Differential sandwich theorems of p -valent analytic functions involving a linear operator

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Abstract

In this paper we derive some subordination and superordination results for certain p -valent analytic functions in the open unit disc, which are acted upon by a class of a linear operator. Some of our results improve and generalize previously known results.

1 Introduction

Let $H(U)$ denotes the class of analytic functions in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $H[a, p]$ denotes the subclass of the functions $f \in H(U)$ of the form:

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \dots (a \in \mathbb{C}; p \in \mathbb{N} = \{1, 2, \dots\}).$$

Also, let $\mathcal{A}(p)$ be the subclass of the functions $f \in H(U)$ of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N}), \quad (1.1)$$

and set $\mathcal{A} \equiv \mathcal{A}(1)$. For functions $f(z) \in \mathcal{A}(p)$, given by (1.1), and $g(z)$ given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k \quad (p \in \mathbb{N}), \quad (1.2)$$

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the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k = (g * f)(z). \quad (1.3)$$

For $f, g \in H(U)$, we say that the function f is subordinate to g , if there exists a Schwarz function w , i.e, $w \in H(U)$ with $w(0) = 0$ and $|w(z)| < 1$, $z \in U$, such that $f(z) = g(w(z))$ for all $z \in U$. This subordination is usually denoted by $f(z) \prec g(z)$. It is well-known that, if the function g is univalent in U , then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$ (see [5] and [9]).

Supposing that h and k are two analytic functions in U , let

$$\phi(r, s, t; z) : \mathbb{C}^3 \times U \rightarrow \mathbb{C}.$$

If h and $\varphi(h(z), zh'(z), z^2h''(z); z)$ are univalent functions in U and if h satisfies the second-order superordination

$$k(z) \prec \varphi(h(z), zh'(z), z^2h''(z); z), \quad (1.4)$$

then h is called to be a solution of the differential superordination (1.4). A function $q \in H(U)$ is called a subordinant of (1.4), if $q(z) \prec h(z)$ for all the functions h satisfying (1.4). A univalent subordinant $\tilde{q}$ that satisfies $q(z) \prec \tilde{q}(z)$ for all of the subordinants q of (1.4), is said to be the best subordinant.

Recently, Miller and Mocanu [10] obtained sufficient conditions on the functions k, q and φ for which the following implication holds:

$$k(z) \prec \varphi(h(z), zh'(z), z^2h''(z); z) \Rightarrow q(z) \prec h(z).$$

Using these results, Bulboacă [3] considered certain classes of first-order differential subordinations, as well as superordination-preserving integral operators [4]. Ali et al. [1], using the results from [3], obtained sufficient conditions for certain normalized analytic functions to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent normalized functions in U .

For complex parameters

$$\alpha_1, \dots, \alpha_q \text{ and } \beta_1, \dots, \beta_s \quad (\beta_j \notin \mathbb{Z}_0^- = \{0, -1, -2, \dots\}; j = 1, 2, \dots, s),$$

we now define the generalized hypergeometric function ${}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)$ by (see, for example, [15, p.19])

$${}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \dots (\alpha_q)_k}{(\beta_1)_k \dots (\beta_s)_k} \cdot \frac{z^k}{k!} \quad (1.5)$$

$$(q \leq s+1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in U),$$

where $(\theta)_\nu$ is the Pochhammer symbol defined, in terms of the Gamma function Γ , by

$$(\theta)_\nu = \frac{\Gamma(\theta + \nu)}{\Gamma(\theta)} = \begin{cases} 1 & (\nu = 0; \theta \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}), \\ \theta(\theta+1)\dots(\theta+\nu-1) & (\nu \in \mathbb{N}; \theta \in \mathbb{C}). \end{cases} \quad (1.6)$$

Let

$$\begin{aligned} h(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) &= z^p {}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) \\ &= z^p + \sum_{k=p+1}^{\infty} \Gamma_{p,q,s}(\alpha_1) z^k, \end{aligned}$$

where

$$\Gamma_{p,q,s}(\alpha_1) = \frac{(\alpha_1)_{k-p} \dots (\alpha_q)_{k-p}}{(\beta_1)_{k-p} \dots (\beta_s)_{k-p} (1)_{k-p}}, \quad (1.7)$$

and using the Hadamard product, El-Ashwah and Aouf [7] defined the following operator

$$I_{p,\lambda}^{m,\ell}(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s) : A(p) \rightarrow A(p)$$

by

$$\begin{aligned} I_{p,\lambda}^{0,\ell}(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s)f(z) &= f(z) * h(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z); \\ I_{p,\lambda}^{1,\ell}(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s)f(z) &= (1-\lambda)(f(z) * h(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)) \\ &\quad + \frac{\lambda}{(p+\ell)z^{\ell-1}}(z^\ell f(z) * h(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z))'; \end{aligned}$$

and

$$I_{p,\lambda}^{m,\ell}(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s)f(z) = I_{p,q,s,\lambda}^{1,\ell}(I_{p,q,s,\lambda}^{m-1,\ell}(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s)f(z)). \quad (1.8)$$

If $f \in A(p)$, then from (1.1) and (1.8), we can easily see that

$$I_{p,\lambda}^{m,\ell}(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s)f(z) = z^p + \sum_{k=p+1}^{\infty} \left[\frac{p+\ell+\lambda(k-p)}{p+\ell} \right]^m \Gamma_{p,q,s}(\alpha_1) a_k z^k, \quad (1.9)$$

$$(p \in \mathbb{N}; m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; \ell \geq 0; \lambda \geq 0; z \in U)$$

It can be easily verified from the definition (1.9) that:

$$z(I_{p,q,s,\lambda}^{m,\ell}(\alpha_1)f(z))' = \alpha_1 I_{p,q,s,\lambda}^{m,\ell}(\alpha_1+1)f(z) - (\alpha_1-p)I_{p,q,s,\lambda}^{m,\ell}(\alpha_1)f(z), \quad (1.10)$$

where

$$I_{p,q,s,\lambda}^{m,\ell}(\alpha_1)f(z) = I_{p,\lambda}^{m,\ell}(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s)f(z).$$

It should be remarked that the linear operator $I_{p,q,s,\lambda}^{m,\ell}(\alpha_1)$ is a generalization of many other linear operators considered earlier. In particular, we have

$$I_{p,q,s,\lambda}^{0,\ell}(\alpha_1)f(z) = H_{p,q,s}(\alpha_1)f(z),$$

where the linear operator $H_{p,q,s}(\alpha_1)$ was investigated by Dziok and Srivastava [8], and also we have

$$I_{p,2,1,\lambda}^{0,\ell}(a, 1; c)f(z) = L_p(a, c)f(z) \quad (a \in \mathbb{R}; c \in \mathbb{R} \setminus \mathbb{Z}_0^-),$$

where the linear operator $L_p(a, c)$ was studied by Saitoh [13] which yields the operator $L(a, c)f(z)$ introduced by Carlson and Shaffer [6] for $p = 1$.

2 Preliminaries

In order to prove our subordination and superordination results, we make use of the following known definition and results.

Definition [10]. Denote by Q the set of all functions $f(z)$ that are analytic and injective on $\bar{U} \setminus E(f)$, where

$$E(f) = \left\{ \zeta : \zeta \in \partial \text{ and } \lim_{z \rightarrow \zeta} f(z) = \infty \right\} \quad (2.1)$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

Lemma 1 [9]. Let the function $q(z)$ be univalent in the unit disc U and let θ and φ be analytic in a domain D containing $q(U)$ with $\varphi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\varphi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that

(i) $Q(z)$ is starlike univalent in U ,

(ii) $\Re \left(\frac{zh'(z)}{Q(z)} \right) > 0$ for $z \in U$.

If p is analytic with $p(0) = q(0)$, $p(U) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)), \quad (2.2)$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Lemma 2 [5]. Let $q(z)$ be convex univalent in the unit disc U and let θ and φ be analytic in a domain D containing $q(U)$. Suppose that

(i) $\Re \left\{ \frac{\theta'(q(z))}{\varphi(q(z))} \right\} > 0$ for $z \in U$;

(ii) $zq'(z)\varphi(q(z))$ is starlike univalent in U .
 If $p(z) \in H[q(0), 1] \cap Q$, with $p(U) \subseteq D$, and $\theta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in U , and

$$\theta(q(z)) + zq'(z)\varphi(q(z)) \prec \theta(p(z)) + zp'(z)\varphi(p(z)), \quad (2.4)$$

then $q(z) \prec p(z)$ and $q(z)$ is the best subordinant

The following lemma gives us a necessary and sufficient condition for the univalence of a special function which will be used in some particular case.

Lemma 3 [12]. *The function $q(z) = (1 - z)^{-2ab}$ ($a, b \in \mathbb{C}^*$) is univalent in the unit disc U if and only if $|2ab - 1| \leq 1$ or $|2ab + 1| \leq 1$.*

3 Main Results

Unless otherwise mentioned, we assume throughout this paper that $p \in \mathbb{N}, m \in \mathbb{N}_0, \ell \geq 0; \lambda \geq 0$ and the power understood as principal values.

Theorem 1. *Let $q(z)$ be univalent in U such that $q(0) = 1, q(z) \neq 0$ and $\frac{zq'(z)}{q(z)}$ is starlike in U . Let $f \in \mathcal{A}(p)$ and suppose that f and q satisfy the next conditions:*

$$\left[\frac{I_{p,q,s,\lambda}^{m,\ell}(\alpha_1)f(z)}{z^p} \right]^\mu \left[\frac{z^p}{I_{p,q,s,\lambda}^{m,\ell}(\alpha_1 + 1)f(z)} \right]^\eta \neq 0 \quad (\mu \in \mathbb{C}^*; \eta \in \mathbb{C}; z \in U), \quad (3.1)$$

and

$$\Re \left\{ 1 + \frac{\zeta}{\gamma} q(z) + \frac{2\delta}{\gamma} [q(z)]^2 - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)} \right\} > 0 \quad (\zeta, \delta \in \mathbb{C}; \gamma \in \mathbb{C}^*; z \in U). \quad (3.2)$$

If

$$\Psi(z) \prec \chi + \zeta q(z) + \delta [q(z)]^2 + \gamma \frac{zq'(z)}{q(z)}, \quad (3.3)$$

where

$$\begin{aligned} \Psi(z) = & \chi + \zeta \left[\frac{I_{p,q,s,\lambda}^{m,\ell}(\alpha_1)f(z)}{z^p} \right]^\mu \left[\frac{z^p}{I_{p,q,s,\lambda}^{m,\ell}(\alpha_1 + 1)f(z)} \right]^\eta \\ & + \delta \left[\frac{I_{p,q,s,\lambda}^{m,\ell}(\alpha_1)f(z)}{z^p} \right]^{2\mu} \left[\frac{z^p}{I_{p,q,s,\lambda}^{m,\ell}(\alpha_1 + 1)f(z)} \right]^{2\eta} \\ & + \gamma \mu \alpha_1 \left[\frac{I_{p,q,s,\lambda}^{m,\ell}(\alpha_1 + 1)f(z)}{I_{p,q,s,\lambda}^{m,\ell}(\alpha_1)f(z)} - 1 \right] \end{aligned}$$

$$+\gamma\eta(\alpha_1+1)\left[1-\frac{I_{p,q,s,\lambda}^{m,\ell}(\alpha_1+2)f(z)}{I_{p,q,s,\lambda}^{m,\ell}(\alpha_1+1)f(z)}\right], \quad (3.4)$$

then

$$\left[\frac{I_{p,q,s,\lambda}^{m,\ell}(\alpha_1)f(z)}{z^p}\right]^\mu \left[\frac{z^p}{I_{p,q,s,\lambda}^{m,\ell}(\alpha_1+1)f(z)}\right]^\eta \prec q(z),$$

and q is the best dominant of (3.3).

Proof. Let

$$h(z) = \left[\frac{I_{p,q,s,\lambda}^{m,\ell}(\alpha_1)f(z)}{z^p}\right]^\mu \left[\frac{z^p}{I_{p,q,s,\lambda}^{m,\ell}(\alpha_1+1)f(z)}\right]^\eta \quad (z \in U). \quad (3.5)$$

According to (3.1) the function $h(z)$ is analytic in U , and differentiating (3.5) logarithmically with respect to z , we obtain

$$\frac{zh'(z)}{h(z)} = \mu \left[\frac{z(I_{p,q,s,\lambda}^{m,\ell}(\alpha_1)f(z))'}{I_{p,q,s,\lambda}^{m,\ell}(\alpha_1)f(z)} - p \right] + \eta \left[p - \frac{z(I_{p,q,s,\lambda}^{m,\ell}(\alpha_1+1)f(z))'}{I_{p,q,s,\lambda}^{m,\ell}(\alpha_1+1)f(z)} \right].$$

By using the identity (1.10), we obtain

$$\frac{zh'(z)}{h(z)} = \mu\alpha_1 \left[\frac{I_{p,q,s,\lambda}^{m,\ell}(\alpha_1+1)f(z)}{I_{p,q,s,\lambda}^{m,\ell}(\alpha_1)f(z)} - 1 \right] + \eta(\alpha_1+1) \left[1 - \frac{I_{p,q,s,\lambda}^{m,\ell}(\alpha_1+2)f(z)}{I_{p,q,s,\lambda}^{m,\ell}(\alpha_1+1)f(z)} \right].$$

In order to prove our result we will use Lemma 1. In this lemma consider

$$\theta(w) = \chi + \zeta w + \delta w^2 \quad \text{and} \quad \varphi(w) = \frac{\gamma}{w},$$

then θ is analytic in $\mathbb{C}$ and $\varphi(w) \neq 0$ is analytic in $\mathbb{C}^*$. Also, if we let

$$Q(z) = zq'(z)\varphi(q(z)) = \gamma \frac{zq'(z)}{q(z)},$$

and

$$g(z) = \theta(q(z)) + Q(z) = \chi + \zeta q(z) + \delta [q(z)]^2 + \gamma \frac{zq'(z)}{q(z)}.$$

We see that $Q(z)$ is starlike function in U . From (3.2), we also have

$$\Re \left\{ \frac{zg'(z)}{Q(z)} \right\} = \Re \left\{ 1 + \frac{\zeta}{\gamma} q(z) + \frac{2\delta}{\gamma} [q(z)]^2 - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)} \right\} > 0 \quad (z \in U),$$

and then, by using Lemma 1 we deduce that the subordination (3.3) implies $h(z) \prec q(z)$, and the function q is the best dominant of (3.3).

Putting $q = 2, s = p = 1, m = 0, \alpha_1 = a (a \in \mathbb{C}), \alpha_2 = 1$ and $\beta_1 = c$ ($c \in \mathbb{C} \setminus \mathbb{Z}_0^-$) in Theorem 1, we obtain the following result which improves the corresponding work of Shammugam et al. [14, Theorem 3.1].

Corollary 1. *Let $q(z)$ be univalent in U such that $q(0) = 1, q(z) \neq 0$ and $\frac{zq'(z)}{q(z)}$ is starlike in U . Let $f \in \mathcal{A}$ such that*

$$\left[\frac{L(a, c) f(z)}{z} \right]^\mu \left[\frac{z}{L(a+1, c) f(z)} \right]^\eta \neq 0 \quad (\mu \in \mathbb{C}^*; z \in U), \quad (3.6)$$

and suppose that q satisfies (3.2). If

$$\Lambda(z) \prec \chi + \zeta q(z) + \delta [q(z)]^2 + \gamma \frac{zq'(z)}{q(z)}, \quad (3.7)$$

where

$$\begin{aligned} \Lambda(z) = & \chi + \zeta \left[\frac{L(a, c) f(z)}{z} \right]^\mu \left[\frac{z}{L(a+1, c) f(z)} \right]^\eta \\ & + \delta \left[\frac{L(a, c) f(z)}{z} \right]^{2\mu} \left[\frac{z}{L(a+1, c) f(z)} \right]^{2\eta} \\ & + \gamma \mu a \left[\frac{L(a+1, c) f(z)}{L(a, c) f(z)} - 1 \right] \\ & + \gamma \eta (a+1) \left[1 - \frac{L(a+2, c) f(z)}{L(a+1, c) f(z)} \right], \end{aligned} \quad (3.8)$$

then

$$\left[\frac{L(a, c) f(z)}{z} \right]^\mu \left[\frac{z}{L(a+1, c) f(z)} \right]^\eta \prec q(z),$$

and q is the best dominant of (3.7).

Putting $q(z) = \frac{1+Az}{1+Bz}$ ($-1 \leq B < A \leq 1$) in Corollary 1, we obtain the following result which improves the corresponding work of Shammugam et al. [14, Corollary 3.2].

Corollary 2. *Assume that*

$$\Re \left\{ \frac{1 - ABz^2}{(1+Az)(1+Bz)} + \frac{\zeta}{\gamma} \left[\frac{1+Az}{1+Bz} \right] + \frac{2\delta}{\gamma} \left[\frac{1+Az}{1+Bz} \right]^2 \right\} > 0$$

$$(\zeta, \delta \in \mathbb{C}; \gamma \in \mathbb{C}^*; z \in U)$$

holds. Let $f \in \mathcal{A}$ such that (3.6) holds. If

$$\Lambda(z) \prec \chi + \zeta \frac{1 + Az}{1 + Bz} + \delta \left[\frac{1 + Az}{1 + Bz} \right]^2 + \frac{\gamma(A - B)z}{(1 + Az)(1 + Bz)}, \quad (3.9)$$

where $\Lambda(z)$ is given by (3.8), then

$$\left[\frac{L(a, c)f(z)}{z} \right]^\mu \left[\frac{z}{L(a + 1, c)f(z)} \right]^\eta \prec \frac{1 + Az}{1 + Bz},$$

and $\frac{1 + Az}{1 + Bz}$ is the best dominant of (3.9).

Putting $q(z) = \left(\frac{1+z}{1-z} \right)^\vartheta$ ($0 < \vartheta \leq 1$) in Corollary 1, we obtain the following result which improves the corresponding work of Shammugam et al. [14, Corollary 3.3].

Corollary 3. Assume that

$$\Re \left\{ \frac{1 - 3z^2}{1 - z^2} + \frac{\zeta}{\gamma} \left[\frac{1 + z}{1 - z} \right]^\vartheta + \frac{2\delta}{\gamma} \left[\frac{1 + z}{1 - z} \right]^{2\vartheta} \right\} > 0 \quad (\zeta, \delta \in \mathbb{C}; \gamma \in \mathbb{C}^*; z \in U)$$

holds. Let $f \in \mathcal{A}$ such that (3.6) holds. If

$$\Lambda(z) \prec \chi + \zeta \left(\frac{1 + z}{1 - z} \right)^\vartheta + \delta \left(\frac{1 + z}{1 - z} \right)^{2\vartheta} + \frac{2\gamma\vartheta z}{(1 - z^2)} \quad (0 < \vartheta \leq 1), \quad (3.10)$$

where $\Lambda(z)$ is given by (3.8), then

$$\left[\frac{L(a, c)f(z)}{z} \right]^\mu \left[\frac{z}{L(a + 1, c)f(z)} \right]^\eta \prec \left(\frac{1 + z}{1 - z} \right)^\vartheta,$$

and $\left(\frac{1 + z}{1 - z} \right)^\vartheta$ is the best dominant of (3.10).

Putting $q(z) = e^{\mu Az}$ ($|\mu A| < \pi$) in Corollary 1, we obtain the following result which improves the corresponding work of Shammugam et al. [14, Corollary 3.4].

Corollary 4. Assume that

$$\Re \left\{ 1 + \frac{\zeta}{\gamma} e^{\mu Az} q(z) + \frac{2\delta}{\gamma} e^{2\mu Az} \right\} > 0 \quad (\zeta, \delta \in \mathbb{C}; \gamma \in \mathbb{C}^*; z \in U)$$

holds and let $f \in \mathcal{A}$ such that (3.6) holds. If

$$\Lambda(z) \prec \chi + \zeta e^{\mu Az} + \delta e^{2\mu Az} + \gamma A \mu z \quad (|\mu A| < \pi), \quad (3.11)$$

where $\Lambda(z)$ is given by (3.8), then

$$\left[\frac{L(a, c) f(z)}{z} \right]^\mu \left[\frac{z}{L(a+1, c) f(z)} \right]^\eta \prec e^{\mu A z},$$

and $e^{\mu A z}$ is the best dominant of (3.11).

Putting $q = s + 1, \alpha_i = 1 (i = 1, \dots, s + 1), \beta_j = 1 (j = 1, \dots, s), m = \zeta = \delta = 0, \chi = p = 1, \gamma = \frac{1}{ab} (a, b \in \mathbb{C}^*), \mu = a, \eta = 0$ and $q(z) = (1 - z)^{-2ab}$ in Theorem 1, then combining this to gather with Lemma 3 we obtain the next result due to Obradovic et al. [11, Theorem 1].

Corollary 5 [11]. Let $a, b \in \mathbb{C}^*$ such that $|2ab - 1| \leq 1$ or $|2ab + 1| \leq 1$. Let $f \in \mathcal{A}$ and suppose that $\frac{f(z)}{z} \neq 0$ for all $z \in U$. If

$$1 + \frac{1}{b} \left(\frac{z f'(z)}{f(z)} - 1 \right) \prec \frac{1 + z}{1 - z},$$

then

$$\left(\frac{f(z)}{z} \right)^a \prec (1 - z)^{-2ab} \quad (3.12)$$

and $(1 - z)^{-2ab}$ is the best dominant of (3.12).

Remark 1. For $a = 1$, Corollary 5 reduces to the recent result of Srivastava and Lashin [16].

Putting $q = s + 1, \alpha_i = 1 (i = 1, \dots, s + 1), \beta_j = 1 (j = 1, \dots, s), m = \zeta = \delta = 0, \chi = p = \gamma = 1, \eta = 0$ and $q(z) = (1 + Bz)^{\frac{\mu(A-B)}{B}}$ in Theorem 2, and using Lemma 2 we obtain the next result.

Corollary 6. Let $-1 \leq A < B \leq 1$ with $B \neq 0$, and suppose that $\left| \frac{\mu(A-B)}{B} - 1 \right| \leq 1$ or $\left| \frac{\mu(A-B)}{B} + 1 \right| \leq 1$. Let $f \in \mathcal{A}$ such that $\frac{f(z)}{z} \neq 0$ for all $z \in U$, and let $\mu \in \mathbb{C}^*$. If

$$1 + \mu \left(\frac{z f'(z)}{f(z)} - 1 \right) \prec \frac{1 + [B + \mu(A - B)]z}{1 + Bz},$$

then

$$\left(\frac{f(z)}{z} \right)^\mu \prec (1 + Bz)^{\frac{\mu(A-B)}{B}}, \quad (3.13)$$

and $(1 + Bz)^{\frac{\mu(A-B)}{B}}$ is the best dominant of (3.13).

Putting $q = s + 1, \alpha_i = 1 (i = 1, \dots, s + 1), \beta_j = 1 (j = 1, \dots, s), m = \zeta = \delta = 0, \chi = p = 1, \gamma = \frac{e^{i\tau}}{ab \cos \tau} (a, b \in \mathbb{C}^*; |\tau| < \frac{\pi}{2}), \mu = a, \eta = 0$ and $q(z) = (1 - z)^{-2ab \cos \tau e^{-i\tau}}$ in Theorem 1, we obtain the following result due to Aouf et al. [2, Theorem 1].

Corollary 7 [2]. Let $a, b \in \mathbb{C}^*$, $|\tau| < \frac{\pi}{2}$ and suppose that $|2ab \cos \tau e^{-i\tau} - 1| \leq 1$ or $|2ab \cos \tau e^{-i\tau} + 1| \leq 1$. Let $f \in \mathcal{A}$ and suppose that $\frac{f(z)}{z} \neq 0$ for all $z \in U$. If

$$1 + \frac{e^{i\tau}}{b \cos \tau} \left(\frac{zf'(z)}{f(z)} - 1 \right) \prec \frac{1+z}{1-z},$$

then

$$\left(\frac{f(z)}{z} \right)^a \prec (1-z)^{-2ab \cos \tau e^{-i\tau}} \quad (3.14)$$

and $(1-z)^{-2ab \cos \tau e^{-i\tau}}$ is the best dominant of (3.14).

Theorem 2. Let q be convex in U such that $q(0) = 1$ and $\frac{zq'(z)}{q(z)}$ is starlike in U . Further assume that

$$\Re \left\{ (\zeta + 2\delta q(z)) \frac{q(z)q'(z)}{\gamma} \right\} > 0 \quad (\zeta, \delta \in \mathbb{C}; \gamma \in \mathbb{C}^*). \quad (3.15)$$

Let $f \in \mathcal{A}(p)$ such that

$$0 \neq \left[\frac{I_{p,q,s,\lambda}^{m,\ell}(\alpha_1)f(z)}{z^p} \right]^\mu \left[\frac{z^p}{I_{p,q,s,\lambda}^{m,\ell}(\alpha_1+1)f(z)} \right]^\eta \in H[q(0), 1] \cap Q. \quad (3.16)$$

If $\Psi(z)$ given by (3.4) is univalent in U and satisfies the following superordination condition

$$\chi + \zeta q(z) + \delta [q(z)]^2 + \gamma \frac{zq'(z)}{q(z)} \prec \Psi(z), \quad (3.17)$$

then

$$q(z) \prec \left[\frac{I_{p,q,s,\lambda}^{m,\ell}(\alpha_1)f(z)}{z^p} \right]^\mu \left[\frac{z^p}{I_{p,q,s,\lambda}^{m,\ell}(\alpha_1+1)f(z)} \right]^\eta,$$

and q is the best subordinator of (3.17).

Putting $q = 2, s = p = 1, m = 0, \alpha_1 = a (a \in \mathbb{C}), \alpha_2 = 1$ and $\beta_1 = c (c \in \mathbb{C} \setminus \mathbb{Z}_0^-)$ in Theorem 2, we obtain the following result which improves the corresponding work of Shammugam et al. [14, Theorem 3.11].

Corollary 8. Let q be convex in U such that $q(0) = 1$ and $\frac{zq'(z)}{q(z)}$ is starlike in U . Further assume that (3.15) holds. Let $f \in \mathcal{A}$ such that

$$0 \neq \left[\frac{L(a, c)f(z)}{z} \right]^\mu \left[\frac{z}{L(a+1, c)f(z)} \right]^\eta \in H[q(0), 1] \cap Q. \quad (3.18)$$

If $\Lambda(z)$ given by (3.8) is univalent in U and satisfies the following superordination condition

$$\chi + \zeta q(z) + \delta [q(z)]^2 + \gamma \frac{zq'(z)}{q(z)} \prec \Lambda(z), \quad (3.19)$$

then

$$q(z) \prec \left[\frac{L(a, c) f(z)}{z} \right]^\mu \left[\frac{z}{L(a+1, c) f(z)} \right]^\eta,$$

and q is the best subordinant of (3.19).

Combining Theorems 1 and 2, we obtain the following two sandwich results:
Theorem 3. Let q_i be two convex functions in U such that $q_i(0) = 1$ and $\frac{zq_i'(z)}{q_i(z)}$ ($i = 1, 2$) is starlike in U . Suppose that $q_1(z)$ satisfies (3.15) and $q_2(z)$ satisfies (3.2). Let $f \in \mathcal{A}(p)$ and suppose that

$$\left[\frac{I_{p,q,s,\lambda}^{m,\ell}(\alpha_1) f(z)}{z^p} \right]^\mu \left[\frac{z^p}{I_{p,q,s,\lambda}^{m,\ell}(\alpha_1 + 1) f(z)} \right]^\eta \in H[q(0), 1] \cap Q.$$

If $\Psi(z)$ given by (3.4) is univalent in U , and

$$\chi + \zeta q_1(z) + \delta [q_1(z)]^2 + \gamma \frac{zq_1'(z)}{q_1(z)} \prec \Psi(z) \prec \chi + \zeta q_2(z) + \delta [q_2(z)]^2 + \gamma \frac{zq_2'(z)}{q_2(z)}, \quad (3.20)$$

then

$$q_1(z) \prec \left[\frac{I_{p,q,s,\lambda}^{m,\ell}(\alpha_1) f(z)}{z^p} \right]^\mu \left[\frac{z^p}{I_{p,q,s,\lambda}^{m,\ell}(\alpha_1 + 1) f(z)} \right]^\eta \prec q_2(z),$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant of (3.20).

Putting $q = 2, s = p = 1, m = 0, \alpha_1 = a (a \in \mathbb{C}), \alpha_2 = 1$ and $\beta_1 = c$ ($c \in \mathbb{C} \setminus \mathbb{Z}_0^-$) in Theorem 3, we obtain the following result which improves the corresponding work of Shammugam et al. [14, Theorem 3.12].

Corollary 9. Let q_i be two convex functions in U such that $q_i(0) = 1$ and $\frac{zq_i'(z)}{q_i(z)}$ ($i = 1, 2$) is starlike in U . Suppose that $q_1(z)$ satisfies (3.15) and $q_2(z)$ satisfies (3.2). Let $f \in \mathcal{A}$ and suppose that $\left[\frac{L(a+1, c) f(z)}{z} \right]^\mu \in H[q(0), 1] \cap Q$. If $\Lambda(z)$ given by (3.8) is univalent in U , and

$$\chi + \zeta q_1(z) + \delta [q_1(z)]^2 + \gamma \frac{zq_1'(z)}{q_1(z)} \prec \Lambda(z) \prec \chi + \zeta q_2(z) + \delta [q_2(z)]^2 + \gamma \frac{zq_2'(z)}{q_2(z)}, \quad (3.21)$$

then

$$q_1(z) \prec \left[\frac{L(a+1, c) f(z)}{z} \right]^\mu \prec q_2(z),$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant of (3.21).

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