On non-commutative Minkowski spheres

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Abstract

The purpose of the following is to try to make sense of the stereographic projection in a non-commutative setup. To this end, we consider the open unit ball of a ternary ring of operators, which naturally comes equipped with a non-commutative version of a hyperbolic metric and ask for a manifold onto which the open unit ball can be mapped so that one might think of this situation as providing a noncommutative analog to mapping the open disk of complex numbers onto the hyperboloid in three space, equipped with the restriction of the Minkowskian metric. We also obtain a related result on the Jordan algebra of self-adjoint operators.

1 Introduction

By definition, the classical Minkowski sphere is the set

$$\mathbf{M} = \mathbf{M}(\mathbb{R}^4) := \{(t, x, y, z) \in \mathbb{R}^4 : t^2 - (x^2 + y^2 + z^2) = 1, t > 0\}.$$

It is straightforward to verify that the Hilbert ball

$$\mathbf{B} = \mathbf{B}(\mathbb{R}^3) := \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1^2 + a_2^2 + a_3^2 = 1\}$$

is mapped injectively onto $\mathbf{M}$ by the transformation

$$\Phi(\mathbf{a}) = \Phi(a_1, a_2, a_3) := \frac{1}{1 - (a_1^2 + a_2^2 + a_3^2)} \left(1 + a_1^2 + a_2^2 + a_3^2, 2a_1, 2a_2, 2a_3 \right).$$

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Namely, we have

$$\Phi : \mathbf{B} \leftrightarrow \mathbf{M}, \quad \Phi^{-1}(t, \mathbf{x}) = (1+t)^{-1}\mathbf{x} \quad \text{at } (t, \mathbf{x}) \in \mathbf{M}.$$

Notice that, by identifying $\mathbb{R}^3$ with $\mathbf{E} := \text{Mat}(1, 3, \mathbb{R})$ the set of all row 3-vectors and $\mathbb{R}^4$ with $\mathbb{R} \times \mathbf{E} \equiv \text{Mat}(1, 1, \mathbb{R}) \times \text{Mat}(1, 3, \mathbb{R})$, respectively, in matrix terms we can write $\Phi(\mathbf{a}) = (\Phi_0(\mathbf{a}), \Phi_1(\mathbf{a}))$ where

$$\Phi_0(\mathbf{a}) = (1 - \mathbf{a}\mathbf{a}^*)^{-1}(1 + \mathbf{a}\mathbf{a}^*), \quad \Phi_1(\mathbf{a}) = 2(1 - \mathbf{a}\mathbf{a}^*)^{-1}\mathbf{a}. \quad (1.1)$$

It is a more interesting fact that Φ lifts the natural hyperbolic geometry of $\mathbf{B}$ to $\mathbf{M}$ in a manner such that vector fields corresponding to hyperbolic translation flows of $\mathbf{B}$ will be mapped to restrictions of $\mathbb{R}^4$ -vector fields to $\mathbf{M}$ depending linearly on the coordinates $t, \mathbf{x}$ and the 3×3 -matrix

$$\tilde{t} := (1 + \mathbf{a}^*\mathbf{a})(1 + \mathbf{a}^*\mathbf{a})^{-1} \quad \text{at } (t, \mathbf{x}) = \Phi(\mathbf{a})$$

of a non-commutative time. That is, for the vector fields

$$v_{\mathbf{u}}(\mathbf{a}) := \mathbf{u} - \mathbf{a}\mathbf{u}^*\mathbf{a} \quad (\mathbf{a} \in \mathbf{B}, \mathbf{u} \in \mathbf{E}) \quad (1.2)$$

we get

$$\begin{aligned} [\Phi^\# v_{\mathbf{u}}](t, \mathbf{x}) &:= \left. \frac{d}{d\tau} \right|_{\tau=0} \Phi \left(\Phi^{-1}(t, \mathbf{x}) + \tau v_{\mathbf{u}}(\Phi^{-1}(t, \mathbf{x})) \right) = \\ &= \left(2\mathbf{u}\mathbf{x}^*, t\mathbf{u} + \mathbf{u}\tilde{t} \right) \quad \text{at } (t, \mathbf{x}) = \Phi(\mathbf{a}). \end{aligned}$$

The appearance of the non-commutative time term suggests that we should regard an embedding of $\mathbf{B}$ instead of $\text{Mat}(1, 1) \times \text{Mat}(1, 3)$ into $\widehat{\mathbf{E}} := \text{Mat}(1, 1) \times \text{Mat}(3, 3) \times \text{Mat}(1, 3) \times \text{Mat}(3, 1)$ by the mapping

$$\begin{aligned} \widehat{\Phi}(\mathbf{a}) &:= (\Phi_0(\mathbf{a}), \tilde{\Phi}_0(\mathbf{a}), \Phi_1(\mathbf{a}), \tilde{\Phi}_1(\mathbf{a})); \\ \tilde{\Phi}_0(\mathbf{a}) &:= \tilde{t}(\mathbf{a}) = (1 + \mathbf{a}^*\mathbf{a})(1 + \mathbf{a}^*\mathbf{a})^{-1}, \\ \tilde{\Phi}_1(\mathbf{a}) &:= \Phi_1(\mathbf{a})^* = 2\mathbf{a}^*(1 - \mathbf{a}\mathbf{a}^*)^{-1} = 2(1 - \mathbf{a}^*\mathbf{a})^{-1}\mathbf{a}^*. \end{aligned} \quad (1.3)$$

In this way, the lifted fields $\widehat{\Phi}^\# v_{\mathbf{u}}$ automatically become the restriction of a real linear vector on $\widehat{\mathbf{M}} := \text{ran}(\widehat{\Phi})$ to a real-linear vector field of $\widehat{\mathbf{E}}$, since

$$\begin{aligned} [\widehat{\Phi}^\# v_{\mathbf{u}}](t, \tilde{t}, \mathbf{x}, \tilde{\mathbf{x}}) &= (\mathbf{u}\mathbf{x}^* + \mathbf{x}\mathbf{u}^*, \mathbf{u}^*\mathbf{x} + \mathbf{x}^*\mathbf{u}, t\mathbf{u} + \mathbf{u}\tilde{t}, \mathbf{u}^*t + \tilde{t}\mathbf{u}^*) \\ \text{if } (t, \tilde{t}, \mathbf{x}, \tilde{\mathbf{x}}) &= \widehat{\Phi}(\mathbf{a}), \quad \mathbf{a} \in \mathbf{B}. \end{aligned} \quad (1.4)$$

Our purpose in this note is to generalize the above considerations to the setting of *ternary rings of operators* (TRO in the sequel). As a by-product of our main theorem, we obtain a result of possible independent interest concerning the Jordan algebra of self-adjoint operators.

2 Results

Henceforth $\mathbf{H}, \mathbf{K}$ will stand for two arbitrarily fixed real or complex Hilbert spaces and $\mathbf{E}$ denotes a TRO in $\mathcal{L}(\mathbf{H}, \mathbf{K}) (= \{\text{bounded linear operators } \mathbf{H} \rightarrow \mathbf{K}\})$. That is $\mathbf{E} \subset \mathcal{L}(\mathbf{H}, \mathbf{K})$ is a closed linear subspace such that $[\mathbf{abc}] := \mathbf{ab}^*\mathbf{c} \in \mathbf{E}$ whenever $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{E}$. We write

$$\mathcal{A}(\mathbf{E}) := \{t \in \mathcal{L}(\mathbf{K}) : t = t^*, t\mathbf{E} \subset \mathbf{E}\}, \quad \tilde{\mathcal{A}}(\mathbf{E}) := \{\tilde{t} \in \mathcal{L}(\mathbf{H}) : \tilde{t} = \tilde{t}^*, \mathbf{E}\tilde{t} \subset \mathbf{E}\}$$

and, by setting also $\tilde{\mathbf{E}} := \mathbf{E}^* = \{\mathbf{z}^* : \mathbf{z} \in \mathbf{E}\} \subset \mathcal{L}(\mathbf{K}, \mathbf{H})$, we define the operator $\hat{\Phi} : \mathbf{B} := \{\mathbf{a} \in \mathbf{E} : \|\mathbf{a}\| < 1\} \rightarrow \hat{\mathbf{E}} := \mathcal{A}(\mathbf{E}) \times \tilde{\mathcal{A}}(\mathbf{E}) \times \mathbf{E} \times \tilde{\mathbf{E}}$ ranging in the linking algebra [2] by (1.1) and (1.3). Indeed $\mathbf{xx}^* \in \mathcal{A}(\mathbf{E})$ and $\mathbf{x}^*\mathbf{x} \in \tilde{\mathcal{A}}(\mathbf{E})$ for any $\mathbf{x} \in \mathbf{E}$ whence, with norm-convergence, also

$$\begin{aligned} \Phi_0(\mathbf{a}) &= 1_{\mathbf{K}} + 2 \sum_{n=1}^{\infty} (\mathbf{aa}^*)^n \in \mathcal{A}(\mathbf{E}), \quad \tilde{\Phi}_0(\mathbf{a}) = 1_{\mathbf{H}} + 2 \sum_{n=1}^{\infty} (\mathbf{a}^*\mathbf{a})^n \in \tilde{\mathcal{A}}(\mathbf{E}), \\ \Phi_1(\mathbf{a}) &= 2 \sum_{n=0}^{\infty} (\mathbf{aa}^*)^n \mathbf{a} = [1_{\mathbf{K}} + \Phi_0(\mathbf{a})]\mathbf{a} = \\ &= 2 \sum_{n=0}^{\infty} \mathbf{a}(\mathbf{a}^*\mathbf{a})^n = \mathbf{a}[1_{\mathbf{H}} + \tilde{\Phi}_0(\mathbf{a})] \in \mathbf{E} \quad \text{for any } \mathbf{a} \in \mathbf{B}. \end{aligned} \tag{2.1}$$

Let us finally define

$$\begin{aligned} \widehat{\mathbf{M}} &:= \left\{ (t, \tilde{t}, \mathbf{x}, \tilde{\mathbf{x}}) \in \hat{\mathbf{E}} : t \in \mathcal{A}_+(\mathbf{E}), t^2 - \mathbf{xx}^* = 1_{\mathbf{K}}, \tilde{\mathbf{x}} = \mathbf{x}^*, \right. \\ &\quad \left. \tilde{t} \in \tilde{\mathcal{A}}_+(\mathbf{E}), \tilde{t}^2 - \tilde{\mathbf{x}}^*\tilde{\mathbf{x}} = 1_{\mathbf{H}}, (1_{\mathbf{K}} + t)^{-1}\mathbf{x} = \mathbf{x}(1_{\mathbf{H}} + \tilde{t})^{-1} \right\}. \end{aligned}$$

Our main result reads as follows.

2.2 Theorem. *In the TRO-setting established above, we have $\hat{\Phi} : \mathbf{B} \leftrightarrow \widehat{\mathbf{M}}$ with*

$$\hat{\Phi}^{-1}(t, \tilde{t}, \mathbf{x}, \tilde{\mathbf{x}}) = (1_{\mathbf{K}} + t)^{-1}\mathbf{x} = \tilde{\mathbf{x}}^*(1_{\mathbf{H}} + \tilde{t})^{-1}, \quad \left((t, \tilde{t}, \mathbf{x}, \tilde{\mathbf{x}}) \in \widehat{\mathbf{M}} \right).$$

The vector fields $v_{\mathbf{u}}$ of infinitesimal hyperbolic parallel shifts on $\mathbf{B}$ defined by (1.2) are lifted to restrictions of linear maps on $\widehat{\mathbf{M}}$ of the form (1.4).

As it is well-known [3], the integration of a vector field $v_{\mathbf{u}}$ provides the flow $[M_{\mathbf{u}}^{\tau} : \tau \in \mathbb{R}]$ of Potapov-Möbius transformations

$$\begin{aligned} M_{\mathbf{u}}^{\tau}(\mathbf{a}) : &= (1_{\mathbf{K}} - \mathbf{u}_{\tau}\mathbf{u}_{\tau}^*)^{-1/2}(\mathbf{a} + \mathbf{u}_{\tau})(1_{\mathbf{H}} + \mathbf{u}_{\tau}^*\mathbf{a})^{-1}(1_{\mathbf{H}} - \mathbf{u}_{\tau}^*\mathbf{u}_{\tau})^{1/2} = \\ &= (1_{\mathbf{K}} - \mathbf{u}_{\tau}\mathbf{u}_{\tau}^*)^{-1/2}(1_{\mathbf{K}} + \mathbf{a}\mathbf{u}_{\tau}^*)^{-1}(\mathbf{a} + \mathbf{u}_{\tau})(1_{\mathbf{H}} - \mathbf{u}_{\tau}^*\mathbf{u}_{\tau})^{1/2}, \quad (\mathbf{a} \in \mathbf{B}) \end{aligned}$$

where, in terms of Kaup's odd functional calculus [1],

$$\mathbf{u}_\tau := \tanh(\tau \mathbf{u}) = \sum_{n=0}^{\infty} \alpha_n \tau^{2n+1} (\mathbf{u} \mathbf{u}^*)^n \mathbf{u} = \sum_{n=0}^{\infty} \alpha_n \tau^{2n+1} \mathbf{u} (\mathbf{u}^* \mathbf{u})^n$$

with the constants $\alpha_0, \alpha_1, \dots \in \mathbb{R}$ of the expansion $\tanh(\xi) = \sum_{n=0}^{\infty} \alpha_n \xi^{2n+1}$.

On the other hand, linear vector fields are integrated simply by taking the exponentials of their multiples with the virtual time parameter τ . Taking into account that (1.4) can be written in the matrix form $\widehat{\Phi}^\# v_{\mathbf{u}} : \widehat{\mathbf{M}} \ni (t, \tilde{t}, \mathbf{x}, \tilde{\mathbf{x}}) \mapsto (t, \tilde{t}, \mathbf{x}, \tilde{\mathbf{x}}) \mathbf{L}_{\mathbf{u}}$ with

$$\mathbf{L}_{\mathbf{u}} := \begin{bmatrix} 0 & 0 & R(\mathbf{u}) & L(\mathbf{u}^*) \\ 0 & 0 & L(\mathbf{u}) & R(\mathbf{u}^*) \\ R(\mathbf{u}^*) & L(\mathbf{u}^*) & 0 & 0 \\ L(\mathbf{u}) & R(\mathbf{u}) & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & S(\mathbf{u}) \\ S(\mathbf{u}^*) & 0 \end{bmatrix}$$

where $L(\cdot)$ and $R(\cdot)$ denote left and right multiplication as usually, we get the following.

2.3 Corollary. $M_{\mathbf{u}}^\tau(\mathbf{a}) = \widehat{\Phi}^{-1} \left(\widehat{\Phi}(\mathbf{a}) \exp(\tau \mathbf{L}_{\mathbf{u}}) \right), \quad (\tau \in \mathbb{R}, \mathbf{a} \in \mathbf{B}).$

Let us restrict ourselves to the case $\mathbf{E} = \mathcal{L}(\mathbf{H}) (= \mathcal{L}(\mathbf{H}, \mathbf{H}))$ and consider the behavior of $\widehat{\Phi}$ on the unit ball $\mathbf{B}^{(s)}$ of the self-adjoint part $\mathcal{L}^{(s)}(\mathbf{H}) := \{\mathbf{a} \in \mathcal{L}(\mathbf{H}) : \mathbf{a} = \mathbf{a}^*\}$. Then $\phi_0(\mathbf{a}) = \widetilde{\phi}_0(\mathbf{a}) = (1_{\mathbf{H}} + \mathbf{a}^2)(1_{\mathbf{H}} - \mathbf{a}^2)^{-1} \in \mathcal{L}^{(s)}(\mathbf{H})$ and $\phi_1(\mathbf{a}) = \widetilde{\phi}_1(\mathbf{a}) = 2\mathbf{a}(1_{\mathbf{H}} - \mathbf{a}^2)^{-1} \in \mathcal{L}^{(s)}(\mathbf{H})$. From (1.4) we see also that

$$[\widehat{\Phi}^\# v_{\mathbf{u}}](t, \mathbf{x}, t, \mathbf{x}) = 2(\mathbf{x} \bullet \mathbf{u}, \mathbf{x} \bullet \mathbf{u}, t \bullet \mathbf{u}, t \bullet \mathbf{u}) \quad \text{if} \quad (t, \mathbf{x}, t, \mathbf{x}) = \widehat{\Phi}(\mathbf{a}), \quad \mathbf{a} \in \mathbf{B}^{(s)} \quad (2.4)$$

in terms of the Jordan product $\mathbf{x} \bullet \mathbf{y} := \frac{1}{2}(\mathbf{x}\mathbf{y} + \mathbf{y}\mathbf{x})$ on $\mathcal{L}^{(s)}(\mathbf{H})$. We get the following explicit linear representation for the Jordan manifold structure of the unit ball of $\mathcal{L}^{(s)}(\mathbf{H})$ discussed in Theorem 2.6 of our paper [4].

2.5 Corollary. *For the transformation $\Phi := [\Phi_0, \Phi_1]$ we have $\Phi : \mathbf{B}^{(s)} \leftrightarrow \mathbf{M}^{(s)} := \{(t, \mathbf{x}) \in \mathcal{L}^{(s)}(\mathbf{H})^2 : t \geq 0, t^2 - \mathbf{x}^2 = 1_{\mathbf{H}}\}$. The Möbius transformations $M_{\mathbf{u}}^\tau$ ($\mathbf{u} \in \mathcal{L}^{(s)}(\mathbf{H})$) map $\mathbf{B}^{(s)}$ onto itself and, in terms of the Jordan multiplication $J(\mathbf{u}) := \frac{1}{2}[L(\mathbf{u}) + R(\mathbf{u})]$,*

$$\begin{aligned} M_{\mathbf{u}}^\tau(\mathbf{a}) &= \Phi^{-1} \left(\Phi(\mathbf{a}) \exp \left(2\tau \begin{bmatrix} J(\mathbf{u}) & 0 \\ 0 & J(\mathbf{u}) \end{bmatrix} \right) \right) = \\ &= \Phi^{-1} \left(\begin{array}{c} \frac{1}{2} \left(e^{\tau \mathbf{u}} [\phi_0(\mathbf{a}) + \phi_1(\mathbf{a})] e^{\tau \mathbf{u}} + e^{\tau \mathbf{u}} [\phi_0(\mathbf{a}) - \phi_1(\mathbf{a})] e^{\tau \mathbf{u}}, \right. \\ \left. e^{\tau \mathbf{u}} [\phi_0(\mathbf{a}) + \phi_1(\mathbf{a})] e^{\tau \mathbf{u}} - e^{\tau \mathbf{u}} [\phi_0(\mathbf{a}) - \phi_1(\mathbf{a})] e^{\tau \mathbf{u}} \right) \end{array} \right). \end{aligned}$$

3 Proof of Theorem 2.2

Theorem 2.2 is an immediate consequence of the following substatements.

3.1 Lemma. *The component Φ_1 of Φ is injective. Moreover $\Phi_1 : \mathbf{B} \leftrightarrow \mathbf{E}$ with*

$$\Phi_1^{-1}(\mathbf{c}) = \left[1_{\mathbf{K}} + \sqrt{1_{\mathbf{K}} + \mathbf{c}\mathbf{c}^*}\right]^{-1} \mathbf{c} = \mathbf{c} \left[1_{\mathbf{H}} + \sqrt{1_{\mathbf{H}} + \mathbf{c}^*\mathbf{c}}\right]^{-1}, \quad (\mathbf{c} \in \mathbf{E}).$$

3.2 Lemma. *For any $\mathbf{a} \in \mathbf{B}$, $\phi_0(\mathbf{a})^2 - \phi_1(\mathbf{a})\phi_1(\mathbf{a})^* = 1_{\mathbf{K}}$ and $\tilde{\phi}_0(\mathbf{a})^2 - \phi_1(\mathbf{a})^*\phi_1(\mathbf{a}) = 1_{\mathbf{H}}$.*

3.3 Lemma. *Let $\mathbf{x} \in \mathbf{E}$, $t \in \mathcal{L}_+(\mathbf{K})$ and $\tilde{t} \in \tilde{\mathcal{L}}_+(\mathbf{H})$ be so given that $t^2 - \mathbf{x}\mathbf{x}^* = 1_{\mathbf{K}}$ and $\tilde{t}^2 - \mathbf{x}^*\mathbf{x} = 1_{\mathbf{H}}$. Then $t \in \mathcal{A}_+(\mathbf{E})$, $\tilde{t} \in \mathcal{A}_+(\tilde{\mathbf{E}}) = (\mathcal{A}_+(\tilde{\mathbf{E}}) := \mathbf{E}^*)$ and $(1_{\mathbf{K}} + t)^{-1}\mathbf{x} = \mathbf{x}(1_{\mathbf{H}} + \tilde{t})^{-1} \in \mathbf{B}$. By writing $\mathbf{a} := (1_{\mathbf{K}} + t)^{-1}\mathbf{x}$ for the common value, we have $t = \Phi_0(\mathbf{a})$, $\tilde{t} = \tilde{\Phi}_0(\mathbf{a})$, $\mathbf{x} = \phi_1(\mathbf{a})$, $\mathbf{x}^* = \tilde{\phi}_1(\mathbf{a})$.*

3.4 Proposition. *Let $\mathbf{M} := \{(t, \mathbf{x}) \in \mathcal{A}_+(\mathbf{E}) \times \mathbf{E} : t^2 - \mathbf{x}\mathbf{x}^* = 1_{\mathbf{K}}\}$ and let $\mathbf{u} \in \mathbf{E}$ be fixed arbitrarily. Then the submap $\Phi := [\Phi_0, \Phi_1]$ of $\hat{\Phi} (= [\Phi_0, \tilde{\Phi}_0, \Phi_1, \tilde{\Phi}_1])$ lifts the vector field $v_{\mathbf{u}}$ to $(t, \mathbf{x}) \mapsto (\mathbf{u}\mathbf{x}^* + \mathbf{x}\mathbf{u}^*, t\mathbf{u} + \mathbf{u}\tilde{t})$ with $\tilde{t} := \sqrt{1_{\mathbf{H}} + \mathbf{x}^*\mathbf{x}}$ on $\mathbf{M}$. That is, given $(t, \mathbf{x}) \in \mathbf{M}$ and, by setting $\mathbf{a} := \Phi^{-1}(t, \mathbf{x}) = (1_{\mathbf{K}} + t)^{-1}\mathbf{x}$, we have*

$$[\Phi^{\#}v_{\mathbf{u}}](t, \mathbf{x}) = \left. \frac{d}{d\tau} \right|_{\tau=0} \Phi(\mathbf{a} + \tau(\mathbf{u} - \mathbf{a}\mathbf{u}^*\mathbf{a})) = (\mathbf{u}\mathbf{x}^* + \mathbf{x}\mathbf{u}^*, t\mathbf{u} + \mathbf{u}\tilde{t}).$$

3.5 Corollary. *If $\tilde{\mathbf{M}} := \{(\tilde{t}, \tilde{\mathbf{x}}) \in \tilde{\mathcal{A}}_+(\mathbf{E}^*) \times \mathbf{E}^* : \tilde{t}^2 - \tilde{\mathbf{x}}\tilde{\mathbf{x}}^* = 1_{\mathbf{H}}\}$ and $\mathbf{u} \in \mathbf{E}$*

is arbitrarily fixed then the submap $\tilde{\Phi} := [\tilde{\Phi}_0, \tilde{\Phi}_1]$ of $\hat{\Phi}$ lifts the vector field $v_{\mathbf{u}}$ to $(\tilde{t}, \tilde{\mathbf{x}}) \mapsto (\mathbf{u}^\tilde{\mathbf{x}}^* + \tilde{\mathbf{x}}\mathbf{u}, \tilde{t}\mathbf{u}^* + \mathbf{u}^*\tilde{t})$ with $t := \sqrt{1_{\mathbf{K}} + \tilde{\mathbf{x}}^*\tilde{\mathbf{x}}}$ to $\tilde{\mathbf{M}}$. That is, given $(\tilde{t}, \tilde{\mathbf{x}}) \in \tilde{\mathbf{M}}$ and, by setting $\mathbf{a} := \tilde{\Phi}^{-1}(\tilde{t}, \tilde{\mathbf{x}}) = \tilde{\mathbf{x}}^*(1_{\mathbf{H}} + \tilde{t})^{-1}$, we have*

$$[\tilde{\Phi}^{\#}v_{\mathbf{u}}](\tilde{t}, \tilde{\mathbf{x}}) = \left. \frac{d}{d\tau} \right|_{\tau=0} \tilde{\Phi}(\mathbf{a} + \tau(\mathbf{u} - \mathbf{a}\mathbf{u}^*\mathbf{a})) = (\mathbf{u}^*\tilde{\mathbf{x}}^* + \tilde{\mathbf{x}}\mathbf{u}, \tilde{t}\mathbf{u}^* + \mathbf{u}^*\tilde{t}).$$

Proof of 3.1. Given any $\mathbf{c} \in \mathbf{E}$ let $t_0(\mathbf{c}) := \psi_0(\mathbf{c}\mathbf{c}^*)$ with the continuous real function $\psi_0(\tau) := 1 + \sqrt{1 + \tau}$. By the Spectral Mapping Theorem, $\text{Sp}(t_0(\mathbf{c})) = \psi_0(\text{Sp}(\mathbf{c}\mathbf{c}^*)) > 0$. Hence $t(\mathbf{c}) : 2[1_{\mathbf{K}} + \sqrt{1 + \mathbf{c}\mathbf{c}^*}] = 2t_0(\mathbf{c})^{-1}$ is well-defined

and, by Sinclair's Theorem*, $\|t(\mathbf{c})\mathbf{c}\|^2 = \|t(\mathbf{c})\mathbf{c}\mathbf{c}^*t(\mathbf{c})\| = \max\{\psi(\tau)^2\tau : \tau \in \text{Sp}(\mathbf{c}\mathbf{c}^*)\} \leq 4\|\mathbf{c}\|^2/[1 + \sqrt{1 + \|\mathbf{c}\|^2}]^2 < 1$. To see that $t(\mathbf{c})\mathbf{c} \in \mathbf{E}$ and hence also $\in \mathbf{B}$, notice that, by Weierstrass' Approximation Theorem, there is a sequence $\pi_1, \pi_2, \dots$ of real polynomials converging uniformly to ψ on $\text{Sp}(\mathbf{c}\mathbf{c}^*)$. By Sinclair's Theorem again, $\pi_n(\mathbf{c}\mathbf{c}^*) \rightarrow t(\mathbf{c})$ in norm. However $\mathbf{c}\mathbf{c}^* \in \mathcal{A}(\mathbf{E})$, whence also $\pi_n(\mathbf{c}\mathbf{c}^*) \in \mathcal{A}(\mathbf{E})$ entailing $t(\mathbf{c}) \in \mathcal{A}(\mathbf{E})$ and $t(\mathbf{c})\mathbf{c} \in \mathbf{B}$.

To complete the proof, we show that $t(\phi_0(\mathbf{a}))\phi_0(\mathbf{a}) = \mathbf{a}$ for any $\mathbf{a} \in \mathbf{B}$. Given $\mathbf{a} \in \mathbf{B}$, we have $1_{\mathbf{K}} + \phi_0(\mathbf{a})\phi_0(\mathbf{a})^* = 1_{\mathbf{K}} + 4(1_{\mathbf{K}} - \mathbf{a}\mathbf{a}^*)^{-1}\mathbf{a}\mathbf{a}^*(1_{\mathbf{K}} - \mathbf{a}\mathbf{a}^*)^{-1} = (1_{\mathbf{K}} - \mathbf{a}\mathbf{a}^*)^{-2}[(1_{\mathbf{K}} - \mathbf{a}\mathbf{a}^*)^2 + 4\mathbf{a}\mathbf{a}^*] = (1_{\mathbf{K}} - \mathbf{a}\mathbf{a}^*)^{-2}(1_{\mathbf{K}} + \mathbf{a}\mathbf{a}^*)^2$. It follows $1_{\mathbf{K}} + \sqrt{1_{\mathbf{K}} + \phi_0(\mathbf{a})\phi_0(\mathbf{a})^*} = (1_{\mathbf{K}} - \mathbf{a}\mathbf{a}^*)^{-1}[(1_{\mathbf{K}} - \mathbf{a}\mathbf{a}^*) + 1_{\mathbf{K}} + \mathbf{a}\mathbf{a}^*] = 2(1_{\mathbf{K}} - \mathbf{a}\mathbf{a}^*)^{-1}$ entailing $t(\phi_0(\mathbf{a}))\phi_0(\mathbf{a}) = \frac{1}{2}(1_{\mathbf{K}} - \mathbf{a}\mathbf{a}^*)[2(1_{\mathbf{K}} - \mathbf{a}\mathbf{a}^*)^{-1}\mathbf{a}] = \mathbf{a}$.

Proof of 3.2. Given any operator $\mathbf{a} \in \mathbf{B}$, we have

$$\begin{aligned} & \phi_0(\mathbf{a})^2 - \phi_1(\mathbf{a})\phi_1(\mathbf{a})^* = \\ &= (1_{\mathbf{K}} - \mathbf{a}\mathbf{a}^*)^{-1}(1_{\mathbf{K}} + \mathbf{a}\mathbf{a}^*)^2(1_{\mathbf{K}} - \mathbf{a}\mathbf{a}^*)^{-1} - (1_{\mathbf{K}} - \mathbf{a}\mathbf{a}^*)^{-1}(4\mathbf{a}\mathbf{a}^*)(1_{\mathbf{K}} - \mathbf{a}\mathbf{a}^*)^{-1} = \\ &= (1_{\mathbf{K}} - \mathbf{a}\mathbf{a}^*)^{-1}[1_{\mathbf{K}} + 2\mathbf{a}\mathbf{a}^* + (\mathbf{a}\mathbf{a}^*)^2 - 4\mathbf{a}\mathbf{a}^*](1_{\mathbf{K}} - \mathbf{a}\mathbf{a}^*)^{-1} = \\ &= (1_{\mathbf{K}} - \mathbf{a}\mathbf{a}^*)^{-1}[1_{\mathbf{K}} - 2\mathbf{a}\mathbf{a}^* + (\mathbf{a}\mathbf{a}^*)^2](1_{\mathbf{K}} - \mathbf{a}\mathbf{a}^*)^{-1} = \\ &= (1_{\mathbf{K}} - \mathbf{a}\mathbf{a}^*)^{-1}[1_{\mathbf{K}} - 2\mathbf{a}\mathbf{a}^* + (\mathbf{a}\mathbf{a}^*)^2](1_{\mathbf{K}} - \mathbf{a}\mathbf{a}^*)^{-1} = \\ &= (1_{\mathbf{K}} - \mathbf{a}\mathbf{a}^*)^{-1}(1_{\mathbf{K}} - \mathbf{a}\mathbf{a}^*)^2(1_{\mathbf{K}} - \mathbf{a}\mathbf{a}^*)^{-1} = 1_{\mathbf{K}}. \end{aligned}$$

The proof of the relationship $\tilde{\phi}_0(\mathbf{a})^2 - \phi_1(\mathbf{a})^*\phi_1(\mathbf{a}) = 1_{\mathbf{H}}$ is analogous with terms $\mathbf{a}^*\mathbf{a}$ replacing $\mathbf{a}\mathbf{a}^*$ and $1_{\mathbf{H}}$ instead of $1_{\mathbf{K}}$.

Proof of 3.3. Since $\mathbf{x}\mathbf{x}^* \in \mathcal{A}_+(\mathbf{E})$, by Sinclair's and Weierstrass' Theorems (as in the proof of 3.1), $t = \sqrt{1_{\mathbf{K}} + \mathbf{x}\mathbf{x}^*} \in \mathcal{A}_+(\mathbf{E})$. Similarly $\tilde{t} = \sqrt{1_{\mathbf{H}} + \mathbf{x}^*\mathbf{x}} \in \mathcal{A}_+(\tilde{\mathbf{E}})$. By Lemma 3.1, $(1_{\mathbf{K}} + t)^{-1}\mathbf{x} = [1_{\mathbf{K}} + \sqrt{1_{\mathbf{K}} + \mathbf{x}\mathbf{x}^*}]^{-1}\mathbf{x} = \Phi_1^{-1}(\mathbf{x}) = \mathbf{x}[1_{\mathbf{H}} + \sqrt{1_{\mathbf{H}} + \mathbf{x}^*\mathbf{x}}]^{-1} = \mathbf{x}(1_{\mathbf{H}} + \tilde{t})^{-1}$. Hence the definition of $\mathbf{a}$ ensures that $\mathbf{x} = \Phi_1(\mathbf{a}) = \tilde{\Phi}_1(\mathbf{a})^*$. Thus, by Lemma 3.2, $t = \sqrt{1_{\mathbf{K}} + \mathbf{x}\mathbf{x}^*} = \sqrt{1_{\mathbf{K}} + \Phi_1(\mathbf{a})\Phi_1(\mathbf{a})^*} = \Phi_0(\mathbf{a})$ and $\tilde{t} = \sqrt{1_{\mathbf{H}} + \mathbf{x}^*\mathbf{x}} = \sqrt{1_{\mathbf{K}} + \Phi_1(\mathbf{a})^*\Phi_1(\mathbf{a})} = \tilde{\Phi}_0(\mathbf{a})$.

Proof of 3.4. Notice that, by 3.3 we have $\mathbf{M} = \text{range}(\Phi)$. Let any point $(t, \mathbf{x}) \in \mathbf{M}$ and $\mathbf{u} \in \mathbf{E}$ be fixed arbitrarily and write

$$\mathbf{a} := \Phi^{-1}(t, \mathbf{x}) = (1_{\mathbf{K}} + t)^{-1}\mathbf{x}, \quad \mathbf{v} := v_{\mathbf{u}}(\mathbf{a}) = \mathbf{a} - \mathbf{a}\mathbf{u}^*\mathbf{a}. \quad (3.6)$$

*The norm of a self-adjoint operator coincides with its spectral radius.

Then, in terms of the Möbius transformations $M_{\mathbf{u}}^\tau := \exp(\tau v_{\mathbf{u}})$ we have

$$[\Phi^\# v_{\mathbf{u}}](t, \mathbf{x}) = \frac{d}{d\tau} \Big|_{\tau=0} \Phi \circ M_{\mathbf{u}}^\tau \circ \Phi^{-1}(t, \mathbf{x}) = \frac{d}{d\tau} \Big|_{\tau=0} \Phi \circ M_{\mathbf{u}}^\tau(\mathbf{a}) = \Phi'(\mathbf{a})\mathbf{v}.$$

We calculate both $\Phi'(\mathbf{a})$ and $\mathbf{v}$ in terms of t, x . For the first component of Φ ,

$$\begin{aligned} \Phi_0(\mathbf{a}) &= (1_{\mathbf{K}} - \mathbf{a}\mathbf{a}^*)^{-1}(1_{\mathbf{K}} + \mathbf{a}\mathbf{a}^*) = (1_{\mathbf{K}} - \mathbf{a}\mathbf{a}^*)^{-1}[(1_{\mathbf{K}} - \mathbf{a}\mathbf{a}^*) + 2\mathbf{a}\mathbf{a}^*] = \\ &= 1_{\mathbf{K}} + 2(1_{\mathbf{K}} - \mathbf{a}\mathbf{a}^*)^{-1}\mathbf{a}\mathbf{a}^* = 1_{\mathbf{K}} + \Phi_1(\mathbf{a})\mathbf{a}^*. \end{aligned}$$

Since, by definition $\Phi_1(\mathbf{a}) = \mathbf{x}$, hence we get

$$\begin{aligned} \Phi'_0(\mathbf{a})\mathbf{v} &= \frac{d}{d\tau} \Big|_{\tau=0} \Phi_1(\mathbf{a} + \tau\mathbf{v})(\mathbf{a} + \tau\mathbf{v})^* = \phi_1(\mathbf{a})\mathbf{v}^* + [\phi'_1(\mathbf{a})\mathbf{v}]\mathbf{a}^* = \\ &= \mathbf{x}\mathbf{v}^* + [\phi'_1(\mathbf{a})\mathbf{v}]\mathbf{a}^*. \end{aligned}$$

We can express $\Phi'_1(\mathbf{a})\mathbf{v}$ in algebraic terms of $\mathbf{a}, \mathbf{v}$ as follows:

$$\begin{aligned} \Phi'_1(\mathbf{a})\mathbf{v} &= \frac{d}{d\tau} \Big|_{\tau=0} \Phi_1(\mathbf{a} + \tau\mathbf{v}) = \frac{d}{d\tau} \Big|_{\tau=0} 2[1_{\mathbf{K}} + (\mathbf{a} + \tau\mathbf{v})(\mathbf{a} + \tau\mathbf{v})^*]^{-1}(\mathbf{a} + \tau\mathbf{v}) = \\ &= 2[1_{\mathbf{K}} - \mathbf{a}\mathbf{a}^*]^{-1}\mathbf{v} + 2[1_{\mathbf{K}} - \mathbf{a}\mathbf{a}^*]^{-1}(\mathbf{a}\mathbf{v}^* + \mathbf{v}\mathbf{a}^*)[1_{\mathbf{K}} - \mathbf{a}\mathbf{a}^*]^{-1}\mathbf{a}. \end{aligned}$$

Since $\mathbf{a} = \Phi^{-1}(t, \mathbf{x}) = (1_{\mathbf{K}} + t)^{-1}\mathbf{x}$ and since $\mathbf{x}\mathbf{x}^* = t^2 - 1_{\mathbf{K}} = (t - 1_{\mathbf{K}})(1_{\mathbf{K}} + t)$, here we have

$$\begin{aligned} 1_{\mathbf{K}} - \mathbf{a}\mathbf{a}^* &= 1_{\mathbf{K}} - (1_{\mathbf{K}} + t)^{-1}\mathbf{x}\mathbf{x}^*(1_{\mathbf{K}} + t)^{-1} = 1_{\mathbf{K}} - (1_{\mathbf{K}} + t)^{-1}(t - 1_{\mathbf{K}}) = \\ &= (1_{\mathbf{K}} + t)^{-1}[(1_{\mathbf{K}} + t) - (t - 1_{\mathbf{K}})] = 2(1_{\mathbf{K}} + t)^{-1}, \\ [1_{\mathbf{K}} - \mathbf{a}\mathbf{a}^*]^{-1} &= \frac{1}{2}(1_{\mathbf{K}} + t). \end{aligned}$$

Hence and with (3.6) we conclude

$$\begin{aligned} \Phi'_1(\mathbf{a})\mathbf{v} &= (1_{\mathbf{K}} + t)\mathbf{v} + 2[1_{\mathbf{K}} - \mathbf{a}\mathbf{a}^*]^{-1}\mathbf{a}\mathbf{v}^*[1_{\mathbf{K}} - \mathbf{a}\mathbf{a}^*]^{-1}\mathbf{a} + \\ &+ 2[1_{\mathbf{K}} - \mathbf{a}\mathbf{a}^*]^{-1}\mathbf{v}\mathbf{a}^*[1_{\mathbf{K}} - \mathbf{a}\mathbf{a}^*]^{-1}\mathbf{a} = \\ &= (1_{\mathbf{K}} + t)\mathbf{v} + \frac{1}{2}\mathbf{x}\mathbf{v}^*\mathbf{x} + \frac{1}{2}(1_{\mathbf{K}} + t)\mathbf{v}\mathbf{x}^*(1_{\mathbf{K}} + t)^{-1}\mathbf{x}. \end{aligned}$$

We can express $\mathbf{v}$ in terms of $t, \mathbf{x}$ as

$$\mathbf{v} = \mathbf{u} - \mathbf{a}\mathbf{u}^*\mathbf{a} = \mathbf{u} - (1_{\mathbf{K}} + t)^{-1}\mathbf{x}\mathbf{u}^*(1_{\mathbf{K}} + t)^{-1}\mathbf{x}.$$

Thus

$$\begin{aligned}\Phi'_1(\mathbf{a})\mathbf{v} &= \overbrace{(1_{\mathbf{K}} + t)\mathbf{u} - \mathbf{x}\mathbf{u}^*(1_{\mathbf{K}} + t)^{-1}\mathbf{x}}^{(1)} + \overbrace{-\mathbf{x}\mathbf{u}^*(1_{\mathbf{K}} + t)^{-1}\mathbf{x}}^{(2)} + \\ &\quad + \overbrace{\frac{1}{2}\mathbf{x}\mathbf{u}^*\mathbf{x} - \frac{1}{2}\mathbf{x}\mathbf{x}^*(1_{\mathbf{K}} + t)^{-1}\mathbf{u}\mathbf{x}^*(1_{\mathbf{K}} + t)^{-1}\mathbf{x}}^{(3)} + \overbrace{-\frac{1}{2}\mathbf{x}\mathbf{x}^*(1_{\mathbf{K}} + t)^{-1}\mathbf{u}\mathbf{x}^*(1_{\mathbf{K}} + t)^{-1}\mathbf{x}}^{(4)} + \\ &\quad + \overbrace{\frac{1}{2}(1_{\mathbf{K}} + t)\mathbf{u}\mathbf{x}^*(1_{\mathbf{K}} + t)^{-1}\mathbf{x} - \frac{1}{2}\mathbf{x}\mathbf{u}^*(1_{\mathbf{K}} + t)^{-1}\mathbf{x}\mathbf{x}^*(1_{\mathbf{K}} + t)^{-1}\mathbf{x}}^{(5)} + \overbrace{-\frac{1}{2}\mathbf{x}\mathbf{u}^*(1_{\mathbf{K}} + t)^{-1}\mathbf{x}\mathbf{x}^*(1_{\mathbf{K}} + t)^{-1}\mathbf{x}}^{(6)} .\end{aligned}$$

The sum (2)+(3)+(6) vanishes because $\mathbf{x}\mathbf{x}^* = (1_{\mathbf{K}} + t)(t - 1_{\mathbf{K}})$ and hence

$$\begin{aligned}(2) + (3) + (6) &= \mathbf{x}\mathbf{u}^* \left[-(1_{\mathbf{K}} + t)^{-1} + \frac{1}{2} \cdot 1_{\mathbf{K}} - \frac{1}{2}(t_{\mathbf{K}} - 1)(1_{\mathbf{K}} + t)^{-1} \right] \mathbf{x} = \\ &= \mathbf{x}\mathbf{u}^* \left[-1_{\mathbf{K}} + \frac{1}{2} \cdot 1_{\mathbf{K}} + \frac{1}{2}t - \frac{1}{2}t + \frac{1}{2} \cdot 1_{\mathbf{K}} \right] (1_{\mathbf{K}} + t)^{-1}\mathbf{x} = 0 .\end{aligned}$$

The sum (4)+(5) can also be simplified as

$$\begin{aligned}(4) + (5) &= \frac{1}{2} \left[-(1_{\mathbf{K}} + t)^{-1} \overbrace{\mathbf{x}\mathbf{x}^*}^{(1_{\mathbf{K}}+t)(t_{\mathbf{K}}-1)} + (1_{\mathbf{K}} + t) \right] \mathbf{u}\mathbf{x}^*(1_{\mathbf{K}} + t)^{-1}\mathbf{x} = \\ &= \frac{1}{2} \left[-(t - 1_{\mathbf{K}}) + (1_{\mathbf{K}} + t) \right] \mathbf{u}\mathbf{x}^*(1_{\mathbf{K}} + t)^{-1}\mathbf{x} = \\ &= \mathbf{u}\mathbf{x}^*(1_{\mathbf{K}} + t)^{-1}\mathbf{x} .\end{aligned}$$

Summing up (1) + ... + (6), we get

$$\begin{aligned}\Phi'_1(\mathbf{a})\mathbf{v} &= (1_{\mathbf{K}} + t)\mathbf{u} + \mathbf{u}\mathbf{x}^*(1_{\mathbf{K}} + t)^{-1}\mathbf{x} = \mathbf{u} + t\mathbf{u} + \mathbf{u}\mathbf{x}^*\mathbf{x}(1_{\mathbf{H}} + \tilde{t})^{-1} = \\ &= \mathbf{u} + t\mathbf{u} + \mathbf{u}(\tilde{t}^2 - 1_{\mathbf{H}})(1_{\mathbf{H}} + \tilde{t})^{-1} = \mathbf{u} + t\mathbf{u} + \mathbf{u}(\tilde{t} - 1_{\mathbf{H}}) = \\ &= t\mathbf{u} + \mathbf{u}\tilde{t} , \\ \Phi'_0(\mathbf{a})\mathbf{v} &= \mathbf{x}\mathbf{v}^* + [\Phi'_1(\mathbf{a})\mathbf{v}]\mathbf{a}^* = \\ &= \mathbf{x}\mathbf{u}^* - (1_{\mathbf{K}} + t)^{-1}\mathbf{u}\mathbf{x}^*(1_{\mathbf{K}} + t)^{-1} + \\ &\quad + [(1_{\mathbf{K}} + t)\mathbf{u} + \mathbf{u}\mathbf{x}^*(1_{\mathbf{K}} + t)^{-1}\mathbf{x}]\mathbf{x}^*(1_{\mathbf{K}} + t)^{-1} = \\ &= \overbrace{\mathbf{x}\mathbf{u}^*}^{(1)} - \overbrace{\mathbf{x}\mathbf{x}^*(1_{\mathbf{K}} + t)^{-1}\mathbf{u}\mathbf{x}^*(1_{\mathbf{K}} + t)^{-1}}^{(2)} + \\ &\quad + \overbrace{(1_{\mathbf{K}} + t)\mathbf{u}\mathbf{x}^*(1_{\mathbf{K}} + t)^{-1}}^{(3)} + \overbrace{\mathbf{u}\mathbf{x}^*(1_{\mathbf{K}} + t)^{-1}\mathbf{x}\mathbf{x}^*(1_{\mathbf{K}} + t)^{-1}}^{(4)} .\end{aligned}$$

Using again the identity $\mathbf{x}\mathbf{x}^* = (1_{\mathbf{K}} + t)(t - 1_{\mathbf{K}})$, here we can write

$$\begin{aligned}
 (2) + (3) &= [-(1_{\mathbf{K}} + t)^{-1}(1_{\mathbf{K}} + t)(t_{\mathbf{K}} - 1) + (1_{\mathbf{K}} + t)]\mathbf{u}\mathbf{x}^*(1_{\mathbf{K}} + t)^{-1} = \\
 &= 2\mathbf{u}\mathbf{x}^*(1_{\mathbf{K}} + t)^{-1}, \\
 (4) &= \mathbf{u}\mathbf{x}^*(1_{\mathbf{K}} + t)^{-1}(1_{\mathbf{K}} + t)(t_{\mathbf{K}} - 1)(1_{\mathbf{K}} + t)^{-1} = \mathbf{u}\mathbf{x}^*(t_{\mathbf{K}} - 1)(1 + t_{\mathbf{K}})^{-1} = \\
 &= -\mathbf{u}\mathbf{x}^*(1_{\mathbf{K}} + t)^{-1}[(1_{\mathbf{K}} + t) - 2t] = -\mathbf{u}\mathbf{x}^* + 2\mathbf{u}\mathbf{x}^*t(1_{\mathbf{K}} + t)^{-1}.
 \end{aligned}$$

Therefore

$$\begin{aligned}
 \Phi'_0(\mathbf{a})\mathbf{v} &= [(1) + (4)] + [(2) + (3)] = \\
 &= \mathbf{x}\mathbf{u}^* - \mathbf{u}\mathbf{x}^* + 2\mathbf{u}\mathbf{x}^*t(1_{\mathbf{K}} + t)^{-1} + 2\mathbf{u}\mathbf{x}^*(1_{\mathbf{K}} + t)^{-1} = \\
 &= \mathbf{x}\mathbf{u}^* + \mathbf{u}\mathbf{x}^*[-(1_{\mathbf{K}} + t) + 2t + 2 \cdot 1_{\mathbf{K}}](1_{\mathbf{K}} + t)^{-1} = \\
 &= \mathbf{x}\mathbf{u}^* + \mathbf{u}\mathbf{x}^*. \quad \text{Q.u.e.d.}
 \end{aligned}$$

Proof of 3.5. By Lemmas 3.1-3 it suffices to see that we have $[\tilde{\Phi}'(\mathbf{a})]v_{\mathbf{u}}(\mathbf{a}) = (\mathbf{u}^*\tilde{\mathbf{x}}^* + \tilde{\mathbf{x}}\mathbf{u}, \tilde{t}\mathbf{u}^* + \mathbf{u}^*t)$ whenever $t = \Phi_0(\mathbf{a})$, $\tilde{t} = \tilde{\Phi}_0(\mathbf{a})$ and $\mathbf{x} := \tilde{\mathbf{x}}^* = \Phi_1(\mathbf{a})$. Let $t := \Phi_0(\mathbf{a})$, $\tilde{t} := \tilde{\Phi}_0(\mathbf{a})$, $\mathbf{x} := \tilde{\mathbf{x}}^* = \Phi_1(\mathbf{a})$. By 3.3 and since $\tilde{\Phi}_1 = [\Phi_1]^*$, we have indeed

$$[\tilde{\Phi}'_1(\mathbf{a})]v_{\mathbf{u}}(\mathbf{a}) = \left[\frac{d}{d\tau} \Big|_{\tau=0} \Phi_1(\mathbf{a} + \tau(\mathbf{u} - \mathbf{a}\mathbf{u}^*\mathbf{a})) \right]^* = [t\mathbf{u} + \mathbf{u}\tilde{t}]^* = \mathbf{u}^*t + \tilde{t}\mathbf{u}^*.$$

We can deduce the expression of $[\tilde{\Phi}'_0(\mathbf{a})]v_{\mathbf{u}}(\mathbf{a})$ by reversing the order of operator multiplications during the proof of the relation $[\Phi'_0(\mathbf{a})]v_{\mathbf{u}}(\mathbf{a}) = \mathbf{u}\mathbf{x}^* + \mathbf{x}\mathbf{u}^*$. Hence we get

$$[\tilde{\Phi}'_0(\mathbf{a})]v_{\mathbf{u}}(\mathbf{a}) = \mathbf{x}^*\mathbf{u} + \mathbf{u}^*\mathbf{x} = \tilde{\mathbf{x}}\mathbf{u} + \mathbf{u}^*\mathbf{x}^*.$$

4 Proof of Corollary 2.5

Henceforth assume $\mathbf{H} = \mathbf{K}$ and consider any $\mathbf{a} \in \mathbf{B}^{(s)}$, $\mathbf{u} \in \mathbf{E}^{(s)} := \mathcal{L}^{(s)}(\mathbf{H})$. By definition $\mathbf{a} = \mathbf{a}^*$ and $\mathbf{u} = \mathbf{u}^*$ whence both the operators

$$\begin{aligned}
 t &:= \Phi_0(\mathbf{a}) = (1_{\mathbf{H}} + \mathbf{a}^2)(1_{\mathbf{H}} - \mathbf{a}^2)^{-1} = (1_{\mathbf{H}} - \mathbf{a}^2)^{-1}(1_{\mathbf{H}} + \mathbf{a}^2) (= \tilde{\Phi}_0(\mathbf{a})), \\
 \mathbf{x} &:= \Phi_1(\mathbf{a}) = 2(1_{\mathbf{H}} - \mathbf{a}^2)^{-1}\mathbf{a} = 2\mathbf{a}(1_{\mathbf{H}} - \mathbf{a}^2)^{-1} (= \tilde{\Phi}_1(\mathbf{a}))
 \end{aligned}$$

are self-adjoint. Thus, since $\Phi : \mathbf{B} \leftrightarrow \mathbf{M}$, also $\Phi : \mathbf{B}^{(s)} \leftrightarrow \mathbf{M}^{(s)}$. On the other hand, the vector field $v_{\mathbf{u}} : \mathbf{b} \mapsto \mathbf{u} - \mathbf{b}\mathbf{u}^*\mathbf{b} = \mathbf{u} - \mathbf{b}\mathbf{u}\mathbf{b}$ is complete in $\mathbf{B}$ and ranges in $\mathbf{E}^{(s)}$ when restricted to $\mathbf{B}^{(s)} = \mathbf{B} \cap \mathbf{E}^{(s)}$. That is for the

Möbius transformations $M_{\mathbf{u}}^\tau = \exp(\tau v_{\mathbf{u}})$ we have $M_{\mathbf{u}}^\tau : \mathbf{B}^{(s)} \leftrightarrow \mathbf{B}^{(s)}$ ($\tau \in \mathbb{R}$) and there lifting $\Phi^\# M_{\mathbf{u}}^\tau = \Phi \circ M_{\mathbf{u}}^\tau \Phi^{-1} : \mathbf{M}^{(s)} \leftrightarrow \mathbf{M}^{(s)}$ can be calculated by taking the exponentials of the vector fields $\tau \Phi^\# v_{\mathbf{u}}$ which are complete in $\mathbf{M}^{(s)} = \mathbf{E}^{(s)} \cap \mathbf{M}$. By 3.4 we have

$$\begin{aligned} [\Phi^\# v_{\mathbf{u}}](t, \mathbf{x}) &= (\mathbf{u}\mathbf{x} + \mathbf{x}\mathbf{u}, \mathbf{u}t + t\mathbf{u}) = (t, \mathbf{x}) \begin{bmatrix} 0 & L(\mathbf{u})+R(\mathbf{u}) \\ L(\mathbf{u})+R(\mathbf{u}) & 0 \end{bmatrix}, \\ [\Phi^\# M_{\mathbf{u}}^\tau](t, \mathbf{x}) &= (t, \mathbf{x}) \exp \left(\tau \begin{bmatrix} 0 & L(\mathbf{u})+R(\mathbf{u}) \\ L(\mathbf{u})+R(\mathbf{u}) & 0 \end{bmatrix} \right). \end{aligned}$$

Straightforward calculations with the power series

$$\exp(\tau \Phi^\# v_{\mathbf{u}}) = \sum_{n=0}^{\infty} \frac{\tau^n}{n!} \begin{bmatrix} 0 & L(\mathbf{u})+R(\mathbf{u}) \\ L(\mathbf{u})+R(\mathbf{u}) & 0 \end{bmatrix}^n$$

yield the following:

$$\begin{aligned} \exp \left(\tau \begin{bmatrix} 0 & L(\mathbf{u})+R(\mathbf{u}) \\ L(\mathbf{u})+R(\mathbf{u}) & 0 \end{bmatrix} \right) &= \sum_{n=0}^{\infty} \frac{\tau^n}{n!} \begin{bmatrix} 0 & L(\mathbf{u})+R(\mathbf{u}) \\ L(\mathbf{u})+R(\mathbf{u}) & 0 \end{bmatrix}^n = \\ &= \sum_{k=0}^{\infty} \frac{\tau^{2k}}{(2k)!} \begin{bmatrix} 0 & L(\mathbf{u})+R(\mathbf{u}) \\ L(\mathbf{u})+R(\mathbf{u}) & 0 \end{bmatrix}^{2k} + \sum_{k=0}^{\infty} \frac{\tau^{2k+1}}{(2k+1)!} \begin{bmatrix} 0 & L(\mathbf{u})+R(\mathbf{u}) \\ L(\mathbf{u})+R(\mathbf{u}) & 0 \end{bmatrix}^{2k+1} = \\ &= \sum_{k=0}^{\infty} \frac{\tau^{2k}}{(2k)!} \begin{bmatrix} [L(\mathbf{u})+R(\mathbf{u})]^{2k} & 0 \\ 0 & [L(\mathbf{u})+R(\mathbf{u})]^{2k} \end{bmatrix} + \sum_{k=0}^{\infty} \frac{\tau^{2k+1}}{(2k+1)!} \begin{bmatrix} 0 & [L(\mathbf{u})+R(\mathbf{u})]^{2k+1} \\ [L(\mathbf{u})+R(\mathbf{u})]^{2k+1} & 0 \end{bmatrix} = \\ &= \begin{bmatrix} \cosh(\tau[L(\mathbf{u})+R(\mathbf{u})]) & 0 \\ 0 & \cosh(\tau[L(\mathbf{u})+R(\mathbf{u})]) \end{bmatrix} + \begin{bmatrix} 0 & \sinh(\tau[L(\mathbf{u})+R(\mathbf{u})]) \\ \sinh(\tau[L(\mathbf{u})+R(\mathbf{u})]) & 0 \end{bmatrix}. \end{aligned}$$

Since left and right multiplications commute (that is $L(\mathbf{g})R(\mathbf{h})\mathbf{z} = \mathbf{g}(\mathbf{zh}) = (\mathbf{gz})\mathbf{h} = R(\mathbf{h})L(\mathbf{g})\mathbf{z}$ for $\mathbf{g}, \mathbf{h}, \mathbf{z} \in \mathbf{E}$), it follows

$$\begin{aligned} \cosh(\tau[L(\mathbf{u})+R(\mathbf{u})]) &= \frac{1}{2} \exp(\tau[L(\mathbf{u})+R(\mathbf{u})]) + \frac{1}{2} \exp(-\tau[L(\mathbf{u})+R(\mathbf{u})]) = \\ &= \frac{1}{2} \exp(\tau L(\mathbf{u})) \exp(\tau R(\mathbf{u})) + \frac{1}{2} \exp(-\tau L(\mathbf{u})) \exp(-\tau R(\mathbf{u})) = \\ &= \frac{1}{2} L(\exp(\tau \mathbf{u})) R(\exp(\tau \mathbf{u})) + \frac{1}{2} L(\exp(-\tau \mathbf{u})) R(\exp(-\tau \mathbf{u})) \end{aligned}$$

with the effect $\cosh(\tau[L(\mathbf{u})+R(\mathbf{u})]) : \mathbf{z} \mapsto \frac{1}{2} \exp(\tau \mathbf{u})\mathbf{z} \exp(\tau \mathbf{u}) + \frac{1}{2} \exp(-\tau \mathbf{u})\mathbf{z} \exp(-\tau \mathbf{u})$. Similarly $\sinh(\tau[L(\mathbf{u})+R(\mathbf{u})]) : \mathbf{z} \mapsto \frac{1}{2} \exp(\tau \mathbf{u})\mathbf{z} \exp(\tau \mathbf{u}) - \frac{1}{2} \exp(-\tau \mathbf{u})\mathbf{z} \exp(-\tau \mathbf{u})$. Q.e.d.

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