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Generalized Broughton polynomials and characteristic varieties

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Abstract

We introduce a family of generalized Broughton polynomials and compute the characteristic varieties of complement of curve arrangements defined by fibers of those generalized Broughton polynomials.

1 Introduction

In [3] Broughton considered the polynomial

$$f(x,y) = x(xy-1).$$

The associated function $f : \mathbb{C}^2 \to \mathbb{C}$ has no critical value, but the fiber $f^{-1}(0)$ is not diffeomorphic to the generic one. This is explained by the existence of the so-called "critical value at infinity", see [10], [3], [4].

In the paper [12] Zahid introduced a family of polynomials:

$$f_{p,q}(x,y) = x^p[xy(x+2)\cdots(x+q)-1],$$

which are called *generalized Broughton polynomials*, where $p \geq 1$ and $q \geq 1$ are integer number, with the convention

$$f_{p,1} = x^p(xy-1).$$

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By computing the characteristic variety $\mathcal{V}_1(M)$, where

$$M = \mathbb{C}^2 \setminus (C_0 \cup C_1)$$

is a complement of a curve arrangement defined by a component of the $0-$fiber:

$$C_0 = \begin{cases} \{xy(x+2)\cdots(x+q)-1=0\} & \text{if } q>1 \\ \{xy-1=0\} & \text{if } q=1 \end{cases}$$

and the generic fiber of $f_{p,q}$:

$$C_1 = \{f_{p,q}(x,y)=1\},$$

the author obtained examples of characteristic varieties with an arbitrary number of translated components for complements of affine curve arrangements consisting of just two curves, see [12].

The aim of this paper is to generalize the Zahid's work in [12]. More precisely, we introduce a family of *generalized Broughton polynomials*, which generalizes the Zahid's one. Namely

$$F(x,y) := p(x)(yq(x)-1)$$

where $p(x), q(x) \in \mathbb{C}[x]$.

Put $f(x,y) := F(x,y)-1$ and $g(x,y) := yq(x)-1$. We denote by M the complement

$$M = \mathbb{C}^2 \setminus \{f(x,y)=0, g(x,y)=0\}.$$

The main result in this note shows how to compute the characteristic variety $\mathcal{V}_1(M)$, for all polynomials $p(x), q(x)$ such that they have at least one common root and $p(x)+1, q(x)$ have no common root.

In Section 2 we recall the definition and the basic properties of the characteristic and resonance varieties. In Section 3 we compute the characteristic variety $\mathcal{V}_1(M)$. In particular, we obtain examples of characteristic varieties with an arbitrary number of translated components (Theorem 3.6). This is an extension for Theorem 4.1 in [12].

2 Characteristic and Resonance varieties

Let M be a smooth, irreducible, quasi-projective complex variety. The character variety of M is defined by

$$\mathbb{T}(M) := Hom(H_1(M), \mathbb{C}^*).$$

This is an algebraic group whose identity irreducible component $\mathbb{T}(M)_1$ is an algebraic torus $(\mathbb{C}^*)^{b_1(M)}$. Consider the exponential mapping

$$\exp : H^1(M, \mathbb{C}) \to H^1(M, \mathbb{C}^*) = \mathbb{T}(M) \tag{1}$$

induced by the usual exponential function $\exp : \mathbb{C} \to \mathbb{C}^*$. Clearly

$$\exp(H^1(M, \mathbb{C})) = \mathbb{T}(M)_1.$$

The *characteristic varieties* of M are the jumping loci for the first cohomology of M, with coefficients in rank one local systems:

$$\mathcal{V}^i_k(M) = \{\rho \in \mathbb{T}(M) : \dim H^i(M, \mathcal{L}_\rho) \geq k\}.$$

When $i = 1$, we use the simpler notation $\mathcal{V}_k(M) = \mathcal{V}^1_k(M)$.

Foundational results on the structure of the cohomology support loci for local systems on quasi-projective algebraic varieties were obtained by Beauville [2], Green and Lazarsfeld [9], Simpson [11] (for the proper case), and Arapura [1] (for the quasi-projective case and first characteristic varieties $\mathcal{V}_1(M)$).

Theorem 2.1. *The strictly positive dimensional irreducible components of the first characteristic variety $\mathcal{V}_1(M)$ are translated subtori in $\mathbb{T}(M)$ by elements of finite order. When M is proper, then all the components of $\mathcal{V}^i_k(M)$ are translated subtori in $\mathbb{T}(M)$ by elements of finite order.*

The strictly positive dimensional irreducible components of the first characteristic variety $\mathcal{V}_1(M)$ are described as follows.

Theorem 2.2. *([2], [1]) Let W be a $d-$dimensional irreducible component of $\mathcal{V}_1(M), d > 0$. Then there is a regular morphism $f : M \to S$ onto a smooth curve S with $b_1(S) = d$ such that the generic fiber F of f is connected, and a torsion character $\rho \in \mathbb{T}(M)$ such that the composition*

$$\pi_1(F) \xrightarrow{i_\#} \pi_1(M) \xrightarrow{\rho} \mathbb{C}^*,$$

where $i : F \to M$ is the inclusion, is trivial and $W = \rho \cdot f^*(\mathbb{T}(S))$.

Remark 2.3. When M is a hypersurface complement in $\mathbb{P}^n$, the curve S in Theorem 2.2 above is obtained from $\mathbb{C}$ by deleting d points, see [7], Theorem 1.11.

If we fix a regular mapping $f : M \to S$ as above, the number of irreducible components $W = \rho \cdot f^*(\mathbb{T}(S))$ obtained by varying the torsion character ρ is given by the following.

Theorem 2.4. *([6]) For a given regular mapping $f : M \to S$ as above, the associated irreducible components $W = \rho \cdot f^*(\mathbb{T}(S))$ are parametrized by the Pontrjagin dual $\hat{T}(f) = Hom(T(f); \mathbb{C}^*)$ of the finite abelian group*

$$T(f) = \frac{ker\{f^* : H_1(M) \to H_1(S)\}}{im\{i^* : H_1(F) \to H_1(M)\}}$$

if $\chi(S) < 0$ and by the non-trivial elements of this Pontrjagin dual $\hat{T}(f)$ if $\chi(S) = 0$.

The group $T(f)$ is determined as follows.

Theorem 2.5. *([6]) Let S is a non-proper smooth curve and $f : M \to S$ be a regular function. Then the group $T(f)$ is computed by the following*

$$T(f) = \oplus_{c \in C(h)} \mathbb{Z}/m_c\mathbb{Z},$$

where m_c is the multiplicity of the divisor $f^{-1}(c)$ and $C(f)$ is the set of bifurcation values of f.

The (first) *resonance varieties* of M are the jumping loci for the first cohomology of the complex $H^*(H^*(M, \mathbb{C}), \alpha\wedge)$, namely

$$\mathcal{R}_k(M) = \{\alpha \in H^1(M, \mathbb{C}) : \dim H^1(H^*(M, \mathbb{C}), \alpha\wedge) \geq k\}.$$

The relation between the resonance and characteristic varieties can be summarized as follows, see [8].

Theorem 2.6. *Assume that M is any hypersurface complement in $\mathbb{P}^n$. Then the irreducible components E of the resonance variety $\mathcal{R}_1(M)$ are linear subspaces in $H^1(M, \mathbb{C})$ and the exponential mapping (1) sends these irreducible components E onto the irreducible components W of $\mathcal{R}_1(M)$ with $1 \in W$.*

3 The Characteristic varieties $\mathcal{V}_1(M)$

Consider from now on the complement $M = \mathbb{C}^2 \setminus C$, where $C = C_0 \cup C_1, C_0 = \{g(x, y) = 0\}$ and $C_1 = \{f(x, y) = 0\}$.

By the same argument as in Section 3 in [12] we can prove the following.

Theorem 3.1. *The integral (co)homology of the surface M is torsion free and*

$$b_1(M) = 2, b_2(M) = s + t,$$

where s and t are the numbers of roots of $q(x)$ and $p(x)q(x)$, respectively. Moreover, the cup-product

$$\cup : H^1(M) \times H^1(M) \to H^2(M)$$

is non-trivial.

Using the definition of the resonance varieties we get the following.

Corollary 3.2. *The resonance varieties of M are trivial, i.e. $\mathcal{R}_k(M) = 0$ for any $k > 0$.*

Since the resonance varieties are trivial, and M is a hypersurface complement, it follows from Theorem 2.6 that the characteristic varieties $\mathcal{V}_1(M)$ can contain only isolated points and 1-dimensional translated components. In this section we determine the latter ones.

In view of Theorem 2.2 and Remark 2.3, any such component comes from a mapping $h : M \to \mathbb{C}^*$. If we regard h as a regular function on the affine variety M, it follows that h should have the form

$$h = \frac{P(x,y)}{f^m g^n}$$

for some polynomial P and some non-negative integers m, n. If P is not in the multiplicative system spanned by f and g, then P vanishes at some point of M and this is a contradiction. It follows that we may assume that

$$h = f^m g^n$$

for some (positive or negative) integers m, n. Now, we are looking for all such maps such that they have multiple fibers and connected generic fiber.

Lemma 3.3. *For all integer numbers $m > 1, n > 1$ and $c \in \mathbb{C} \setminus \{0\}$, then the generic fiber of the polynomial $f^m(x,y) + cg^n(x,y)$ is connected.*

We need the following fact.

Lemma 3.4. *([6]) For any polynomial map $P : \mathbb{C}^n \to \mathbb{C}$ the followings are equivalent:*

(1) The generic fiber of P is connected;

(2) There do not exist polynomials $H : \mathbb{C} \to \mathbb{C}$ and $Q : \mathbb{C}^n \to \mathbb{C}$ such that $\deg(H) > 1$ *and $P = H(Q)$.*

Proof of Lemma 3.3. Let $\Phi : \mathbb{C}^2 \to \mathbb{C}^2$ be given by $\Phi(x,y) = (x, g(x,y))$. We have

$$f^m(x,y) + cg^n(x,y) = h \circ \Phi,$$

where $h(u,v) := (p(u)v - 1)^m + cv^n$.

It is easy to see that the restriction of Φ on $\mathbb{C}^2 \setminus A$ is a homeomorphism, where $A = \{(a,y) : q(a) = 0, y \in \mathbb{C}\}$. Then, the generic fiber of $f^m(x,y) + cg^n(x,y)$ is connected if and only if the generic fiber of $h(u,v)$ is connected.

Now, we assume by contradiction that the generic fiber of $h(u,v)$ is not connected. According to Lemma 3.4, there are polynomials $H : \mathbb{C} \to \mathbb{C}$ and $Q : \mathbb{C}^2 \to \mathbb{C}$ such that $\deg(H) > 1$ and

$$(p(u)v-1)^m + cv^n = H(Q(u,v)).$$

We consider the singular locus of the polynomials in the above equality. Since $\deg(H) > 1$ then the singular locus of $H(Q(u,v))$ has dimension at least one. In particular, there are infinitely many points. However, singular points of $h(u,v)$ are roots of the following systems.

$$\begin{cases} p^{'}(u) = 0, \\ mp(u)(p(u)v-1)^{m-1} + cnv^{n-1} = 0 \end{cases}$$

or

$$\begin{cases} p(u) = 0. \\ v = 0 \end{cases}$$

It is easy to see that the above systems have only finitely many points. Contradiction. $\square$

Lemma 3.5. *Assume that the map $h = f^m g^n : M \to \mathbb{C}^*$ has connected generic fiber and a multiple fiber. Then $n = 0$ and $m = \pm 1$.*

Proof. If $n = 0$ then $m = \pm 1$, because h has connected generic fiber. Similarly, if $m = 0$ then $n = \pm 1$. However, since $\deg(f) > \deg(g)$, it is easy to show that the function $g : \mathbb{C}^2 \setminus \{fg = 0\} \to \mathbb{C}^*$ has not any multiple fiber.

Now, we assume that $mn \neq 0$. Since $M = \mathbb{C}^2 \setminus \{fg = 0\}$ and f, g are two irreducible polynomial then the map $h : M \to \mathbb{C}^*$ has multiple fiber if and only if, there exist $c \in \mathbb{C}^*, h_1 \in \mathbb{C}[x,y], h_1 \not\vdots f, h_1 \not\vdots g$ and integer numbers $s, l, k, |s| > 1$, such that

$$f^m g^n = c + h_1^s f^l g^k. \tag{2}$$

Since f, g, h_1 are pairwise relatively prime then $ml \geq 0$ and $nk \geq 0$. There are four cases.

a) $m, l, n, k \geq 0$: This implies that $l = k = 0$ and hence, the generic fiber of h has at least $|s| > 1$ connected components which is a contradiction.

b) $m, l, n, k \leq 0$: By dividing two sides of the equality (2) by the lowest powers of f and g, one can prove that $m = l$ and $n = k$. It means

$$(f^m g^n)^{-1} = \frac{1}{c}(1 - h_1^s).$$

So the generic fiber of $f^m g^n$ is not connected.

c) $m, l \geq 0$ and $n, k \leq 0$: Similarly, we get $l = 0$ and $n = k$. Hence $f^m = cg^{-n} + h_1^s$. Therefore, the generic fiber of the polynomial $f^m - cg^{-n}$ is not connected, contradicts to Lemma 3.3.

d) $m, l \leq 0$ and $n, k \geq 0$: By the same argument, we also obtain the contradiction. □

The main result in this paper is the following.

Theorem 3.6. *Let $p(x)$ and $q(x) \in \mathbb{C}[x]$ be two polynomials such that they have at least one common root and $p(x)+1, q(x)$ have no common root. Then, if there exist an integer number $s > 1$ and a polynomial $p_1 \in \mathbb{C}[x]$ such that*

$$p(x) = p_1(x)^s,$$

the strictly positive dimensional components of $\mathcal{V}_1(M)$ are the translated 1-dimensional sub-tori

$$W_j = \epsilon_j \times \mathbb{C}^*,$$

where d is the maximum of the exponent s above and $\epsilon_j = \exp(2\pi i j/d)$ for $j = 1, 2, \ldots, d-1$. Moreover, for a local system $\mathcal{L} \in W_j$ one has $\dim H^1(M, \mathcal{L}) \geq 1$ and equality holds with finitely many exceptions.

Otherwise, there do not exist strictly positive dimensional components of $\mathcal{V}_1(M)$.

Proof. According to Theorem 2.2 and Remark 2.3, any translated positive dimensional component of $\mathcal{V}_1(M)$ comes from a map $h : M \to \mathbb{C}^*$ which has connected generic fibers.

According to Lemma 3.5, the only morphisms associated to strictly positive dimensional components of $\mathcal{V}_1(M)$ are $f : M \to \mathbb{C}^*$ and $f^{-1} : M \to \mathbb{C}^*, z \mapsto f(z)^{-1}$, but they give the same associated component of $\mathcal{V}_1(M)$. Thus all translated positive dimensional components of $\mathcal{V}_1(M)$ are associated to the map $f : M \to \mathbb{C}^*$.

On the other hand, it is easy to see that the only possibly multiple fiber of f is $f^{-1}(-1)$. Hence, according to Theorem 2.5, if $p(x)$ is not a power of a polynomial then $T(f) = 0$ and there does not exist strictly positive dimensional components of $\mathcal{V}_1(M)$; unless $T(f) = \mathbb{Z}/d\mathbb{Z}$, where

$$d = \max\{s \in \mathbb{N} : p(x) = p_1(x)^s, p_1 \in \mathbb{C}[x]\}.$$

We now consider the later case. It is deduced from Theorem 2.6 that there are exactly $d-1$ associated 1-dimensional translated components. If we identify $\mathbb{T}(M) = \mathbb{C}^*$ by associating to a local system $\mathcal{L} \in \mathbb{T}(M)$ the two monodromies (λ_0, λ_1) about the curves C_0 and C_1, and in a similar way $\mathbb{T}(\mathbb{C}^*) = \mathbb{C}^*$, then the induced morphism

$$f^* : \mathbb{T}(\mathbb{C}^*) \to \mathbb{T}(M)$$

is just $\lambda \mapsto (1, \lambda)$.

With these identifications, the above $d-1$ associated 1-dimensional translated components of $\mathcal{V}_1(M)$ are given by $W_j = \epsilon_j \times \mathbb{C}^*$, where $\epsilon_j = \exp(2\pi i j/d)$ for $j = 1, 2, \ldots, d-1$.

The inequality on dimension of cohomology group of M is the direct consequence of Corollary 5.9 in [6]. □

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