambda C'$. It follows from the definition that for a g-frame, this operator is positive and invertible and

$$AI \leq L_{C\Lambda C'} \leq BI.$$

Also, if C and C' commute with each other, then C', C'^{-1}, C, C^{-1} commute with $L_{C\Lambda C'}, C S_\Lambda, S_\Lambda C'$.

The following proposition shows that any g-frame is a controlled g-frame and versa. This is the most important advantage of weighted and controlled g-frame in the sense of precondition.

Proposition 3.4. *Let $C \in \mathcal{GL}^+(\mathcal{H})$. The family $\Lambda = \{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ is a g-frame if and only if Λ is a C^2-controlled g-frame.*

Proof 3.5. *Suppose that Λ is a C^2-controlled g-frame with bounds A, B. Then*

$$A\|f\|^2 \leq \sum_{i \in I} \|\Lambda_i C f\|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H},$$

For $f \in \mathcal{H}$

$$A\|f\|^2 = A\|CC^{-1}f\|^2 \leq A\|C\|^2\|C^{-1}f\|^2 \leq \|C\|^2 \sum_{i \in I} \|\Lambda_i C C^{-1} f\|^2$$

$$= \|C\|^2 \sum_{i \in I} \|\Lambda_i f\|^2.$$

Hence

$$A\|C\|^{-2}\|f\|^2 \leq \sum_{i\in I}\|\Lambda_i f\|^2, \quad \forall f \in \mathcal{H}.$$

On the other hand for every $f \in \mathcal{H}$,

$$\sum_{i\in I}\|\Lambda_i f\|^2 = \sum_{i\in I}\|\Lambda_i CC^{-1}f\|^2 \leq B\|C^{-1}f\|^2 \leq B\|C^{-1}\|^2\|f\|^2.$$

These inequalities yields that Λ *is a g-frame with bounds* $A\|C\|^{-2}, B\|C^{-1}\|^2$. *For the converse assume that* Λ *is g-frame with bounds* A', B'. *Then for all* $f \in \mathcal{H}$,

$$A'\|f\|^2 \leq \sum_{i\in I}\|\Lambda_i f\|^2 \leq B'\|f\|^2.$$

So for $f \in \mathcal{H}$,

$$\sum_{i\in I}\|\Lambda_i Cf\|^2 \leq B'\|Cf\|^2 \leq B'\|C\|^2\|f\|^2.$$

For lower bound, the g-frameness of Λ *shows that for any if* $f \in \mathcal{H}$,

$$A'\|f\|^2 = A'\|C^{-1}Cf\|^2 \leq A'\|C^{-1}\|^2\|Cf\|^2 \leq \|C^{-1}\|^2\sum_{i\in I}\|\Lambda_i Cf\|^2.$$

Therefore Λ *is a* C^2*-controlled g-frame with bounds* $A'\|C^{-1}\|^{-2}, B'\|C\|^2$.

Proposition 3.6. *Assume that* $\Lambda = \{\Lambda_i : i \in I\}$ *is a g-frame and* $C, C' \in \mathcal{GL}^+(\mathcal{H})$, *which commute with each other and commute with* S_Λ. *Then* $\Lambda = \{\Lambda_i : i \in I\}$ *is a* (C, C')*-controlled g-frame.*

Proof 3.7. *Let* Λ *be g-frame with bounds* A, B *and* $m, m' > 0, M, M' < \infty$ *so that*

$$m \leq C \leq M, \quad m' \leq C' \leq M'.$$

Then

$$mA \leq CS_\Lambda \leq MB,$$

because C *commute with* S_Λ. *Again* C' *commutes with* CS_Λ *and then*

$$mm'A \leq L_{C\Lambda C'} \leq MM'B.$$

4 Multipliers of Controlled g-frames

Extending the concept of multipliers of frames, in this section, we will define controlled g-frame's multiplier for C-controlled g-frames in Hilbert spaces. The definition of general case (C, C')-controlled g-frames goes smooth.

Lemma 4.1. *Let $C, C' \in \mathcal{GL}^+(\mathcal{H})$ and $\Lambda = \{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$, $\Theta = \{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be ${C'}^2$ and C^2-controlled g-Bessel sequences for $\mathcal{H}$, respectively. Let $m \in \ell^\infty$. The operator*

$$\boldsymbol{M}_{mC\Theta\Lambda C'} : \mathcal{H} \to \mathcal{H}$$

defined by

$$\boldsymbol{M}_{mC\Theta\Lambda C'} f := \sum_{i\in I} m_i C\Theta_i^* \Lambda_i C' f$$

is a well-defined bounded operator.

Proof 4.2. *Assume $\Lambda = \{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$, $\Theta = \{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be ${C'}^2$ and C^2-controlled g-Bessel sequences for $\mathcal{H}$ with bounds B, B', respectively. For any $f, g \in \mathcal{H}$ and finite subset $J \subset I$,*

$$\begin{aligned}
\|\sum_{i\in J} m_i C\Theta_i^* \Lambda_i C' f\| &= \sup_{g\in\mathcal{H}, \|g\|=1} \|\sum_{i\in J} m_i \langle \Lambda_i C' f, \Theta_i C^* g\rangle\| \\
&\leq \sup_{g\in\mathcal{H}, \|g\|=1} \sum_{i\in J} |m_i| \|\Lambda_i C' f\| \|\Theta_i C^* g\| \\
&\leq \sup_{g\in\mathcal{H}, \|g\|=1} \|m\|_\infty \big(\sum_{i\in I} \|\Theta_i C^* g\|^2\big)^{\frac{1}{2}} \big(\sum_{i\in J} \|\Lambda_i C' f\|^2 \\
&\leq \|m\|_\infty \sqrt{BB'} \|f\|
\end{aligned}$$

This shows that $\boldsymbol{M}_{mC\Theta\Lambda C'}$ is well-defined and

$$\|\boldsymbol{M}_{mC\Theta\Lambda C'}\| \leq \|m\|_\infty \sqrt{BB'}.$$

Above Lemma ia a motivation to define the following definition.

Definition 4.3. *Let $C, C' \in \mathcal{GL}^+(\mathcal{H})$ and $\Lambda = \{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$, $\Theta = \{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be ${C'}^2$ and C^2-controlled g-Bessel sequences for $\mathcal{H}$, respectively. Let $m \in \ell^\infty$. The operator*

$$\boldsymbol{M}_{mC\Theta\Lambda C'} : \mathcal{H} \to \mathcal{H}$$

defined by

$$\boldsymbol{M}_{mC\Theta\Lambda C'} f := \sum_{i\in I} m_i C\Theta_i^* \Lambda_i C' f, \tag{10}$$

is called the (C, C')-controlled multiplier operator with symbol m.

By using representations (6) and (8), we have

$$\mathbf{M}_{mC\Theta\Lambda C'} = C\mathbf{M}_{m\Theta\Lambda}C' = CT_\Theta D_m T_\Lambda^* C'.$$

The proof of Proposition 4.7. of [28] shows that if $m = (m_i) \in \ell^p$ and $(dim\mathcal{H}_i)_{i\in I} \in \ell^\infty$, then the diagonal operator D_m is a Schatten p-class operator. Since $\mathcal{S}_p$ is a $*$-ideal of $\mathcal{L}(\mathcal{H})$ so we have:

Theorem 4.4. *Let* $\Lambda = \{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ *and* $\Theta = \{ \Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ *be controlled g-Bessel sequences for* $\mathcal{H}$. *If* $m = (m_i) \in \ell^p$ *and* $(dim\mathcal{H}_i)_{i\in I} \in \ell^\infty$, *then* $\boldsymbol{M}_{mC\Theta\Lambda C'}$ *is a Schatten p-class operator.*

And

Corollary 4.5. *Let* $\Lambda = \{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ *and* $\Theta = \{ \Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ *be controlled g-Bessel sequences for* $\mathcal{H}$.

1. *If* $m = (m_i) \in \ell^1$ *and* $(dim\mathcal{H}_i)_{i\in I} \in \ell^\infty$, *then* $\boldsymbol{M}_{mC\Theta\Lambda C'}$ *is a trace- class operator.*

2. *If* $m = (m_i) \in \ell^2$ *and* $(dim\mathcal{H}_i)_{i\in I} \in \ell^\infty$, *then* $\boldsymbol{M}_{mC\Theta\Lambda C'}$ *is a Hilbert-Schmit operator.*

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