Strong convergence of a hybrid method for pseudomonotone variational inequalities and fixed point problems

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Abstract

In this paper, we suggest a hybrid method for finding a common element of the set of solution of a pseudomonotone, Lipschitz-continuous variational inequality problem and the set of common fixed points of an infinite family of nonexpansive mappings. The proposed iterative method combines two well-known methods: extragradient method and CQ method. We derive a necessary and sufficient condition for the strong convergence of the sequences generated by the proposed method.

1 Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H . Let $A : C \rightarrow H$ be a nonlinear operator. By definition, the variational inequality problem $VI(C, A)$ is to find $u \in C$ such that

$$(VI(C, A)): \quad \langle Au, v - u \rangle \geq 0, \quad \forall v \in C.$$

The set of solutions of the variational inequality is denoted by Ω .

Variational inequality theory has emerged as an important tool in studying a wide class of obstacle, unilateral and equilibrium problems, which arise in

Key Words: Variational inequality problem; Fixed point problems; Pseudomonotone mapping; Nonexpansive mapping; Extragradient method; CQ method; Projection.

2010 Mathematics Subject Classification: 47H05; 47H09; 47H10; 47J05; 47J25.

Received: March, 2011.

Revised: April, 2011.

Accepted: February, 2012.

several branches of pure and applied sciences in a unified and general framework. Several numerical methods have been developed for solving variational inequalities and related optimization problems, see [1], [8], [9], [11]-[14], [21]-[24], [28]-[31] and the references therein. Let us start with Korpelevich's extragradient method which was introduced by Korpelevich [13] in 1976 and which generates a sequence $\{x_n\}$ via the recursion:

$$\begin{cases} y_n = P_C[x_n - \lambda Ax_n], \\ x_{n+1} = P_C[x_n - \lambda Ay_n], n \geq 0, \end{cases} \quad (1)$$

where P_C is the metric projection from R^n onto C , $A : C \rightarrow H$ is a monotone operator and λ is a constant. Korpelevich [13] proved that the sequence $\{x_n\}$ converges strongly to a solution of $VI(C, A)$. Note that the setting of the problem is the Euclidean space R^n .

Korpelevich's extragradient method has extensively been studied in the literature for solving a more general problem that consists of finding a common point that lies in the solution set of a variational inequality and the set of fixed points of a nonexpansive mapping. This type of problem arises in various theoretical and modeling contexts, see e.g., [2], [4]-[7], [15], [25], [26] and references therein. Especially, Nadezhkina and Takahashi [17] introduced the following iterative method which combines Korpelevich's extragradient method and a CQ method:

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= P_C[x_n - \lambda_n Ax_n], \\ z_n &= \alpha_n x_n + (1 - \alpha_n)SP_C[x_n - \lambda_n Ay_n], \\ C_n &= \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x, n \geq 0, n \geq 0, \end{aligned}$$

where P_C is the metric projection from H onto C , $A : C \rightarrow H$ is a monotone k -Lipschitz-continuous mapping, $S : C \rightarrow C$ is a nonexpansive mapping, $\{\lambda_n\}$ and $\{\alpha_n\}$ are two real number sequences. They proved the strong convergence of the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ to the same element in $Fix(S) \cap \Omega$. We note that Nadezhkina and Takahashi [17] employed the monotonicity and Lipschitz-continuity of A to define a maximal monotone operator T as follows:

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

where $N_C v = \{w \in H : \langle v - u, w \rangle \geq 0, \forall u \in C\}$ is the normal cone to C at $v \in C$ (see, [19]). However, if the mapping A is a pseudomonotone Lipschitz-

continuous, then T is not necessarily a maximal monotone operator. This fact implies that the approach used in [17] cannot be applied. To overcome this difficulty, Ceng, Teboulle and Yao [3] suggested a new iterative method as follows:

$$\begin{aligned} y_n &= P_C[x_n - \lambda_n A x_n], \\ z_n &= \alpha_n x_n + (1 - \alpha_n) S_n P_C[x_n - \lambda_n A y_n], \\ C_n &= \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ \text{find } x_{n+1} &\in C_n \text{ such that} \\ \langle x_n - x_{n+1} + e_n - \sigma_n A x_{n+1}, x_{n+1} - x \rangle &\geq -\epsilon_n, \quad \forall x \in C_n, \end{aligned}$$

where $A : C \rightarrow H$ is a pseudomonotone, k -lipschitz-continuous and (w, s) -sequentially-continuous mapping, $\{S_i\}_{i=1}^N : C \rightarrow C$ are N nonexpansive mappings. Under some mild conditions, they proved that the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge weakly to the same element of $\bigcap_{i=1}^N \text{Fix}(S_i) \cap \Omega$ if and only if $\liminf_n \langle A x_n, x - x_n \rangle \geq 0, \forall x \in C$. Note that Ceng, Teboulle and Yao's method has only weak convergence. So, we may ask of whether *i*) a strong convergence property is available, *ii*) a denumerable family of maps $(S_i; i \geq 1)$ is allowed.

Motivated and inspired by the works of Nadezhkina and Takahashi [17] and Ceng, Teboulle and Yao [3], in this paper we suggest a hybrid method for finding a common element of the set of solution of a pseudomonotone, Lipschitz-continuous variational inequality problem and the set of common fixed points of an infinite family of nonexpansive mappings. The proposed iterative method combines two well-known methods: extragradient method and CQ method. We derive a necessary and sufficient condition for the strong convergence of the sequences generated by the proposed method.

2 Preliminaries

In this section, we will recall some basic notations and collect some conclusions that will be used in the next section.

Let C be a nonempty closed convex subset of a real Hilbert space H . A mapping $A : C \rightarrow H$ is called monotone if

$$\langle Au - Av, u - v \rangle \geq 0, \forall u, v \in C.$$

A mapping $A : C \rightarrow H$ is called pseudomonotone if, for all $u, v \in C$,

$$\langle Au, v - u \rangle \geq 0 \Rightarrow \langle Av, v - u \rangle \geq 0.$$

It is clear that if a mapping A is monotone, then it is pseudomonotone.

Recall that a mapping $S : C \rightarrow C$ is said to be nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \forall x, y \in C.$$

Denote by $Fix(S)$ the set of fixed points of S ; that is, $Fix(S) = \{x \in C : Sx = x\}$.

It is well known that, for any $u \in H$, there exists a unique $u_0 \in C$ such that

$$\|u - u_0\| = \inf\{\|u - x\| : x \in C\}.$$

We denote u_0 by $P_C[u]$, where P_C is called the *metric projection* of H onto C . The metric projection P_C of H onto C has the following basic properties:

- (i) $\|P_C[x] - P_C[y]\| \leq \|x - y\|$ for all $x, y \in H$.
- (ii) $\langle x - P_C[x], y - P_C[x] \rangle \leq 0$ for all $x \in H, y \in C$.
- (iii) The property (ii) is equivalent to

$$\|x - P_C[x]\|^2 + \|y - P_C[x]\|^2 \leq \|x - y\|^2, \forall x \in H, y \in C.$$

- (iv) In the context of the variational inequality problem, the characterization of the projection implies that

$$u \in \Omega \Leftrightarrow u = P_C[u - \lambda Au], \forall \lambda > 0.$$

Recall that H satisfies the Opial condition [27]; i.e., for any sequence $\{x_n\}$ with x_n converges weakly to x , the inequality

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\{S_i\}_{i=1}^{\infty}$ be infinite family of nonexpansive mappings of C into itself and let $\{\xi_i\}_{i=1}^{\infty}$ be real number sequences such that $0 \leq \xi_i \leq 1$ for every $i \in \mathbf{N}$. For

any $n \in \mathbb{N}$, define a mapping W_n of C into itself as follows:

$$\begin{aligned} U_{n,n+1} &= I, \\ U_{n,n} &= \xi_n S_n U_{n,n+1} + (1 - \xi_n)I, \\ U_{n,n-1} &= \xi_{n-1} S_{n-1} U_{n,n} + (1 - \xi_{n-1})I, \\ &\vdots \\ U_{n,k} &= \xi_k S_k U_{n,k+1} + (1 - \xi_k)I, \\ U_{n,k-1} &= \xi_{k-1} S_{k-1} U_{n,k} + (1 - \xi_{k-1})I, \\ &\vdots \\ U_{n,2} &= \xi_2 S_2 U_{n,3} + (1 - \xi_2)I, \\ W_n &= U_{n,1} = \xi_1 S_1 U_{n,2} + (1 - \xi_1)I. \end{aligned} \quad (2)$$

Such W_n is called the W -mapping generated by $\{S_i\}_{i=1}^\infty$ and $\{\xi_i\}_{i=1}^\infty$.

We have the following crucial Lemmas 3.1 and 3.2 concerning W_n which can be found in [20]. Now we only need the following similar version in Hilbert spaces.

Lemma 2.1. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $S_1, S_2, \dots$ be nonexpansive mappings of C into itself such that $\bigcap_{n=1}^\infty \text{Fix}(S_n)$ is nonempty, and let $\xi_1, \xi_2, \dots$ be real numbers such that $0 < \xi_i \leq b < 1$ for any $i \in \mathbb{N}$. Then, for every $x \in C$ and $k \in \mathbb{N}$, the limit $\lim_{n \rightarrow \infty} U_{n,k}x$ exists.*

Lemma 2.2. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $S_1, S_2, \dots$ be nonexpansive mappings of C into itself such that $\bigcap_{n=1}^\infty \text{Fix}(S_n)$ is nonempty, and let $\xi_1, \xi_2, \dots$ be real numbers such that $0 < \xi_i \leq b < 1$ for any $i \in \mathbb{N}$. Then, $\text{Fix}(W) = \bigcap_{n=1}^\infty \text{Fix}(S_n)$.*

Lemma 2.3. *(see [27]) Using Lemmas 2.1 and 2.2, one can define a mapping W of C into itself as: $Wx = \lim_{n \rightarrow \infty} W_n x = \lim_{n \rightarrow \infty} U_{n,1}x$, for every $x \in C$. If $\{x_n\}$ is a bounded sequence in C , then we have*

$$\lim_{n \rightarrow \infty} \|Wx_n - W_n x_n\| = 0.$$

We also need the following well-known lemmas for proving our main results.

Lemma 2.4. *([10]) Let C be a nonempty closed convex subset of a real Hilbert space H . Let $S : C \rightarrow C$ be a nonexpansive mapping with $\text{Fix}(S) \neq \emptyset$. Then S is demiclosed on C , i.e., if $y_n \rightarrow z \in C$ weakly and $y_n - Sy_n \rightarrow y$ strongly, then $(I - S)z = y$.*

Lemma 2.5. ([16]) *Let C be a closed convex subset of H . Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_C[u]$. If $\{x_n\}$ is such that $\omega_w(x_n) \subset C$ and satisfies the condition*

$$\|x_n - u\| \leq \|u - q\| \text{ for all } n.$$

Then $x_n \rightarrow q$.

We adopt the following notation:

- For a given sequence $\{x_n\} \subset H$, $\omega_w(x_n)$ denotes the weak ω -limit set of $\{x_n\}$; that is,

$$\omega_w(x_n) := \{x \in H : \{x_{n_j}\} \text{ converges weakly to } x \text{ for some subsequence } \{n_j\} \text{ of } \{n\}\}.$$

- $x_n \rightharpoonup x$ stands for the weak convergence of (x_n) to x ;
- $x_n \rightarrow x$ stands for the strong convergence of (x_n) to x .

3 Main results

In this section we will state and prove our main results.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $A : C \rightarrow H$ be a pseudomonotone, k -Lipschitz-continuous and (w, s) -sequentially-continuous mapping and let $\{S_n\}_{n=1}^\infty$ be an infinite family of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^\infty \text{Fix}(S_n) \cap \Omega \neq \emptyset$. Let $x_1 = x_0 \in C$. For $x_1 \in C$, $C_1 = C$, let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{C_n\}$ be sequences generated as:*

$$\begin{aligned} y_n &= P_{C_n}[x_n - \lambda_n A x_n], \\ z_n &= \alpha_n x_n + (1 - \alpha_n) W_n P_{C_n}[x_n - \lambda_n A y_n], \\ C_{n+1} &= \{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} &= P_{C_{n+1}}[x_0], n \geq 1, \end{aligned} \tag{3}$$

where $\{W_n; n \geq 1\}$ are W -mappings of (2). Assume that

- (i) $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/k)$;
- (ii) $\{\alpha_n\} \subset [0, c]$ for some $c \in [0, 1)$.

Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ generated by (3) converge strongly to the same point $P_{\bigcap_{n=1}^\infty \text{Fix}(S_n) \cap \Omega}[x_0]$ if and only if $\liminf_n \langle A x_n, x - x_n \rangle \geq 0$, $\forall x \in C$.

The proof will be divided into several conclusions. Assume in the sequel that all assumptions of Theorem 3.1 are satisfied.

Conclusion 3.2. (1) Every C_n is closed and convex, $n \geq 1$;

(2) $\bigcap_{n=1}^{\infty} \text{Fix}(S_n) \cap \Omega \subset C_{n+1}, \forall n \geq 1$;

(3) $\{x_{n+1}\}$ is well-defined.

Proof. First we note that $C_1 = C$ is closed and convex. Assume that C_k is closed and convex. From (3), we can rewrite C_{k+1} as

$$C_{k+1} = \{z \in C_k : \langle z - \frac{x_k + z_k}{2}, z_k - x_k \rangle \geq 0\}.$$

It is clear that C_{k+1} is a half space. Hence, C_{k+1} is closed and convex. By induction, we deduce that C_n is closed and convex for all $n \geq 1$. Next we show that $\bigcap_{n=1}^{\infty} \text{Fix}(S_n) \cap \Omega \subset C_{n+1}, \forall n \geq 1$.

Set $t_n = P_{C_n}[x_n - \lambda A y_n]$ for all $n \geq 1$. Pick up $u \in \bigcap_{n=1}^{\infty} \text{Fix}(S_n) \cap \Omega$. From property (iii) of P_C , we have

$$\begin{aligned} \|t_n - u\|^2 &\leq \|x_n - \lambda_n A y_n - u\|^2 - \|x_n - \lambda_n A y_n - t_n\|^2 \\ &= \|x_n - u\|^2 - \|x_n - t_n\|^2 + 2\lambda_n \langle A y_n, u - t_n \rangle \\ &= \|x_n - u\|^2 - \|x_n - t_n\|^2 + 2\lambda_n \langle A y_n, u - y_n \rangle + 2\lambda_n \langle A y_n, y_n - t_n \rangle \end{aligned}$$

Since $u \in \Omega$ and $y_n \in C_n \subset C$, we get

$$\langle A u, y_n - u \rangle \geq 0.$$

This together with the pseudomonotonicity of A imply that

$$\langle A y_n, y_n - u \rangle \geq 0. \quad (5)$$

Combine (4) with (5) to deduce

$$\begin{aligned} \|t_n - u\|^2 &\leq \|x_n - u\|^2 - \|x_n - t_n\|^2 + 2\lambda_n \langle A y_n, y_n - t_n \rangle \\ &= \|x_n - u\|^2 - \|x_n - y_n\|^2 - 2\langle x_n - y_n, y_n - t_n \rangle - \|y_n - t_n\|^2 \\ &\quad + 2\lambda_n \langle A y_n, y_n - t_n \rangle \\ &= \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 \\ &\quad + 2\langle x_n - \lambda_n A y_n - y_n, t_n - y_n \rangle. \end{aligned} \quad (6)$$

Note that $y_n = P_{C_n}[x_n - \lambda_n A x_n]$ and $t_n \in C_n$. Then, by using the property (ii) of P_C , we have

$$\langle x_n - \lambda_n A x_n - y_n, t_n - y_n \rangle \leq 0.$$

Hence,

$$\begin{aligned}
 \langle x_n - \lambda_n A y_n - y_n, t_n - y_n \rangle &= \langle x_n - \lambda_n A x_n - y_n, t_n - y_n \rangle + \langle \lambda_n A x_n - \lambda_n A y_n, t_n - y_n \rangle \\
 &\leq \langle \lambda_n A x_n - \lambda_n A y_n, t_n - y_n \rangle \\
 &\leq \lambda_n k \|x_n - y_n\| \|t_n - y_n\|.
 \end{aligned} \tag{7}$$

From (6) and (7), we get

$$\begin{aligned}
 \|t_n - u\|^2 &\leq \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\lambda_n k \|x_n - y_n\| \|t_n - y_n\| \\
 &\leq \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + \lambda_n^2 k^2 \|x_n - y_n\|^2 + \|y_n - t_n\|^2 \\
 &= \|x_n - u\|^2 + (\lambda_n^2 k^2 - 1) \|x_n - y_n\|^2 \\
 &\leq \|x_n - u\|^2.
 \end{aligned} \tag{8}$$

Therefore, from (8), together with $z_n = \alpha_n x_n + (1 - \alpha_n) W_n t_n$ and $u = W_n u$, we get

$$\begin{aligned}
 \|z_n - u\|^2 &= \|\alpha_n(x_n - u) + (1 - \alpha_n)(W_n t_n - u)\|^2 \\
 &\leq \alpha_n \|x_n - u\|^2 + (1 - \alpha_n) \|W_n t_n - u\|^2 \\
 &\leq \alpha_n \|x_n - u\|^2 + (1 - \alpha_n) \|t_n - u\|^2 \\
 &\leq \|x_n - u\|^2 + (1 - \alpha_n) (\lambda_n^2 k^2 - 1) \|x_n - y_n\|^2 \\
 &\leq \|x_n - u\|^2,
 \end{aligned} \tag{9}$$

which implies that

$$u \in C_{n+1}.$$

Therefore,

$$\bigcap_{n=1}^{\infty} \text{Fix}(S_n) \cap \Omega \subset C_{n+1}, \forall n \geq 1.$$

This implies that $\{x_{n+1}\}$ is well-defined. □

Conclusion 3.3. *The sequences $\{x_n\}$, $\{z_n\}$ and $\{t_n\}$ are all bounded and $\lim_{n \rightarrow \infty} \|x_n - x_0\|$ exists.*

Proof. From $x_{n+1} = P_{C_{n+1}}[x_0]$, we have

$$\langle x_0 - x_{n+1}, x_{n+1} - y \rangle \geq 0, \forall y \in C_{n+1}.$$

Since $\bigcap_{n=1}^{\infty} \text{Fix}(S_n) \cap \Omega \subset C_{n+1}$, we also have

$$\langle x_0 - x_{n+1}, x_{n+1} - u \rangle \geq 0, \forall u \in \bigcap_{n=1}^{\infty} \text{Fix}(S_n) \cap \Omega.$$

So, for $u \in \bigcap_{n=1}^{\infty} \text{Fix}(S_n) \cap \Omega$, we have

$$\begin{aligned} 0 &\leq \langle x_0 - x_{n+1}, x_{n+1} - u \rangle \\ &= \langle x_0 - x_{n+1}, x_{n+1} - x_0 + x_0 - u \rangle \\ &= -\|x_0 - x_{n+1}\|^2 + \langle x_0 - x_{n+1}, x_0 - u \rangle \\ &\leq -\|x_0 - x_{n+1}\|^2 + \|x_0 - x_{n+1}\| \|x_0 - u\|. \end{aligned}$$

Hence,

$$\|x_0 - x_{n+1}\| \leq \|x_0 - u\|, \forall u \in \bigcap_{n=1}^{\infty} \text{Fix}(S_n) \cap \Omega, \quad (10)$$

which implies that $\{x_n\}$ is bounded. From (8) and (9), we can deduce that $\{z_n\}$ and $\{t_n\}$ are also bounded.

From $x_n = P_{C_n}[x_0]$ and $x_{n+1} = P_{C_{n+1}}[x_0] \in C_{n+1} \subset C_n$, we have

$$\langle x_0 - x_n, x_n - x_{n+1} \rangle \geq 0. \quad (11)$$

Hence,

$$\begin{aligned} 0 &\leq \langle x_0 - x_n, x_n - x_{n+1} \rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1} \rangle \\ &= -\|x_0 - x_n\|^2 + \langle x_0 - x_n, x_0 - x_{n+1} \rangle \\ &\leq -\|x_0 - x_n\|^2 + \|x_0 - x_n\| \|x_0 - x_{n+1}\|, \end{aligned}$$

and therefore

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\|.$$

This together with the boundedness of the sequence $\{x_n\}$ imply that $\lim_{n \rightarrow \infty} \|x_n - x_0\|$ exists. $\square$

Conclusion 3.4. $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \rightarrow \infty} \|x_n - y_n\| = \lim_{n \rightarrow \infty} \|x_n - z_n\| = \lim_{n \rightarrow \infty} \|x_n - t_n\| = 0$ and $\lim_{n \rightarrow \infty} \|x_n - W_n x_n\| = \lim_{n \rightarrow \infty} \|x_n - W x_n\| = 0$.

Proof. It is well-known that in Hilbert spaces H , the following identity holds:

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y \rangle, \quad \forall x, y \in H.$$

Therefore,

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\ &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0 \rangle. \end{aligned}$$

It follows from (11) that

$$\|x_{n+1} - x_n\|^2 \leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2.$$

Since $\lim_{n \rightarrow \infty} \|x_n - x_0\|$ exists, we get $\|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 \rightarrow 0$. Therefore,

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0.$$

Since $x_{n+1} \in C_n$, we have

$$\|z_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|,$$

and hence

$$\begin{aligned} \|x_n - z_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| \\ &\leq 2\|x_{n+1} - x_n\| \\ &\rightarrow 0. \end{aligned}$$

For each $u \in \bigcap_{n=1}^{\infty} \text{Fix}(S_n) \cap \Omega$, from (9), we have

$$\begin{aligned} \|x_n - y_n\|^2 &\leq \frac{1}{(1 - \alpha_n)(1 - \lambda_n^2 k^2)} (\|x_n - u\|^2 - \|z_n - u\|^2) \\ &\leq \frac{1}{(1 - \alpha_n)(1 - \lambda_n^2 k^2)} (\|x_n - u\| + \|z_n - u\|) \|x_n - z_n\|. \end{aligned}$$

Since $\|x_n - z_n\| \rightarrow 0$ and the sequences $\{x_n\}$ and $\{z_n\}$ are bounded, we obtain $\|x_n - y_n\| \rightarrow 0$.

We note that

$$\begin{aligned} \|t_n - u\|^2 &\leq \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\lambda_n k \|x_n - y_n\| \|t_n - y_n\| \\ &\leq \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + \|x_n - y_n\|^2 + \lambda_n^2 k^2 \|y_n - t_n\|^2 \\ &= \|x_n - u\|^2 + (\lambda_n^2 k^2 - 1) \|y_n - t_n\|^2. \end{aligned}$$

Hence,

$$\begin{aligned} \|z_n - u\|^2 &\leq \alpha_n \|x_n - u\|^2 + (1 - \alpha_n) \|t_n - u\|^2 \\ &\leq \alpha_n \|x_n - u\|^2 + (1 - \alpha_n) (\|x_n - u\|^2 + (\lambda_n^2 k^2 - 1) \|y_n - t_n\|^2) \\ &= \|x_n - u\|^2 + (1 - \alpha_n) (\lambda_n^2 k^2 - 1) \|y_n - t_n\|^2. \end{aligned}$$

It follows that

$$\begin{aligned} \|t_n - y_n\|^2 &\leq \frac{1}{(1 - \alpha_n)(1 - \lambda_n^2 k^2)} (\|x_n - u\|^2 - \|z_n - u\|^2) \\ &\leq \frac{1}{(1 - \alpha_n)(1 - \lambda_n^2 k^2)} (\|x_n - u\| + \|z_n - u\|) \|x_n - z_n\| \\ &\rightarrow 0. \end{aligned}$$

Since A is k -Lipschitz-continuous, we have $\|Ay_n - At_n\| \rightarrow 0$. From

$$\|x_n - t_n\| \leq \|x_n - y_n\| + \|y_n - t_n\|,$$

we also have

$$\|x_n - t_n\| \rightarrow 0.$$

Since $z_n = \alpha_n x_n + (1 - \alpha_n)W_n t_n$, we have

$$(1 - \alpha_n)(W_n t_n - t_n) = \alpha_n(t_n - x_n) + (z_n - t_n).$$

Then,

$$\begin{aligned} (1 - c)\|W_n t_n - t_n\| &\leq (1 - \alpha_n)\|W_n t_n - t_n\| \\ &\leq \alpha_n\|t_n - x_n\| + \|z_n - t_n\| \\ &\leq (1 + \alpha_n)\|t_n - x_n\| + \|z_n - x_n\| \end{aligned}$$

and hence $\|t_n - W_n t_n\| \rightarrow 0$. Observe also that

$$\begin{aligned} \|x_n - W_n x_n\| &\leq \|x_n - t_n\| + \|t_n - W_n t_n\| + \|W_n t_n - W_n x_n\| \\ &\leq \|x_n - t_n\| + \|t_n - W_n t_n\| + \|t_n - x_n\| \\ &\leq 2\|x_n - t_n\| + \|t_n - W_n t_n\|. \end{aligned}$$

So, we have $\|x_n - W_n x_n\| \rightarrow 0$. On the other hand, since $\{x_n\}$ is bounded, from Lemma 2.3, we have $\lim_{n \rightarrow \infty} \|W_n x_n - W x_n\| = 0$. Therefore, we have

$$\lim_{n \rightarrow \infty} \|x_n - W x_n\| = 0.$$

□

Proof. Proof of Theorem 3.1, continued. First, we prove the necessity. Suppose that $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to the same element $\tilde{u} \in \bigcap_{n=1}^{\infty} \text{Fix}(S_n) \cap \Omega$. From the (w, s) -sequential continuity of A , we have $Ax_n \rightarrow A\tilde{u}$. Observe that, for every $x \in C$,

$$\begin{aligned} |\langle Ax_n, x - x_n \rangle - \langle A\tilde{u}, x - \tilde{u} \rangle| &\leq |\langle Ax_n, x - x_n \rangle - \langle A\tilde{u}, x - x_n \rangle| \\ &\quad + |\langle A\tilde{u}, x - x_n \rangle - \langle A\tilde{u}, x - \tilde{u} \rangle| \\ &= |\langle Ax_n - A\tilde{u}, x - x_n \rangle| + |\langle A\tilde{u}, \tilde{u} - x_n \rangle| \\ &\leq \|Ax_n - A\tilde{u}\| \|x - x_n\| + |\langle A\tilde{u}, \tilde{u} - x_n \rangle|. \end{aligned}$$

This implies that

$$\liminf_{n \rightarrow \infty} \langle Ax_n, x - x_n \rangle = \lim_{n \rightarrow \infty} \langle Ax_n, x - x_n \rangle = \langle A\tilde{u}, x - \tilde{u} \rangle, \forall x \in C.$$

Consequently, the necessity holds.

Next, we prove the sufficiency. By Conclusions 3.3-3.5, we have

$$\lim_{n \rightarrow \infty} \|x_n - Wx_n\| = 0.$$

Furthermore, since $\{x_n\}$ is bounded, it has a subsequence $\{x_{n_j}\}$ which converges weakly to some $\tilde{u} \in C$; hence, we have $\lim_{j \rightarrow \infty} \|x_{n_j} - Wx_{n_j}\| = 0$. Note that, from Lemma 2.4, it follows that $I - W$ is demiclosed at zero. Thus $\tilde{u} \in \text{Fix}(W)$. Observe that, for every $x \in C$,

$$\begin{aligned} & |\langle Ax_{n_j}, x - x_{n_j} \rangle - \langle A\tilde{u}, x - \tilde{u} \rangle| \\ & \leq |\langle Ax_{n_j}, x - x_{n_j} \rangle - \langle A\tilde{u}, x - x_{n_j} \rangle| + |\langle A\tilde{u}, x - x_{n_j} \rangle - \langle A\tilde{u}, x - \tilde{u} \rangle| \\ & = |\langle Ax_{n_j} - A\tilde{u}, x - x_{n_j} \rangle| + |\langle A\tilde{u}, \tilde{u} - x_{n_j} \rangle| \\ & \leq \|Ax_{n_j} - A\tilde{u}\| \|x - x_{n_j}\| + |\langle A\tilde{u}, \tilde{u} - x_{n_j} \rangle|. \end{aligned}$$

From the (w, s) -sequential continuity of A , it follows that $\lim_{j \rightarrow \infty} \|Ax_{n_j} - A\tilde{u}\| = 0$. Hence, we have

$$\langle A\tilde{u}, x - \tilde{u} \rangle = \lim_{j \rightarrow \infty} \langle Ax_{n_j}, x - x_{n_j} \rangle \geq \liminf_{n \rightarrow \infty} \langle Ax_n, x - x_n \rangle \geq 0, \forall x \in C.$$

This implies that $\tilde{u} \in \Omega$. Consequently, $\tilde{u} \in \bigcap_{n=1}^{\infty} \text{Fix}(S_n) \cap \Omega$. That is, $\omega_w(x_n) \subset \bigcap_{n=1}^{\infty} \text{Fix}(S_n) \cap \Omega$.

In (10), if we take $u = P_{\bigcap_{n=1}^{\infty} \text{Fix}(S_n) \cap \Omega}[x_0]$, we get

$$\|x_0 - x_{n+1}\| \leq \|x_0 - P_{\bigcap_{n=1}^{\infty} \text{Fix}(S_n) \cap \Omega}[x_0]\|. \quad (12)$$

Notice that $\omega_w(x_n) \subset \bigcap_{n=1}^{\infty} \text{Fix}(S_n) \cap \Omega$. Then, (12) and Lemma 2.5 ensure the strong convergence of $\{x_{n+1}\}$ to $P_{\bigcap_{n=1}^{\infty} \text{Fix}(S_n) \cap \Omega}[x_0]$. Consequently, $\{y_n\}$ and $\{z_n\}$ also converge strongly to $P_{\bigcap_{n=1}^{\infty} \text{Fix}(S_n) \cap \Omega}[x_0]$. This completes the proof. $\square$

Acknowledgments

The authors thank the anonymous referee for helpful comments and suggestions which improved the presentation of this paper.

Yonghong Yao was supported in part by NSFC 11071279 and NSFC 71161001-G0105. Yeong-Cheng Liou was partially supported by the Taiwan NSC grant 100-2221-E-230-012.

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