DOI: 10.2478/v10309-012-0006-7
An. Şt. Univ. Ovidius Constanţa **Vol. 20(1), 2012, 73–90**

Seiberg-Witten Equations on Pseudo-Riemannian Spin^c Manifolds With Neutral Signature

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Abstract

Pseudo-Riemannian spin^c manifolds were introduced by Ikemakhen in [7]. In the present work we consider pseudo-Riemannian 4−manifolds with neutral signature whose structure groups are $SO_+(2,2)$. We prove that such manifolds have pseudo-Riemannian spin^c structure. We construct spinor bundle S and half-spinor bundles S^+ and S^- on these manifolds. For the first Seiberg-Witten equation we define Dirac operator on these bundles. Due to the neutral metric self-duality of a 2−form is meaningful and it enables us to write down second Seiberg-Witten equation. Lastly we write down the explicit forms of these equations on 4−dimensional flat space.

1 Introduction

Spinors are geometric objects living around manifolds. They are important for the investigation of manifolds (see [6, 9]). Seiberg-Witten Monopole equations were defined by E. Witten on 4−dimensional Riemannian manifolds by using the spinors [16]. The solution space of these equations provides new invariants for 4−manifolds, namely Seiberg-Witten invariants ([1, 12, 13]). Similar equations were written down on 4−dimensional Lorentzian manifolds [3]. Pseudo-Riemannian 4−manifolds with neutral signature are being studied by various authors from different point of view (see [2, 4, 8, 10, 11]).

Key Words: Neutral metric, Pseudo-Riemannian spin^c−structure, Dirac operator, Seiberg-Witten equations.
2010 Mathematics Subject Classification: 53C50, 53C27.
Received: January, 2011.
Accepted: February, 2012.

Pseudo-Riemannian spin^c spinors are introduced by Ikemakhen in [7] recently. The aim of this article is to write down similar equations to Seiberg-Witten equations on 4−dimensional Pseudo-Riemannian spin^c manifolds with neutral signature.

2 Some Preliminaries

On $\mathbb{R}^4$, we consider the pseudo-Riemannian metric $g(x,y) = x_1y_1 + x_2y_2 - x_3y_3 - x_4y_4$, where $x = (x_1, x_2, x_3, x_4)$, $y = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4$. When this metric considered on the 4-dimensional space is denoted by $\mathbb{R}^{2,2}$. The isometry group of this space is denoted by $O(2,2)$, that is

$$O(2,2) = \left\{A \in GL(4,\mathbb{R}) : g(A(x), A(y)) = g(x,y), \text{ where } x, y \in \mathbb{R}^{2,2}\right\}.$$

The group $O(2,2)$ has four connected components. The special orthogonal subgroup of $O(2,2)$ is denoted by

$$SO(2,2) = \{A \in O(2,2) : \det A = 1\}.$$

The subgroup $SO(2,2)$ has two connected components and the connected component to the identity of $SO(2,2)$ is denoted by $SO_+(2,2)$. In this work we mainly deal with the group $SO_+(2,2)$. $Spin_+(2,2)$ lives in the Clifford algebra $Cl_{2,2} = Cl\left(\mathbb{R}^4, g\right)$ and it is isomorphic to $SU(1,1) \times SU(1,1)$ (see [11]).

The covering map $\lambda : Spin_+(2,2) \to SO_+(2,2)$ is a $2:1$ group homomorphism given by $\lambda(g)(x) = g \cdot x \cdot g^{-1}$ for any $x \in \mathbb{R}^4$, $g \in Spin_+(2,2)$.

Remark 1. *Contrary to the Euclidean and Lorentzian cases the fundamental group of $SO_+(2,2)$ is not $\mathbb{Z}_2$ and $Spin_+(2,2)$ is not simply connected.*

One can define a new group which lies in the complex Clifford algebra $\mathbb{C}l_{2,2} \cong \mathbb{C}l_4$ by

$$Spin^c_+(2,2) = Spin_+(2,2) \times S^1/\mathbb{Z}_2.$$

The elements of $Spin^c_+(2,2)$ are the equivalence classes $[g,z]$ of pairs $(g,z) \in Spin_+(2,2) \times S^1$ under the equivalence relation $(g,z) \sim (-g,-z)$. From the definitions of $Spin_+(2,2)$ and $Spin^c_+(2,2)$ the following sequences are exact:

$$1 \to \mathbb{Z}_2 \to Spin_+(2,2) \xrightarrow{\lambda} SO_+(2,2) \to 1,$$

$$1 \to \mathbb{Z}_2 \to Spin^c_+(2,2) \xrightarrow{\xi} SO_+(2,2) \times S^1 \to 1,$$

where $\xi([g,z]) = \left(\lambda(g), z^2\right)$.

Since the complex Clifford algebra $\mathbb{C}l_{2,2}$ is isomorphic to the endomorphism algebra $End(\mathbb{C}^4)$, there is a natural representation $\kappa : \mathbb{C}l_{2,2} \to End(\mathbb{C}^4)$. For example, we can define κ on the basis elements as follows:

$$\kappa(e_1) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \qquad \kappa(e_2) = \begin{pmatrix} 0 & i\sigma_3 \\ i\sigma_3 & 0 \end{pmatrix}$$

$$\kappa(e_3) = \begin{pmatrix} 0 & -\sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}, \qquad \kappa(e_4) = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}$$

where I_2 is 2×2 unit matrix and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

are Pauli-spin matrices.

The complex vector space $\mathbb{C}^4$ is called the space of spinors and denoted by $\Delta_{2,2}$. The spinor space $\Delta_{2,2}$ carries a non-degenerate indefinite Hermitian inner product $\langle \ , \ \rangle_{\Delta_{2,2}}$which is invariant under the action of $Spin^c_+(2,2)$ is given by

$$\langle \Psi_1, \Psi_2 \rangle_{\Delta_{2,2}} = \langle \kappa(e_1)\kappa(e_2)\Psi_1, \Psi_2 \rangle,$$

where $\langle \ , \ \rangle$ denotes the positive definite Hermitian inner product on $\mathbb{C}^4$ (see [7]). We can restrict the map κ to $Spin^c_+(2,2)$ and we obtain a group representation

$$\kappa : Spin^c_+(2,2) \to Aut(\Delta_{2,2}).$$

The restricted map κ is called spinor representation of $Spin^c_+(2,2)$. The spinor space $\Delta_{2,2}$ decomposes two parts

$$\Delta_{2,2} = \Delta^+_{2,2} \oplus \Delta^-_{2,2},$$

where $\Delta^\pm_{2,2}$ are the eigenspaces of $f = \kappa(e_1e_2e_3e_4)$, f^2 is the identity map. The elements of $\Delta^+_{2,2}$ are called the positive spinors. Since the spinor representation of $Spin^c_+(2,2)$ preserves these eigenspaces, we obtain the following representations by restriction

$$\kappa^\pm : Spin^c_+(2,2) \to Aut(\Delta^\pm_{2,2}).$$

The spinor representation κ has the following properties:

Proposition 1.

i) $\kappa(Spin_+(2,2)) \cong SU(1,1) \times SU(1,1)$.

ii) $\kappa^+(Spin_+(2,2)) \cong SU(1,1)$

iii) $\kappa^-(Spin_+(2,2)) \cong SU(1,1)$

iv) $\kappa(Spin_+^c(2,2)) \cong \{(A,B) \in U(1,1) \times U(1,1) : det(A) = det(B)\}$

v) $\kappa^+(Spin_+^c(2,2)) \subset U(1,1)$

vi) $\kappa(v)$ *maps* $\Delta_{2,2}^-$ *to* $\Delta_{2,2}^+$ *and* $\Delta_{2,2}^+$ *to* $\Delta_{2,2}^-$ *for each* $v \in \mathbb{R}^4$.

vii) $\kappa(v)^2 = -g(v,v)I_4$ *for each* $v \in \mathbb{R}^4$, *where* I_4 *is* 4×4 *identity matrix.*

The Lie algebras of the groups $Spin_+(2,2)$ and $Spin_+^c(2,2)$ are

$$spin_+(2,2) = \{e_i \cdot e_j \ ; \ 1 \leq i < j \leq 4\}$$

and

$$spin_+^c(2,2) = spin_+(2,2) \oplus i\mathbb{R},$$

respectively, where $e_i \cdot e_j$ is the second order element of the $\mathbb{C}l_{2,2}$. The derivative of $\xi = \lambda \times l$ is a Lie algebra isomorphism and given by

$$\xi_* \left(e_i \cdot e_j, it\right) = \left(\lambda_* \left(e_i \cdot e_j\right), l_* \left(it\right)\right) = \left(2E_{ij}, 2it\right),$$

where E_{ij} denotes the basis elements of the Lie algebra $so_+(2,2)$ and

$$\lambda : Spin_+^c(2,2) \to SO_+(2,2), \quad \lambda\left([g,z]\right) = \lambda\left(g\right)$$

and $l : Spin_+^c(2,2) \to S^1, \quad l([g,z]) = z^2$ are group homomorphisms.

3 Pseudo-Riemannian Manifolds of Metric Signature $(++--)$

3.1 Existence of Neutral Metric

Let M be a $4-$dimensional space and time oriented smooth manifold with the pseudo-Riemannian metric g of signature $(2,2)$ (that is of type $(+,+,-,-)$). Such a metric is called neutral metric. Existence conditions of neutral metric on a $4-$dimensional differentiable manifold M were given in [11] in detail form. In the present work we focus on the completely orientiable case, i.e., the structure group of the tangent bundle TM is $SO_+(2,2)$. It is pointed out in [11] that the structure group of M is $SO_+(2,2)$ if and only if it admits a fields of orientiable tangent $2-$planes. Following theorem will be useful for our discussion on the existence of pseudo-Riemannian spin$^c-$structure.

Theorem 1. *Existence of neutral metric on a compact manifold* M *with structure group* $SO_+(2,2)$ *is equivalent to the existence of a pair* (J,J') *of an almost complex structure* J *and an opposite almost complex structure* J' *on the manifold, where* J *and* J' *are orthogonal with respect to the metric* g *and they commutes, that is;* $JJ' = J'J$ *[11].*

The family of manifolds which have neutral metric is rather large, for example; K3 surfaces, Enriques surfaces, Kodaria surfaces, Ruled surfaces of genus $g \geq 1$ and see [11] for others.

3.2 Self-Duality

Neutral metric shares some properties of the Riemannian metric. For example, the Hodge star operator $*$ is an involution on the space of two forms $\Lambda^2(M)$. Since $*^2 = id$, $*$ induces a splitting of $\Lambda^2(M) = \Lambda^+ \oplus \Lambda^-$, where Λ^+ and Λ^- denote the space of *self-dual* and *anti-self-dual* $2-$forms

$$\Lambda^+ = \left\{\eta \in \Lambda^2(M) : *\eta = \eta\right\}, \quad \Lambda^- = \left\{\eta \in \Lambda^2(M) : *\eta = -\eta\right\}.$$

The projection of a 2-form $\eta \in \Lambda^2(M)$ onto the subspace Λ^+ is called the self-dual part of η and we denote it by η^+, similarly the projection of η onto the subspace Λ^- is called the anti-self-dual part of η and we denote it by η^-. Note that $\eta = \eta^+ + \eta^-$ and the self-dual and anti-self-dual parts can be expressed in terms of the Hodge star operator $*$ by the following way:

$$\eta^+ = \frac{1}{2}(\eta + *\eta) \text{ and } \eta^- = \frac{1}{2}(\eta - *\eta).$$

Let $\{e_1, e_2, e_3, e_4\}$ be a local pseudo-orthonormal frame on the open set $U \subset M$ and $\left\{e^1, e^2, e^3, e^4\right\}$ be the corresponding dual frame. Then the vectors $f_1 = e^1 \wedge e^2 + e^3 \wedge e^4$, $f_2 = e^1 \wedge e^3 + e^2 \wedge e^4$, $f_3 = e^1 \wedge e^4 - e^2 \wedge e^3$ form a basis for self-dual $2-$forms, that is

$$\Lambda^+ = span\left\{f_1, f_2, f_3\right\}.$$

Similarly the vectors $g_1 = e^1 \wedge e^2 - e^3 \wedge e^4$, $g_2 = e^1 \wedge e^3 - e^2 \wedge e^4$, $g_3 = e^1 \wedge e^4 + e^2 \wedge e^3$ form a basis for anti-self-dual $2-$forms, that is

$$\Lambda^- = span\left\{g_1, g_2, g_3\right\}.$$

The componentwise expression of these two parts is given by

$$\eta^+ = \frac{1}{2}\left[\left((\eta_{12} + \eta_{34}\right) f_1 + (-\eta_{13} - \eta_{24}) f_2 + (-\eta_{14} + \eta_{23}) f_3\right]$$

$$\eta^- = \frac{1}{2}\left[\left((\eta_{12} - \eta_{34}\right) f_1 + (-\eta_{13} + \eta_{24}) f_2 + (-\eta_{14} - \eta_{23}) f_3\right].$$

Similar to the Riemannian case self-duality and anti-self-duality of a neutral metric can be defined in terms of the Weyl tensor. Such structures are also related with the geometry of underlying manifolds (see [4, 8]).

3.3 Pseudo-Riemannian spin^c Structure

The definitions of a pseudo-Riemannian spin and spin^c structures on M can be given similar to the Riemannian cases as follows:

Since the structure group of M is $SO_+(2,2)$, there are an open covering $\{U_\alpha\}_{\alpha\in A}$ and transition functions $g_{\alpha\beta}: U_\alpha \cap U_\beta \to SO_+(2,2)$ for M. If there exists another collection of transition functions

$$\widetilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to Spin_+(2,2)$$

such that the following diagram commutes

$$\begin{array}{ccc} & & Spin_+(2,2) \\ & \overset{\widetilde{g}_{\alpha\beta}}{\nearrow} & \Big\downarrow \lambda \ 2{:}1 \\ U_\alpha \cap U_\beta & \xrightarrow[g_{\alpha\beta}]{} & SO_+(2,2) \end{array}$$

that is, $\lambda\circ\widetilde{g}_{\alpha\beta} = g_{\alpha\beta}$ and the cocycle condition $\widetilde{g}_{\alpha\beta}\widetilde{g}_{\beta\gamma} = \widetilde{g}_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$ is satisfied, then M is called a pseudo-Riemannian spin manifold. Then one can construct a principal $Spin_+(2,2)-$bundle $P_{Spin_+(2,2)}$ on M and a $2:1$ bundle map $\Lambda : P_{Spin_+(2,2)} \to P_{SO_+(2,2)}$ such that the following diagram commutes:

$$\begin{array}{ccccc} P_{Spin_+(2,2)} \times Spin_+(2,2) & \longrightarrow & P_{Spin_+(2,2)} & & \\ \Big\downarrow (\Lambda,\lambda) & & \Big\downarrow \Lambda & \searrow & \\ P_{SO_+(2,2)} \times SO_+(2,2) & \longrightarrow & P_{SO_+(2,2)} & \longrightarrow & M \end{array}$$

Similarly pseudo-Riemannian spin^c structures on M can be defined by a collection of transition functions. There are an open covering $\{U_\alpha\}_{\alpha\in A}$ of M and transition functions $g_{\alpha\beta}: U_\alpha \cap U_\beta \to SO_+(2,2)$ for M. If there exists another collection of transition functions

$$\widetilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to Spin^c_+(2,2)$$

such that the following diagram commutes

$$\begin{array}{ccc} & & Spin^c_+(2,2) \\ & \overset{\widetilde{g}_{\alpha\beta}}{\nearrow} & \Big\downarrow \lambda \\ U_\alpha \cap U_\beta & \xrightarrow[g_{\alpha\beta}]{} & SO_+(2,2) \end{array}$$

that is, $\lambda\circ\widetilde{g}_{\alpha\beta} = g_{\alpha\beta}$ and the cocycle condition $\widetilde{g}_{\alpha\beta}\widetilde{g}_{\beta\gamma} = \widetilde{g}_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$ is satisfied then M is called a pseudo-Riemannian *spin*^{c} manifold. Then one can construct a principal $Spin^c_+(2,2)-$bundle $P_{Spin^c_+(2,2)}$ on M and a $2:1$ bundle map $\Lambda : P_{Spin^c_+(2,2)} \to P_{SO_+(2,2)}$ such that the following diagram commutes:

$$\begin{array}{ccccc}
P_{Spin^c_+(2,2)} \times Spin^c_+(2,2) & \longrightarrow & P_{Spin^c_+(2,2)} & & \\
\downarrow{\scriptstyle (\Lambda,\lambda)} & & \downarrow{\scriptstyle \Lambda} & \searrow & \\
P_{SO_+(2,2)} \times SO_+(2,2) & \longrightarrow & P_{SO_+(2,2)} & \longrightarrow & M
\end{array}$$

Remark 2. *Since $Spin^c_+(2,2)$ is isomorphic to the group*

$$H = \{(A,B) \in U(1,1) \times U(1,1) : det(A) = det(B)\},$$

one can define spin^c structure on M by the existence of transition functions

$$\widetilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to H$$

such that $Ad \circ \widetilde{g}_{\alpha\beta} = g_{\alpha\beta}$ and the cocycle condition $\widetilde{g}_{\alpha\beta}\widetilde{g}_{\beta\gamma} = \widetilde{g}_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$ is satisfied. The covering map $Ad : H \to SO_+(2,2)$ is defined by $Ad(A,B)(V) = AVB^{-1}$ for each $(A,B) \in H$ and $V \in \mathbb{C}^2 \cong \mathbb{R}^4$. In the definition of the map Ad we use the one to one correspondence between the vectors $V = (v_1, v_2, v_3, v_4)$ in $\mathbb{R}^{2,2}$ and the 2 by 2 complex matrices by the following way

$$V = (v_1, v_2, v_3, v_4) = v_1 I + v_2 i\sigma_3 - v_3\sigma_2 + v_4\sigma_1 = \begin{pmatrix} v_1 + iv_2 & v_4 + iv_3 \\ v_4 - iv_3 & v_1 - iv_2 \end{pmatrix}.$$

Note that the equality $det(V) = v_1^2 + v_2^2 - v_3^2 - v_4^2 = g(V,V)$ holds. From this equality we obtain $g(Ad(A,B)(V), Ad(A,B)(V)) = det(AVB^{-1}) = det(V) = g(V,V)$, so $Ad(A,B)$ belongs to the group $SO_+(2,2)$ for each $(A,B) \in H$.

If M has a pseudo-Riemannian spin (spin^c) structure, then M is called pseudo-Riemannian spin (spin^c) manifold. It is known that each pseudo-Riemannian spin structure on M induces a pseudo-Riemannian spin^c structure, hence every pseudo-Riemannian spin manifold is a pseudo-Riemannian spin^c manifold.

Theorem 2. *If M is a 4-dimensional compact differentiable manifold with structure group $SO_+(2,2)$, then M is a pseudo-Riemannian spin^c manifold.*

Proof. By Theorem 1 there is a $g-$orthogonal almost complex structure J on M. Then the structure group of M can be reduced from $SO_+(2,2)$ to $U(1,1)$. That is, there are an open covering $\{U_\alpha\}_{\alpha \in A}$ and transition functions

$g_{\alpha\beta} : U_\alpha \cap U_\beta \to U(1,1)$ for M. The canonical action of $U(1,1)$ on $\mathbb{R}^{2,2} \cong \mathbb{C}^2$ is given by ordinary matrix product AV for each $A \in U(1,1)$ and $V = (v_1 + iv_2, v_4 - iv_3)$. This action can also be interpreted as follows: Think the vector V as the following 2 by 2 matrix

$$\begin{pmatrix} v_1 + iv_2 & .. \\ v_4 - v_3 & .. \end{pmatrix}$$

whose first column is the components of V and second column may be anything. Multiply A with this matrix, consider the first column of the resulting matrix. The map $j : SU(1,1) \to H$ by $j(A) = (A, B)$ is an injective group homomorphism, where $B = \begin{pmatrix} 1 & 0 \\ 0 & det(A) \end{pmatrix}$.

Define new transition functions $\widetilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to H$ by $\widetilde{g}_{\alpha\beta} = j \circ g_{\alpha\beta}$. It is clear that these functions satisfy cocycle condition. On the other hand, let $x \in U_\alpha \cap U_\beta$ be any point and say $A = g_{\alpha\beta}(x)$. We obtain following identity

$$Ad(j(A))(V) = A \begin{pmatrix} v_1 + iv_2 & v_3 + iv_4 \\ v_4 - v_3 & v_1 + iv_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & det(A) \end{pmatrix}^{-1} = A \begin{pmatrix} v_1 + iv_2 & .. \\ v_3 - iv_4 & .. \end{pmatrix}.$$

It means that the commuting relation $Ad \circ \widetilde{g}_{\alpha\beta}(x) = g_{\alpha\beta}(x)$ holds for each $x \in U_\alpha \cap U_\beta$. This completes the proof. □

For the Riemannian analogy of this theorem and other concepts see [13].

3.4 Connection 1−Form on $P_{Spin^c_+(2,2)}$

If M is a pseudo-Riemannian spin^c manifold, then by using the map

$$l : Spin^c_+(2,2) \to S^1, \; l([g,z]) = z^2,$$

we can construct an associated principal S^1-bundle

$$P_{S^1} = P_{Spin^c_+(2,2)} \times_l S^1.$$

Let ∇ be the Levi-Civita covariant derivative associated to g on M. Then it is known that the Levi-Civita covariant derivative ∇ determines an $so(2,2)-$valued connection $1-$form ω on the principal bundle $P_{SO_+(2,2)}$. The connection $1-$form ω can be expressed locally

$$\omega_U = \sum_{i<j} \omega_{ij} E_{ij},$$

where $\{e_1, e_2, e_3, e_4\}$ is a local orthonormal frame on open set $U \subset M$ and $\omega_{ij} = \varepsilon_i g(\nabla e_i, e_j)$. Take an $i\mathbb{R}-$valued connection $1-$form A on S^1-principal bundle P_{S^1}. Now we can define an $so(2,2) \oplus i\mathbb{R}-$valued connection $1-$form on the principal bundle $P_{SO_+(2,2)} \widetilde{\times} P_{S^1}$ (the fibre product bundle):

$$\omega \times A : T\left(P_{SO_+(2,2)} \widetilde{\times} P_{S^1}\right) \to so(2,2) \oplus i\mathbb{R}.$$

This connection can be lift to a connection $1-$form Z^A in the principal bundle $P_{Spin^c_+(2,2)}$ via the $2-$fold covering $\pi : P_{Spin^c_+(2,2)} \to P_{SO_+(2,2)} \widetilde{\times} P_{S^1}$ and the following diagram commutes:

$$\begin{array}{ccc}
T\left(P_{Spin^c_+(2,2)}\right) & \xrightarrow{Z^A} & Lie(Spin^c_+(2,2)) \cong spin(2,2) \oplus i\mathbb{R} \\
{\scriptstyle d\pi}\downarrow & & \downarrow{\scriptstyle \xi_*} \\
T\left(P_{SO_+(2,2)} \widetilde{\times} P_{S^1}\right) & \xrightarrow{\omega \times A} & so(2,2) \oplus i\mathbb{R}
\end{array}$$

where $\xi_* : Lie(Spin^c_+(2,2)) \to so(2,2) \oplus i\mathbb{R}$ is the differential of the $2-$fold covering
$\xi = (\lambda, l) : Spin^c_+(2,2) \to SO_+(2,2) \times S^1$.

4 Spinor bundle

Let $(P_{Spin^c_+(2,2)}, \Lambda)$ be a pseudo-Riemannian spin^c structure on M. If we consider the $Spin^c_+(2,2)$ representation

$$\kappa : Spin^c_+(2,2) \to Aut(\Delta_{2,2})$$

then we can construct a new associated complex vector bundle

$$S = P_{Spin^c_+(2,2)} \times_\kappa \Delta_{2,2}.$$

This complex vector bundle is called spinor bundle for a given spin^c structure on M and sections of S are called spinor fields. One can obtain a covariant derivative operator ∇^A on S by using the connection $1-$form Z^A and a local expression of ∇^A is

$$\nabla^A \Psi = d\Psi + \frac{1}{2}\sum_{i<j} \varepsilon_i \varepsilon_j w_{ij} \kappa\left(e_i e_j\right)\Psi + \frac{1}{2} A\Psi,$$

where $\varepsilon_i = g\left(e_i, e_i\right)$ and Ψ is a local section of S over the open set $U \subset M$ (see [5, 7]).

The composite map $\tau \circ \lambda : Spin_{+}^{c}(2,2) \to Aut\left(\mathbb{R}^4\right)$ is a representation of $Spin_{+}^{c}(2,2)$ on $\mathbb{R}^4$ and gives

$$TM \cong P_{Spin_{+}^{c}(2,2)} \times_{\tau\circ\lambda} \mathbb{R}^4,$$

where $\tau : SO_{+}(2,2) \to Aut(\mathbb{R}^4)$ is the canonical representation. Such interpretations of tangent bundle enable us to product the elements of spinor bundle with tangent vectors by the formula

$$[p,v] \cdot [p,\psi] = [p,\kappa\left(v\right)\psi]$$

where $p \in P_{Spin_{+}^{c}(2,2)}$, $v \in \mathbb{R}^4$, $\psi \in \mathbb{C}^4$. This product is bilinear and we extend it to the tensor product space

$$\begin{array}{rcl} TM \otimes S & \to & S \\ {[p,v]} \otimes [p,\psi] & \mapsto & [p,\kappa\left(v\right)\psi]\,, \end{array}$$

and denote it as a map $\kappa : TM \otimes S \to S$ and call it Clifford multiplication. Also we obtain a bundle map

$$\kappa : TM \to End(S).$$

Some authors call the bundle map κ as the spin^c structure ([15]). Generally the Clifford multiplication $\kappa\left(X\right)\left(\Psi\right)$ is denoted by $X \cdot \Psi$. One can endow S with an indefinite Hermitian inner product by using the inner product on $\Delta_{2,2}$ and denote it again by $\langle\ ,\ \rangle_{\Delta_{2,2}}$. The covariant derivative operator ∇^A is compatible with $\langle\ ,\ \rangle_{\Delta_{2,2}}$ and Clifford multiplication in the following sense (see [7]):

Proposition 2. *For all* $X, Y \in \Gamma\left(TM\right)$ *and* $\Psi, \Psi_1, \Psi_2 \in \Gamma\left(S\right)$,

1. $\left\langle X \cdot \Psi_1, \Psi_2\right\rangle_{\Delta_{2,2}} = (-1)\left\langle \Psi_1, X \cdot \Psi_2\right\rangle_{\Delta_{2,2}}$,
2. $\nabla_Y^A\left(X \cdot \Psi\right) = X \cdot \nabla_Y^A\left(\Psi\right) + \left(\nabla_Y X\right) \cdot \Psi$,
3. $X\left\langle \Psi_1, \Psi_2\right\rangle_{\Delta_{2,2}} = \left\langle \nabla_X^A \Psi_1, \Psi_2\right\rangle_{\Delta_{2,2}} + \left\langle \Psi_1, \nabla_X^A \Psi_2\right\rangle_{\Delta_{2,2}}$.

5 Seiberg-Witten Like Equations on pseudo-Riemannian spin^c manifolds

The spinor bundle S splits into the sum of subbundles S^+, S^-:

$$S = S^+ \oplus S^-, \quad S^{\pm} = P_{Spin_{+}^{c}(2,2)} \times_{\kappa^{\pm}} \Delta_{2,2}^{\pm}.$$

The subbundles $S^{\pm}$ can be endowed with indefinite Hermitian inner product by Proposition 2. The indefinite Hermitian inner product on

$$S^{+}=P_{Spin_{+}^{c}(2,2)}\times_{\kappa^{+}}\Delta_{2,2}^{+}$$

is crucial for the interpretation of second Seiberg-Witten equation on M. Since κ^{+} takes value in $U(1,1)$, we can endow S^{+} with an indefinite Hermitian inner product of type $(1,1)$ and we denote it by $<,>_{1,1}$.

Moreover the covariant derivative operator ∇^{A} on S preserves the subbundles S^{+} and S^{-}. So ∇^{A} induces covariant derivative operators on these subbundles and we denote both of them with same symbol ∇^{A}.

5.1 The Dirac Equation

Now we want to define a Dirac operator on S. Note that the covariant derivative operator ∇^{A} can be thought as a linear map

$$\nabla^{A}:\Gamma\left(S\right)\rightarrow\Gamma\left(T^{*}M\otimes S\right)$$

satisfying the Leibnitz rule:

$$\nabla^{A}\left(f\Psi\right)=\left(df\right)\otimes\Psi+f\nabla^{A}\Psi.$$

Definition 1. *The composite map*

$$D_{A}=\kappa\circ\nabla^{A}:\Gamma\left(S\right)\overset{\nabla^{A}}{\rightarrow}\Gamma\left(T^{*}M\otimes S\right)\overset{g}{\cong}\Gamma\left(TM\otimes S\right)\overset{\kappa}{\rightarrow}\Gamma\left(S\right)$$

is called Dirac operator on pseudo-Riemannian spin^c manifold M.

In a space and time oriented local orthonormal frame $\{e_1,e_2,e_3,e_4\}$, the covariant derivative operator ∇^{A} can be written as

$$\nabla^{A}\Psi=\sum_{i=1}^{4}\varepsilon_{i}e_{i}^{*}\otimes\nabla_{e_{i}}^{A}\Psi.$$

Then a local expression of D_A is

$$D_{A}\Psi=\sum_{i=1}^{4}\varepsilon_{i}e_{i}\cdot\nabla_{e_{i}}^{A}\Psi.$$

Obviously the operator D_A is first order differential operator. The Dirac operator splits into two pieces $D_A=D_A^{+}\oplus D_A^{-}$ with respect to the decomposition $S=S^{+}\oplus S^{-}$, where $D_A^{+}:\Gamma(S^{+})\rightarrow\Gamma(S^{-})$ and $D_A^{-}:\Gamma(S^{-})\rightarrow\Gamma(S^{+})$.

We are ready to express the first Seiberg-Witten equation, the Dirac equation, on a pseudo-Riemannian manifold with neutral metric. The first Seiberg-Witten equation associated to the pair (A, Ψ) is

$$D_A^+ \Psi = 0 \tag{2}$$

where A is an $i\mathbb{R}-$valued connection $1-$form on the principal bundle P_{S^1} and Ψ is a positive spinor field on M, i.e. a section of S^+.

5.2 The Curvature Equation

We need some other concepts for the second Seiberg-Witten equation. We consider the situation in local form firstly. We can define an action of the space of $2-$forms $\Lambda^2(\mathbb{R}^{2,2})^*$ on the spinor space S. Let C_2 be the set of the second order elements of the Clifford algebra $Cl_{2,2}$ and consider the linear map

$$\begin{array}{rcl} \Lambda^2(\mathbb{R}^{2,2})^* & \to & C_2 \\ \eta = \sum\limits_{i<j} \eta_{ij} e^i \wedge e^j & \mapsto & \sum\limits_{i<j} \varepsilon_i \varepsilon_j \eta_{ij} e_i e_j \end{array}$$

where $\varepsilon_i = g(e_i, e_i)$. If we compose this map with the spinor representation κ, then we obtain a map $\rho : \Lambda^2(\mathbb{R}^{2,2})^* \to End(\mathbb{C}^4)$ by

$$\rho(\sum_{i<j} \eta_{ij} e^i \wedge e^j) = \sum_{i<j} \varepsilon_i \varepsilon_j \eta_{ij} \kappa(e_i) \kappa(e_j).$$

The half-spinor spaces $\triangle_{2,2}^{\pm}$ are invariant under $\rho(\eta)$ for every $\eta \in \Lambda^2(\mathbb{R}^{2,2})^*$, so we obtain the following maps by restriction

$$\rho^{\pm}(\eta) = \rho(\eta)|_{S^{\pm}}.$$

Now we calculate the explicit forms of the maps $\rho(\eta)$ and $\rho(\eta)^{\pm}$ for arbitrary 2-form $\eta \in \Lambda^2(\mathbb{R}^{2,2})^*$.

$$\begin{array}{rcl} \rho(\eta) & = & \rho(\sum\limits_{i<j} \eta_{ij} e^i \wedge e^j) \\ & = & \eta_{12}\kappa(e_1)\kappa(e_2) - \eta_{13}\kappa(e_1)\kappa(e_3) - \eta_{14}\kappa(e_1)\kappa(e_4) - \eta_{23}\kappa(e_2)\kappa(e_3) \\ & & -\eta_{24}\kappa(e_2)\kappa(e_4) + \eta_{34}\kappa(e_3)\kappa(e_4 \end{array}$$

The left upper block of $\rho(\eta)$ represents $\rho^+(\eta)$, so it is given by

$$\begin{array}{rl} \rho^+(\eta) = & (\eta_{12} + \eta_{34})(i\sigma_3) + (-\eta_{13} - \eta_{24})(-\sigma_2) + (-\eta_{14} + \eta_{23})\sigma_1 \\ = & \begin{pmatrix} i(\eta_{12} + \eta_{34}) & -\eta_{14} + \eta_{23} - i(\eta_{13} + \eta_{24}) \\ -\eta_{14} + \eta_{23} + i(\eta_{13} + \eta_{24}) & -i(\eta_{12} + \eta_{34}) \end{pmatrix}, \end{array}$$

similarly the right lower block of $\rho(\eta)$ represents $\rho^-(\eta)$, so it is given by

$$\rho^-(\eta) = (-\eta_{12} + \eta_{34})(i\sigma_3) + (\eta_{13} - \eta_{24})(-\sigma_2) + (\eta_{14} + \eta_{23})\sigma_1$$

$$= \begin{pmatrix} i(-\eta_{12} + \eta_{34}) & \eta_{14} + \eta_{23} + i(\eta_{13} - \eta_{24}) \\ \eta_{14} + \eta_{23} - i(\eta_{13} - \eta_{24}) & -i(-\eta_{12} + \eta_{34}) \end{pmatrix}.$$

Proposition 3. *Let $\eta \in \Lambda^2(\mathbb{R}^{2,2})^*$ be a 2-form, then*
i) η is anti-self-dual if and only if $\rho^+(\eta) = 0$.
ii) η is self-dual if and only if $\rho^-(\eta) = 0$.
iii) The space of self-dual $2-$forms Λ^+ is isomorphic to $su(1,1)$
iv)The space of complex valued self-dual $2-$forms $\Lambda^+ \otimes \mathbb{C}$ is isomorphic to $End_0(\triangle^+_{2,2})$

Since M is a spin^c manifold, globalizing above concepts is possible. We pointed out the global map, a bundle map, $\kappa : TM \to End(S)$ in Section 4, similarly we can define bundle map

$$\rho : \Lambda^2(M) \to End(S)$$

and complexified map

$$\rho : \Lambda^2(M) \otimes \mathbb{C} \to End(S).$$

The restriction of this map to the complex valued self-dual $2-$forms gives the following bundle map

$$\rho^+ : \Lambda^+ \otimes \mathbb{C} \to End_0(S^+)$$

where $End_0(S)$ denotes the space of traceless endomorphisms of the bundle S^+. Now we can write down the second Seiberg-Witten equation. Let A be an $i\mathbb{R}-$valued connection $1-$form on the S^1 principal bundle P_{S^1} and F_A be its curvature $2-$form, which is $i\mathbb{R}$ valued $2-$form on P_{S^1}. It is known that such curvature $2-$forms are in one to one correspondence with the $i\mathbb{R}-$valued $2-$forms on M (see [5]). We denote the corresponding $2-$form on M with the same symbol F_A. Let F_A^+ be the self-dual part of F_A, then $\rho^+(F_A^+)$ is a traceless endomorphism of the bundle S^+. On the other hand any positive spinor field Ψ determines an endomorphism $\Psi\Psi^*$ of S^+ by the formula

$$(\Psi\Psi^*)(\Phi) = < \Psi, \Phi >_{1,1} \Psi$$

where $<, >_{1,1}$ is indefinite Hermitian inner product on S^+ and Φ is a spinor field on S^+. The traceless part of $\Psi\Psi^*$ is denoted by $(\Psi\Psi^*)_0$. Then the second Seiberg-Witten equation for the pair (A, Ψ) is

$$\rho^+(F_A^+) = (\Psi\Psi^*)_0. \tag{3}$$

5.3 Seiberg-Witten Equations on $\mathbb{R}^{2,2}$

Now we write down Seiberg-Witten equations on $4-$dimensional flat space with neutral metric. Explicit interpretations of original Seiberg-Witten equations on flat Euclidean flat space $\mathbb{R}^4$ and some properties of them can be found in [14] and [15]). For the explicit interpretations of these equations in the neutral case we use the spinor representation κ given in Section 2. In this case $S=\mathbb{R}^{2,2}\times\triangle_{2,2}$, $S^+=\mathbb{R}^{2,2}\times\triangle_{2,2}^+$ and $S^-=\mathbb{R}^{2,2}\times\triangle_{2,2}^-$. The sections of the subbundles $S^\pm$ can be expressed as follows

$$\begin{aligned}\Gamma(S^+) &= \left\{(\psi_1,\psi_2,0,0)\ \mid \psi_1,\psi_2\in C^\infty\left(\mathbb{R}^{2,2},\mathbb{C}\right)\right\},\\ \Gamma(S^-) &= \left\{(0,0,\psi_3,\psi_4)\ \mid \psi_3,\psi_4\in C^\infty\left(\mathbb{R}^{2,2},\mathbb{C}\right)\right\}.\end{aligned}$$

Since $P_{S^1}=\mathbb{R}^{2,2}\times S^1$, the $i\mathbb{R}-$valued connection $1-$form on P_{S^1} is given by

$$A=\sum_{j=1}^{4}A_j dx_j\in\Omega^1\left(\ \mathbb{R}^{2,2},i\mathbb{R}\right)$$

where $A_j:\mathbb{R}^{2,2}\longrightarrow i\mathbb{R}$ are smooth maps. The associated spin^c connection $\nabla=\nabla^A$ on $\mathbb{R}^{2,2}$ is given by

$$\nabla_j\Psi=\frac{\partial\Psi}{\partial x_j}+A_j\Psi,$$

where $\Psi:\mathbb{R}^{2,2}\longrightarrow\mathbb{C}^2$. Then the Dirac equation in flat case is given by

$$\begin{aligned}D_A\Psi &= e_1\cdot\nabla_{e_1}\Psi+e_2\cdot\nabla_{e_2}\Psi-e_3\cdot\nabla_{e_3}\Psi-e_4\cdot\nabla_{e_4}\Psi\\ &= \sum_{i=1}^{4}\varepsilon_i\kappa(e_i)(\nabla_{e_i}\Psi)\\ &= \sum_{i=1}^{4}\kappa(e_i)\begin{pmatrix}\frac{\partial\psi_1}{\partial x_i}+A_i\psi_1\\ \frac{\partial\psi_2}{\partial x_i}+A_i\psi_2\end{pmatrix}\\ &= \begin{pmatrix}\frac{\partial\psi_1}{\partial x_1}+A_1\psi_1+i(\frac{\partial\psi_2}{\partial x_2}+A_2\psi_2)+i(\frac{\partial\psi_2}{\partial x_3}+A_3\psi_2)+\frac{\partial\psi_2}{\partial x_4}+A_4\psi_2\\ \frac{\partial\psi_2}{\partial x_1}+A_1\psi_2-i(\frac{\partial\psi_1}{\partial x_2}+A_2\psi_1)-i(\frac{\partial\psi_1}{\partial x_3}+A_3\psi_1)+\frac{\partial\psi_1}{\partial x_4}+A_4\psi_1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.\end{aligned}$$

The explicit form of Dirac equation:

$$\frac{\partial\psi_1}{\partial x_1}+A_1\psi_1+i(\frac{\partial\psi_2}{\partial x_2}+A_2\psi_2)+i(\frac{\partial\psi_2}{\partial x_3}+A_3\psi_2)+\frac{\partial\psi_2}{\partial x_4}+A_4\psi_2=0$$

$$\frac{\partial\psi_2}{\partial x_1}+A_1\psi_2-i(\frac{\partial\psi_1}{\partial x_2}+A_2\psi_1)-i(\frac{\partial\psi_1}{\partial x_3}+A_3\psi_1)+\frac{\partial\psi_1}{\partial x_4}+A_4\psi_1=0.$$

The curvature 2-form F_A is given by

$$F_A = dA = \sum_{i<j} F_{ij} e^i \wedge e^j \in \Omega^2\left(\mathbb{R}^{2,2}, i\mathbb{R}\right),$$

where $F_{ij} = \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}$ for $i, j = 1, ..., 4$.

The matrix form of $\Psi\Psi^*$ with respect to the frame $E_1 = (1,0)$, $E_2 = (0,1)$ is given by

$$(\Psi\Psi^*) = \begin{pmatrix} |\psi_1|^2 & -\psi_1\overline{\psi_2} \\ \psi_2\overline{\psi_1} & -|\psi_2|^2 \end{pmatrix}.$$

The traceless part of $\Psi\Psi^*$ is

$$\begin{aligned}(\Psi\Psi^*)_0 &= \begin{pmatrix} |\psi_1|^2 & -\psi_1\overline{\psi_2} \\ \psi_2\overline{\psi_1} & -|\psi_2|^2 \end{pmatrix} - \frac{1}{2}|\psi|^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}(|\psi_1|^2 + |\psi_2|^2) & -\psi_1\overline{\psi_2} \\ \psi_2\overline{\psi_1} & -\frac{1}{2}(|\psi_1|^2 + |\psi_2|^2) \end{pmatrix}\end{aligned}$$

Now we can interpret the curvature equation by $\rho^+(F^+) = (\Psi\Psi^*)_0$. Then we obtain following set of equations

$$\begin{aligned} F_{12} + F_{34} &= -\frac{i}{2}(|\psi_1|^2 + |\psi_2|^2) \\ F_{23} - F_{14} &= \frac{1}{2}(\overline{\psi_1}\psi_2 - \psi_1\overline{\psi_2}) \\ F_{13} + F_{34} &= -\frac{i}{2}(\overline{\psi_1}\psi_2 + \psi_1\overline{\psi_2}) \end{aligned} \tag{4}$$

which are consistent and similar to the classical Seiberg-Witten equations on $\mathbb{R}^4$ with Euclidean metric.

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