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A necessary and sufficient condition for some steady Ricci solitons to have positive asymptotic volume ratio

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Abstract

In this paper, we firstly establish a useful ODE relationship between $\mathrm{R}_1(\mathrm{c})$ and $\mathrm{V}_1(\mathrm{c})$ on the steady Ricci soliton. Based on this, we obtain a necessary and sufficient condition for some complete noncompact steady gradient Ricci solitons to have positive asymptotic volume ratio.

1 Introduction and Main Results

Recall that a complete Riemannian manifold (M^n, g) is called a steady gradient Ricci soliton if there exists a smooth function $f : M^n \to \mathbb{R}$, called the potential function such that

$$R_{ij} + \nabla_i \nabla_j f = 0. \tag{1}$$

Moreover for the steady gradient Ricci soliton, we actually have

$$R + |\nabla f|^2 = C \tag{2}$$

holds on M^n, where C is a constant.

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In [2], it was proved by B.-L. Chen that the complete ancient solutions to the Ricci flow, and in particular the steady Ricci soliton, must have nonnegative scalar curvature. As a consequence, the potential function f satisfies the following estimate:

$$-\sqrt{C}r(x) + f(O) \leq f(x) \leq \sqrt{C}r(x) + f(O), \tag{3}$$

where $r(x)$ denotes the distance function from x to a fixed point O in M^n.

Moreover, if we assume that the Ricci curvature is positive and the scalar curvature R approaches 0 towards spatial infinity, then by the following Lemma 1.1 proved by H. X. Guo in [5], we can derive that there is one point where R obtains its maximum, and the point of maximum is unique.

Lemma 1.1 (Guo). *Let (M^n, g) be a steady gradient Ricci soliton with positive* (*or negative*) *Ricci curvature, then there is at most one critical point of R.*

Thus we can denote O the unique point of maximum of R, called the origin, and assume $f(O) = 0$ by adding a constant. Calculating the constant in (2) at O we have

$$R + |\nabla f|^2 = R(O) = R_0. \tag{4}$$

Based on these, H. X. Guo [5] also proved a more precise estimate for the potential function of a complete steady gradient Ricci soliton as follows:

Theorem 1.2 (Guo). *Assume (M^n, g) is a complete steady gradient Ricci soliton with positive Ricci curvature, and the scalar curvature approaches 0 towards infinity. Then for any $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that when $r(x) \geq r_\varepsilon$ we have*

$$\left(\sqrt{R_0} - \varepsilon\right) r(x) \leq -f(x) \leq \sqrt{R_0}r(x), \tag{5}$$

where $r(x) = d(x, O)$ and R_0 is the maximum of R.

Then define the functions

$$\mathrm{V} : \mathbb{R} \to [0, \infty), \qquad \mathrm{R} : \mathbb{R} \to [0, \infty)$$

by

$$\mathrm{V}(c) = \int_{\{f<c\}} d\mu, \qquad \mathrm{R}(c) = \int_{\{f<c\}} R\, d\mu.$$

In [1], the following ODE relating $\mathrm{V}(c)$ and $\mathrm{R}(c)$ was established for the shrinking Ricci soliton

$$0 \leq \frac{n}{2}\mathrm{V}(c) - \mathrm{R}(c) = c\,\mathrm{V}'(c) - \mathrm{R}'(c). \tag{6}$$

In this paper, to prove our main result, we establish a similar result to (6) for the steady Ricci soliton as follows:

Theorem 1.3. *Define*

$$D_1(c) = \{x \in M^n \,|{-f(x)} < c\}, \quad \mathrm{R}_1(\mathrm{c}) = \int_{D_1(c)} R\,d\mu, \quad \mathrm{V}_1(\mathrm{c}) = \int_{D_1(c)} d\mu,$$

then

$$\mathrm{R}_1(\mathrm{c}) + \mathrm{R}_1'(\mathrm{c}) - \mathrm{R}_0\mathrm{V}_1'(\mathrm{c}) = 0. \tag{7}$$

Recall that the asymptotic volume ratio (AVR) of a complete noncompact Riemannian manifold (N^n, h) is defined by

$$\mathrm{AVR}(h) = \lim_{r\to\infty} \frac{\mathrm{Vol}_h\, B(p,r)}{\omega_n r^n} \tag{8}$$

if the limit exists, where $B(p,r)$ denotes the geodesic ball in N^n with center p and radius r and ω_n is the volume of the unit Euclidean n-ball. It is easy to check that the AVR(h) is independent of the choice of p. Moreover, if (N^n, h) has nonnegative Ricci curvature, then this limit (8) exists by the Bishop-Gromov volume comparison theorem.

For the case of shrinking Ricci solitons, H.-D. Cao and D.-T. Zhou [1] proved the following result aided by an observation of Munteanu [6].

Theorem 1.4 (Cao-Zhou)**.** *Any complete noncompact shrinking gradient linebreak Ricci soliton must have at most Euclidean volume growth, i.e.,*

$$\limsup_{r\to\infty} \frac{\mathrm{Vol}\, B(O,r)}{\omega_n r^n} < \infty. \tag{9}$$

For the case of steady Ricci solitons, by using Theorem 1.3, we can prove the following estimate.

Theorem 1.5. *For the steady gradient Ricci soliton we have*

$$\mathrm{V}_1(\mathrm{c}) \geq \frac{\mathrm{R}_1(\mathrm{c})}{R_0} + \frac{R_0\mathrm{V}_1(\mathrm{c}_0) - \mathrm{R}_1(\mathrm{c}_0)}{R_0}. \tag{10}$$

In particular, more recently, observing the results in [1], [2], [4] and [8], B. Chow, P. Lu and B. Yang [3] derived a necessary and sufficient condition for noncompact shrinking Ricci soliton to have positive AVR as follows:

Theorem 1.6 (Chow-Lu-Yang)**.** *Let* (M^n, g) *be a complete noncompact shrinking gradient Ricci soliton, then* $\mathrm{AVR}(g)$ *exists* (*and is finite*)*. Moreover,* $\mathrm{AVR}(g) > 0$ *if and only if*

$$\int_{n+2}^{\infty} \frac{\mathrm{R}(c)}{c\,\mathrm{V}(c)} dc < \infty. \tag{11}$$

In this paper, for the case of steady Ricci solitons, we prove a similar necessary and sufficient condition for some noncompact steady solitons to have positive AVR:

Theorem 1.7. *Let* (M^n, g) *be a complete noncompact steady gradient Ricci soliton such that the average scalar curvature*

$$0 < \overline{\mathrm{R}}\,(g) = \lim_{r \to \infty} \frac{\int_{B(O,r)} R d\mu}{\mathrm{Vol}_g\,(B\,(O,r))} < \infty, \tag{12}$$

then $\mathrm{AVR}(g)$ *exists* (*and is finite*)*. Moreover,* $\mathrm{AVR}(g) > 0$ *if and only if*

$$\int_{c_0}^{\infty} \left(\frac{\mathrm{R}_1(c)}{\int_0^c \mathrm{R}_1(s) ds} - \frac{n+1}{c} \right) dc > -\infty. \tag{13}$$

The paper is organized as follows. In section 2, we prove Theorem 1.3 by calculating, and then obtain Theorem 1.5 applying Theorem 1.3. Based on these, in section 3, we prove our main result Theorem 1.7.

2 Proof of Theorem 1.3 and 1.5

Proof of Theorem 1.3. Firstly, by Theorem 1.2, when $r(x)$ is greater than some constant r_ε, we have

$$\left(\sqrt{R_0} - \varepsilon\right) r(x) \leq -f(x) \leq \sqrt{R_0} r(x).$$

Denote by

$$D_1(c) = \{x \in M^n \,|{-f(x)} < c\} \quad \text{and} \quad \mathrm{V}_1(\mathrm{c}) = \int_{D(c)} dV, \tag{14}$$

then by the Co-Area formula (cf. [7]), we have

$$\mathrm{V}_1(\mathrm{c}) = \int_0^c ds \int_{\partial D_1(s)} \frac{1}{|\nabla(-f)|} dA. \tag{15}$$

Hence

$$\mathrm{V}_1'(\mathrm{c}) = \int_{\partial D_1(c)} \frac{1}{|\nabla f|} dA. \tag{16}$$

Then taking the trace in

$$R_{ij} + \nabla_i \nabla_j f = 0,$$

we have

$$R + \Delta f = 0. \tag{17}$$

Thus by using the Divergence Theorem and (4)

$$\begin{aligned}
-\int_{D_1(c)} R d\mu &= \int_{D_1(c)} \Delta f d\mu \\
&= \int_{\partial D_1(c)} \nabla f \cdot \frac{-\nabla f}{|\nabla f|} dA \\
&= -\int_{\partial D_1(c)} |\nabla f| dA \\
&= \int_{\partial D_1(c)} \frac{R - R_0}{|\nabla f|} dA \\
&= \int_{\partial D_1(c)} \frac{R}{|\nabla f|} dA - R_0 \mathrm{V}_1'(\mathrm{c})
\end{aligned}$$

Then by using the Co-Area formula again, we have

$$\mathrm{R}_1(\mathrm{c}) = \int_{D_1(c)} R \, d\mu = \int_0^c ds \int_{\partial D_1(s)} \frac{R}{|\nabla f|} dA.$$

Hence

$$\mathrm{R}_1'(\mathrm{c}) = \int_{\partial D_1(c)} \frac{R}{|\nabla f|} dA. \tag{18}$$

Therefore, we have

$$-\mathrm{R}_1(c) = -\int_{D_1(c)} R d\mu = \int_{\partial D_1(c)} \frac{R}{|\nabla f|} dA - R_0 \mathrm{V}_1'(\mathrm{c}) = \mathrm{R}_1'(\mathrm{c}) - \mathrm{R}_0 \mathrm{V}_1'(\mathrm{c}).$$

□

Now we turn to prove Theorem 1.5 by using Theorem 1.3.

Proof of Theorem 1.5. Integrate the identity (7) from c_0 to c we get

$$\begin{aligned} R_0\left(\mathrm{V}_1(\mathrm{c})-\mathrm{V}_1(\mathrm{c}_0)\right) &= \int_{c_0}^{c} R_0 \mathrm{V}'{}_1(\mathrm{s}) d\mathrm{s} \\ &= \int_{c_0}^{c} \left(\mathrm{R}'{}_1(\mathrm{s})+\mathrm{R}_1(\mathrm{s})\right) d\mathrm{s} \\ &= \mathrm{R}_1(\mathrm{c})-\mathrm{R}_1(\mathrm{c}_0)+\int_{c_0}^{c} \mathrm{R}_1(\mathrm{s}) d\mathrm{s} \end{aligned}$$

Therefore, (10) follows from the observation that $\mathrm{R}_1(\mathrm{c})$ is nonnegative, because the scalar curvature $R \geq 0$. □

3 Proof of Theorem 1.7

In this section, we prove Theorem 1.7 by using Theorem 1.2 and 1.3.

Proof of Theorem 1.7. Let

$$\mathrm{P(c)}=\frac{\mathrm{R}_1(\mathrm{c})-R_0\mathrm{V}_1(\mathrm{c})}{\mathrm{c}^{\mathrm{n}+1}} \quad \text{and} \quad \mathrm{N}(\mathrm{c})=\frac{\mathrm{R}_1(\mathrm{c})}{R_0\mathrm{V}_1(\mathrm{c})}, \tag{19}$$

then

$$\frac{\mathrm{N}(\mathrm{c})}{\mathrm{N}(\mathrm{c})-1}=\frac{\frac{\mathrm{R}_1(\mathrm{c})}{R_0\mathrm{V}_1(\mathrm{c})}}{\frac{\mathrm{R}_1(\mathrm{c})}{R_0\mathrm{V}_1(\mathrm{c})}-1}=\frac{\mathrm{R}_1(\mathrm{c})}{\mathrm{R}_1(\mathrm{c})-R_0\mathrm{V}_1(\mathrm{c})}=\frac{\mathrm{R}_1(\mathrm{c})}{\mathrm{c}^{\mathrm{n}+1}\mathrm{P}(\mathrm{c})}. \tag{20}$$

Note that $\frac{\mathrm{R}_1(\mathrm{c})}{\mathrm{V}_1(\mathrm{c})}$ is the average scalar curvature over the set $D(\mathrm{c})$, and the ODE (7) implies

$$\begin{aligned} \mathrm{P}'(\mathrm{c}) &= \frac{\mathrm{R}_1'(\mathrm{c})\,\mathrm{c}^{\mathrm{n}+1}-(\mathrm{n}+1)\,\mathrm{R}_1(\mathrm{c})\mathrm{c}^{\mathrm{n}}-R_0\mathrm{V}_1'(\mathrm{c})\,\mathrm{c}^{\mathrm{n}+1}+(\mathrm{n}+1)\,R_0\mathrm{V}_1(\mathrm{c})\,\mathrm{c}^{\mathrm{n}}}{\mathrm{c}^{2\mathrm{n}+2}} \\ &= \frac{\left(\mathrm{R}_1'(\mathrm{c})-R_0\mathrm{V}_1'(\mathrm{c})\right)\mathrm{c}^{\mathrm{n}+1}-(\mathrm{n}+1)\,\mathrm{c}^{\mathrm{n}}\left(\mathrm{R}_1(\mathrm{c})-R_0\mathrm{V}_1(\mathrm{c})\right)}{\mathrm{c}^{2\mathrm{n}+2}} \\ &= \frac{-\mathrm{R}_1(\mathrm{c})\mathrm{c}^{\mathrm{n}+1}-(\mathrm{n}+1)\,\mathrm{c}^{2\mathrm{n}+1}\mathrm{P}(\mathrm{c})}{\mathrm{c}^{2\mathrm{n}+2}} \\ &= \frac{-\frac{\mathrm{N}(\mathrm{c})}{\mathrm{N}(\mathrm{c})-1}\mathrm{c}^{2\mathrm{n}+2}\mathrm{P}(\mathrm{c})-(\mathrm{n}+1)\,\mathrm{c}^{2\mathrm{n}+1}\mathrm{P}(\mathrm{c})}{\mathrm{c}^{2\mathrm{n}+2}} \\ &= -\left(\frac{\mathrm{N}(\mathrm{c})}{\mathrm{N}(\mathrm{c})-1}+\frac{n+1}{\mathrm{c}}\right)\mathrm{P}(\mathrm{c}). \end{aligned}$$

Then we choose c_0 such that $\mathrm{P}(\mathrm{c}_0) \neq 0$, and integrate

$$\mathrm{P}'(\mathrm{c})=-\left(\frac{\mathrm{N}(\mathrm{c})}{\mathrm{N}(\mathrm{c})-1}+\frac{n+1}{\mathrm{c}}\right)\mathrm{P}(\mathrm{c}). \tag{21}$$

from c_0 to c we have

$$\mathrm{P(c)} = \mathrm{P(c_0)} e^{-\int_{c_0}^{c} \left(\frac{\mathrm{N(c)}}{\mathrm{N(c)}-1} + \frac{\mathrm{n}+1}{\mathrm{c}}\right) \mathrm{dc}}. \tag{22}$$

From ODE (7) it is easy to see that

$$\mathrm{R}_1(\mathrm{c}) - R_0 \mathrm{V}_1(\mathrm{c}) = -\int_0^{\mathrm{c}} \mathrm{R}_1(\mathrm{s}) \mathrm{ds},$$

which implies

$$\frac{\mathrm{R}_1(\mathrm{c})}{\mathrm{c}^{\mathrm{n}+1}\mathrm{P(c)}} = \frac{\mathrm{N(c)}}{\mathrm{N(c)}-1} = \frac{\mathrm{R}_1(\mathrm{c})}{\mathrm{R}_1(\mathrm{c}) - R_0\mathrm{V}_1(\mathrm{c})} = -\frac{\mathrm{R}_1(\mathrm{c})}{\int_0^c \mathrm{R}_1(\mathrm{s})\mathrm{ds}}. \tag{23}$$

Note that (23) implies $\mathrm{P(c)} \leq 0$. Furthermore, by the following bounds

$$\left(\sqrt{R_0} - \varepsilon\right) r(x) \leq -f(x) \leq \sqrt{R_0} r(x)$$

we have

$$\begin{aligned}
\lim_{\mathrm{c}\to\infty} \mathrm{P(c)} &= \lim_{\mathrm{c}\to\infty} \frac{\mathrm{R}_1(\mathrm{c}) - R_0\mathrm{V}_1(\mathrm{c})}{\mathrm{c}^{\mathrm{n}+1}} \\
&= \lim_{\mathrm{c}\to\infty} \frac{\mathrm{R}_1'(\mathrm{c}) - R_0\mathrm{V}_1'(\mathrm{c})}{(\mathrm{n}+1)\mathrm{c}^{\mathrm{n}}} \\
&= -\lim_{\mathrm{c}\to\infty} \frac{\mathrm{R}_1(\mathrm{c})}{(\mathrm{n}+1)\mathrm{c}^{\mathrm{n}}} \\
&= -\frac{1}{\mathrm{n}+1} \lim_{\mathrm{c}\to\infty} \frac{\mathrm{R}_1(\mathrm{c})}{\mathrm{V}_1(\mathrm{c})} \lim_{\mathrm{c}\to\infty} \frac{\mathrm{V}_1(\mathrm{c})}{\mathrm{c}^{\mathrm{n}}} \\
&= -\frac{1}{\mathrm{n}+1} \overline{\mathrm{R}}(g) \lim_{\mathrm{c}\to\infty} \frac{\mathrm{Vol}(B\left(O, \frac{c}{\sqrt{R_0}}\right))}{\mathrm{c}^{\mathrm{n}}} \\
&= -\frac{\omega_n}{(\mathrm{n}+1) R_0^{n/2}} \overline{\mathrm{R}}(g)\, \mathrm{AVR}(g),
\end{aligned}$$

that is to say

$$-\frac{\omega_n}{(\mathrm{n}+1) R_0^{n/2}} \overline{\mathrm{R}}(g)\, \mathrm{AVR}(g) = \lim_{\mathrm{c}\to\infty} \mathrm{P(c)}, \tag{24}$$

which exists by (21). Since the average scalar curvature

$$\overline{\mathrm{R}}(g) = \lim_{r\to\infty} \frac{\int_{B(O,r)} R d\mu}{\mathrm{Vol}_g(B(O,r))} > 0,$$

we have $\mathrm{AVR}(g)$ exists (and is finite).

Moreover, by using (22) and (23) we have

$$\begin{aligned}\mathrm{AVR}(g) &= -\frac{(\mathrm{n}+1)\, R_0^{n/2}\mathrm{P}(\mathrm{c}_0)}{\omega_n \overline{\mathrm{R}}\,(g)} \mathrm{e}^{-\int_{\mathrm{c}_0}^{\infty}\left(\frac{\mathrm{N}(\mathrm{c})}{\mathrm{N}(\mathrm{c})-1}+\frac{\mathrm{n}+1}{\mathrm{c}}\right)\mathrm{dc}} \\ &= -\frac{(\mathrm{n}+1)\, R_0^{n/2}\mathrm{P}(\mathrm{c}_0)}{\omega_n \overline{\mathrm{R}}\,(g)} \mathrm{e}^{\int_{\mathrm{c}_0}^{\infty}\left(\frac{\mathrm{R}_1(\mathrm{c})}{\int_0^{\mathrm{c}}\mathrm{R}_1(\mathrm{s})\mathrm{ds}}-\frac{\mathrm{n}+1}{\mathrm{c}}\right)\mathrm{dc}}.\end{aligned}$$

Note that $\mathrm{P}(\mathrm{c}_0) < 0$, and by using (12) we obtain that $\mathrm{AVR}(g) > 0$ if and only if

$$\int_{c_0}^{\infty}\left(\frac{\mathrm{R}_1(c)}{\int_0^c \mathrm{R}_1(s)ds} - \frac{n+1}{c}\right)dc > -\infty.$$

□

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