ON DISCRETE FOURIER SPECTRUM OF RANDOMLY MODULATED SIGNALS

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ABSTRACT. In this work the problem of characterization of Discrete Fourier Transform (DFT) spectrum of an original complex-valued signal modulated by random fluctuations of amplitude and phase is investigated. It is assumed that the amplitude and phase of signal values at discrete time moments of observations are distorted by adding realizations of independent and identically distributed random variables. The obtained results of theoretical analysis of such distorted signal spectra show that only in the case of amplitude modulation the DFT spectrum of the modulated bounded signal can be similar to the original signal spectrum, although there occur random deviations. On the other hand, if phase modulation is present, then the DFT spectrum of the modulated bounded signal not only shows random deviations but also amplitudes of peaks existing in the original spectrum are diminished, and consequently similarity to the original signal spectrum can be significantly blurred.

Keywords: Discrete Fourier Transform, signal spectrum, amplitude modulation, phase modulation, random fluctuations

1. INTRODUCTION

The Discrete Fourier Transform (DFT) based periodogram is a widely used tool for analyzing time series that can be decomposed as a sum of monochromatic oscillations plus noise. Important applications of the periodogram include detection of hidden periodicities and estimation of unknown oscillation parameters (amplitude and frequency). For example, it is well known that very accurate frequency estimates of the sinusoidal components can be obtained from the local maxima of a periodogram (Walker 1971).

If time series of the analyzed signal observation values at discrete time moments x_t , $t = 0, 1, \dots, N-1$, is available, then its corresponding DFT is computed as follows (Gasquet and Witomski 1999)

$$\tilde{x}_k = \frac{1}{N} \sum_{t=0}^{N-1} x_t \exp(-i2\pi kt / N) \quad , \quad (1)$$

for $k = 0, 1, \dots, N-1$. Frequently, the well-known Fast Fourier Transform procedures are used to perform the relevant calculations (Cooley and Tukey 1965), (Press *et al.* 1992), (Singleton 1969). Theoretical as well as numerical properties of the DFT are described in time series analysis textbooks (Blackledge 2003), (Bloomfield 2000), (Bremaud 2002), (Brillinger 1975), (Koopmans 1974). Certain statistical properties of spectrum estimation using the DFT are investigated in the work of (Foster 1996a,b) and some other aspects like periodogram smoothing are considered in (Speed 1985).

The present work is a continuation of the author's previous publication (Popiński 1997) on theoretical, statistical, as well as numerical properties of the DFT spectrum. It deals with the problem of applicability of such technique to spectrum estimation of signals which are subject to random amplitude and phase modulation. The proposed approach is justified by the fact that all signals usually referred to as "periodic" have some amplitude and phase variation from period to period. For example an active sonar system transmits a periodic pulse train to detect targets. The received pulses are not perfectly periodic due to random modulation of the pulses from scattering and attenuation (Hinich 2003). Also El Niño signals are recognized as amplitude and phase modulated (Allen and Robertson 1996).

The assumed concept of random modulation modeling is described in section 2. In section 3 the theoretical results related to the modulated signal DFT spectrum are presented both in the case of a noiseless signal and in the case of signal observations corrupted by random errors.

2. MODULATION MODELING

Let us consider finite duration time series of complex-valued signal observations at discrete time moments $o_t = \text{Re}[o_t] + i \text{Im}[o_t]$, $t = 0, 1, \dots, N-1$. We assume that the analyzed signal is of deterministic character and comprises some regular oscillations. Such a signal can represent for example the monochromatic oscillation $o_{0t} = A_0 \exp(i\omega_0 t + i\varphi_0)$, $t = 0, 1, \dots, N-1$, with frequency ω_0 , constant amplitude A_0 , and constant phase φ_0 .

Now, let us assume that the signal values amplitudes and phases are distorted by random additive fluctuations a_t and φ_t , $t = 0, 1, \dots, N-1$, respectively, which are realizations of independent and identically distributed random variables. We also assume that amplitude distortions are independent of phase distortions. Hence, we deal with an amplitude and phase modulated signal satisfying the mathematical model

$$r_t = o_t(1 + a_t)\exp(i\varphi_t) = o_t \exp(i\varphi_t) + o_t a_t \exp(i\varphi_t), \quad (2)$$

where $\exp(i\varphi_t)$ and $a_t \exp(i\varphi_t)$, $t = 0, 1, \dots, N-1$, are realizations of independent and identically distributed random variables. Let the distribution of φ_t be uniform on the interval $(-\Phi, \Phi)$ ($\varphi_t \sim U(-\Phi, \Phi)$) with $0 < \Phi \leq \pi$, which gives immediately for $t = 0, 1, \dots, N-1$,

$$m_\varphi = E_\varphi \exp(i\varphi_t) = \frac{1}{2\Phi} \int_{-\Phi}^{\Phi} \exp(iu) du = \frac{\sin(\Phi)}{\Phi}, \quad (3)$$

$$\sigma_\varphi^2 = E_\varphi |\exp(i\varphi_t) - m_\varphi|^2 = E_\varphi |\exp(i\varphi_t)|^2 - m_\varphi^2 = \frac{1}{2\Phi} \int_{-\Phi}^{\Phi} |\exp(iu)|^2 du - m_\varphi^2 = 1 - m_\varphi^2.$$

As one can clearly see the mean value m_φ satisfies $0 \leq m_\varphi < 1$ and is equal to zero only for $\Phi = \pi$, so that we also have $0 < \sigma_\varphi^2 \leq 1$.

About the distribution of the real-valued random variable a_t we assume only that

$$E_a a_t = 0 \quad \text{and} \quad E_a |a_t|^2 = \sigma_a^2 \geq 0 \quad (4)$$

(if $\sigma_a = 0$ there is only phase modulation of the signal), which further implies for $t = 0, 1, \dots, N-1$,

$$m_{a\varphi} = E_a E_\varphi (1 + a_t) \exp(i\varphi_t) = E_a (1 + a_t) E_\varphi \exp(i\varphi_t) = m_\varphi, \quad (5)$$

$$\sigma_{a\varphi}^2 = E_a E_\varphi \left| (1 + a_t) \exp(i\varphi_t) - m_{a\varphi} \right|^2 = E_a (1 + a_t)^2 E_\varphi \left| \exp(i\varphi_t) \right|^2 - m_{a\varphi}^2 = 1 + \sigma_a^2 - m_\varphi^2 ,$$

and obviously we have $\sigma_a^2 < \sigma_{a\varphi}^2 \leq 1 + \sigma_a^2$. For example, a_t can be uniformly distributed on the interval $[-A, A]$, where $A > 0$, and then $\sigma_a^2 = A^2/3$.

3. MODULATED SIGNAL SPECTRA

It follows immediately from (2) that in order to analyze the DFT spectrum of our model signal we shall consider the case of signals $r_t = z_t o_t$, $t = 0, 1, \dots, N-1$, modulated by series of independent and identically distributed complex-valued random variables z_t . Indeed, the case of $z_t = \exp(i\varphi_t)$ corresponds to phase modulation, the case of $z_t = (1 + a_t) \exp(i\varphi_t)$ to phase and amplitude modulation, and the case of real-valued modulation series $z_t = a_t$ is needed for amplitude modulation analysis, respectively. In all three mentioned cases the random variables z_t have constant mean value and variance according to (3), (4) and (5).

In order to compute the DFT of the modulated signals of the form $r_t = z_t o_t$, $t = 0, 1, \dots, N-1$, we apply the well-known convolution formula (Gasquet and Witomski 1999)

$$\tilde{r}_k = \sum_{j=0}^k \tilde{o}_{k-j} \tilde{z}_j + \sum_{j=k+1}^{N-1} \tilde{o}_{N-j+k} \tilde{z}_j \quad (6)$$

for $k = 0, 1, \dots, N-1$.

Hence, if we want to analyze the DFT of the modulated signal $\tilde{r}_k$, it is necessary to characterize the statistical properties of the random modulating series DFT $\tilde{z}_j$, $j = 0, 1, \dots, N-1$. Such a characterization is given in the following lemma.

Lemma 1.

If a series z_t , $t = 0, 1, \dots, N-1$, represents realizations of uncorrelated complex-valued random variables with constant mean value $E_z z_t = m_z$ and finite variance $E_z |z_t - m_z|^2 = \sigma_z^2 > 0$, then its DFT satisfies

$$E_z \tilde{z}_0 = m_z, \quad E_z \tilde{z}_k = 0 \quad \text{for } k = 1, \dots, N-1,$$

$$\text{and } E_z (\tilde{z}_k - E_z \tilde{z}_k)(\tilde{z}_l - E_z \tilde{z}_l) = \frac{1}{N} \sigma_z^2 \delta_{kl} \quad \text{for } k, l = 0, 1, \dots, N-1,$$

where the symbol δ_{kl} denotes the Kronecker delta.

Proof.

The first property $E_z \tilde{z}_0 = m_z$ follows immediately from the assumptions of the lemma and the definition of the DFT. In order to prove the equality $E_z \tilde{z}_k = 0$ for $k = 1, \dots, N-1$, observe that

$$\begin{aligned} E_z \tilde{z}_k &= \frac{1}{N} E_z \sum_{t=0}^{N-1} z_t \exp(-i2\pi kt / N) = \frac{1}{N} \sum_{t=0}^{N-1} E_z z_t \exp(-i2\pi kt / N) = \frac{1}{N} \sum_{t=0}^{N-1} m_z \exp(-i2\pi kt / N) \\ &= \frac{m_z}{N} \sum_{t=0}^{N-1} \exp(-i2\pi kt / N) = \frac{m_z}{N} \frac{1 - \exp(-i2\pi k)}{1 - \exp(-i2\pi k / N)} = 0. \end{aligned}$$

Also from the definition of the DFT and from the assumed zero correlation of the random variables considered we have for $k, l = 0, 1, \dots, N-1$,

$$E_z (\tilde{z}_k - E_z \tilde{z}_k)(\tilde{z}_l - E_z \tilde{z}_l) = \frac{1}{N^2} E_z \sum_{t=0}^{N-1} (z_t - E_z z_t) \exp(-i2\pi kt / N) \sum_{s=0}^{N-1} (\bar{z}_s - E_z \bar{z}_s) \exp(i2\pi ls / N)$$

$$\begin{aligned}
&= \frac{1}{N^2} \sum_{t=0}^{N-1} \sum_{s=0}^{N-1} E_z(z_t - E_z z_t)(\bar{z}_s - E_z \bar{z}_s) \exp(-i2\pi(kt - ls)/N) = \frac{1}{N^2} \sum_{t=0}^{N-1} \sigma_z^2 \exp(-i2\pi(k-l)t/N) \\
&= \frac{1}{N} \sigma_z^2 \delta_{kl},
\end{aligned}$$

since $\sum_{t=0}^{N-1} \exp(-i2\pi(k-l)t/N) = \frac{1 - \exp(-i2\pi(k-l))}{1 - \exp(-i2\pi(k-l)/N)} = 0$ for $k \neq l$, which proves the second property. $\square$

Now, as we know the mean values of the random variables $\tilde{z}_j$, $j = 0, 1, \dots, N-1$, according to the above lemma, we easily obtain from (6)

$$E_z \tilde{r}_k = \tilde{o}_k m_z \quad \text{for } k = 0, 1, \dots, N-1, \quad (7)$$

and further since $E_z(\tilde{z}_k - E_z \tilde{z}_k)(\tilde{z}_l - E_z \tilde{z}_l) = \sigma_z^2 \delta_{kl}/N$ for $k, l = 0, 1, \dots, N-1$, the same formula yields for $k = 0, 1, \dots, N-1$,

$$\begin{aligned}
E_z |\tilde{r}_k - E_z \tilde{r}_k|^2 &= E_z \left| \sum_{j=0}^k \tilde{o}_{k-j} (\tilde{z}_j - E_z \tilde{z}_j) + \sum_{j=k+1}^{N-1} \tilde{o}_{N-j+k} (\tilde{z}_j - E_z \tilde{z}_j) \right|^2 \\
&= \frac{\sigma_z^2}{N} \left[\sum_{j=0}^k |\tilde{o}_{k-j}|^2 + \sum_{j=k+1}^{N-1} |\tilde{o}_{N-j+k}|^2 \right] = \frac{\sigma_z^2}{N} \sum_{k=0}^{N-1} |\tilde{o}_k|^2 = \frac{\sigma_z^2}{N^2} \sum_{t=0}^{N-1} |o_t|^2,
\end{aligned} \quad (8)$$

where the last equality follows from the well-known property of the DFT $\sum_{k=0}^{N-1} |\tilde{o}_k|^2 = \sum_{t=0}^{N-1} |o_t|^2 / N$ (Gasquet and Witomski 1999).

From the Schwartz inequality (Bremaud 2002) we also have the following estimate related to the covariance of the random variables $\tilde{r}_k$ and $\tilde{r}_l$ for $k, l = 0, 1, \dots, N-1$,

$$\begin{aligned}
|E_z(\tilde{r}_k - E_z \tilde{r}_k)(\tilde{r}_l - E_z \tilde{r}_l)| &\leq E_z |(\tilde{r}_k - E_z \tilde{r}_k)(\tilde{r}_l - E_z \tilde{r}_l)| \leq \sqrt{E_z |\tilde{r}_k - E_z \tilde{r}_k|^2} \sqrt{E_z |\tilde{r}_l - E_z \tilde{r}_l|^2} \\
&= \frac{\sigma_z^2}{N^2} \sum_{t=0}^{N-1} |o_t|^2.
\end{aligned} \quad (9)$$

Let us remark that for bounded signals satisfying $|o_t| \leq B$, $t = 0, 1, \dots, N-1$, where $B > 0$ is a real number, we immediately obtain from (8) and (9),

$$E_z |\tilde{r}_k - E_z \tilde{r}_k|^2 \leq \frac{\sigma_z^2}{N^2} \sum_{k=0}^{N-1} B^2 = \frac{\sigma_z^2 B^2}{N}, \quad |E_z(\tilde{r}_k - E_z \tilde{r}_k)(\tilde{r}_l - E_z \tilde{r}_l)| \leq \frac{\sigma_z^2 B^2}{N}. \quad (10)$$

In such a case the random variables $\tilde{r}_k$, $k = 0, 1, \dots, N-1$, representing the spectrum of the modulated signal, have equal variances decreasing asymptotically to zero as $N \rightarrow \infty$ (which implies that their covariances also decrease asymptotically to zero).

Bounded signals are of course of primary interest in this work since we intend to investigate spectra of regular oscillations of stationary character, modulated by random amplitude and phase fluctuations.

In view of the equalities in (3), (4), (5) and lemma 1, the equalities (7), (8) and inequality (9) hold in the case of phase modulation series $z_t = \exp(i\varphi_t)$ with $m_z = m_\varphi$ and $\sigma_z^2 = 1 - m_\varphi^2$, as well as in the case of amplitude and phase modulation series $z_t = (1 + a_t) \exp(i\varphi_t)$ with

$m_z = m_\varphi$ and $\sigma_z^2 = 1 + \sigma_a^2 - m_\varphi^2$. Thus, taking into account the equality (7) and the inequalities in (10), we see that if a bounded signal is phase or amplitude and phase modulated according to our model, then its DFT spectrum will have random character. The modulated signal DFT spectrum has mean values $E_z \tilde{r}_k = \tilde{o}_k m_z$, $k = 0, 1, \dots, N-1$, with random deviations $\tilde{r}_k - E_z \tilde{r}_k$ imposed, i.e. random phase or amplitude and phase modulation diminishes the amplitudes of the original bounded signal spectrum by the factor $0 \leq m_\varphi < 1$ and corrupts the spectrum by additive deviations of random character, which have zero mean and known variance. In consequence, any peaks present in the spectrum of the original bounded signal may be less distinguishable in the modulated signal spectrum. Distortion of the bounded signal spectrum in the case of random phase modulation is small for Φ values close to 0 (i.e. if phase fluctuations are small), since then $m_z \approx 1$ and $E_z |\tilde{r}_k - E_z \tilde{r}_k|^2 \approx 0$. In the other extreme case of $\Phi = \pi$ we have $m_z = 0$, and then the modulated signal spectrum will have purely stochastic character without any frequencies distinguished, so that any peaks present in the original signal spectrum will be completely blurred.

In accordance with our model (2), if an original signal is only amplitude modulated, then $r_t = o_t + r_{at}$, where $r_{at} = a_t o_t$, $t = 0, 1, \dots, N-1$, so we have $\tilde{r}_k = \tilde{o}_k + \tilde{r}_{ak}$, $k = 0, 1, \dots, N-1$. Hence, the spectrum of the amplitude modulated signal differs from the original signal spectrum only by the additive distortion terms $\tilde{r}_{ak}$, $k = 0, 1, \dots, N-1$. Having in view the assumptions in (4) and putting $z_t = a_t$ with $\sigma_z^2 = \sigma_a^2$ we can apply again lemma 1, which further yields $E_z \tilde{r}_k = \tilde{o}_k$, as well as equality (8) and inequality (9) for random variables $\tilde{r}_{ak} = \tilde{r}_k - E_z \tilde{r}_k$, $k = 0, 1, \dots, N-1$. Consequently, if the original signal is bounded, then by (10) the distortion terms will have equal variances tending to zero as $N \rightarrow \infty$, so that they will also have asymptotically zero covariances. This means that they will not distort completely the original bounded signal spectrum on which they are superimposed. Some small peaks present in the original signal spectrum can be smoothed due to amplitude modulation but possibly larger ones will be still distinguishable.

In Figure 1. the above described effects of amplitude and/or phase signal modulation on the DFT spectrum of a model signal (sum of 7 monochromatic complex harmonics with constant amplitudes and phases) is shown. The random variables representing amplitude and phase fluctuations were distributed according to uniform distribution $a_t \sim U(-A, A)$ and $\varphi_t \sim U(-\Phi, \Phi)$ with parameters $A = 1.0$ and $\Phi = \pi/2$, respectively.

Assume now that the time series of the original signal values o_t , $t = 0, 1, \dots, N-1$, is corrupted by random errors η_t , according to the model

$$y_t = o_t + \eta_t, \quad t = 0, 1, \dots, N-1, \quad (11)$$

where η_t are realizations of uncorrelated complex-valued random variables having zero mean $E_\eta \eta_t = 0$ and finite second moment $E_\eta |\eta_t|^2 = \sigma_\eta^2 < \infty$.

Let us see what happens if the corrupted signal values are submitted to random modulation $r_{yt} = z_t y_t$, $t = 0, 1, \dots, N-1$, of the same kind as considered above. For a modulating series z_t , $t = 0, 1, \dots, N-1$, satisfying the assumptions of lemma 1, we obtain on the basis of the DFT convolution formula (Gasquet and Witomski 1999)

$$\tilde{r}_{yk} = \sum_{j=0}^k \tilde{y}_{k-j} \tilde{z}_j + \sum_{j=k+1}^{N-1} \tilde{y}_{N-j+k} \tilde{z}_j \quad (12)$$

for $k = 0, 1, \dots, N-1$, which by assertion of the lemma yields

$$E_z \tilde{r}_{yk} = \tilde{y}_k m_z ,$$

(13)

$$E_\eta E_z \tilde{r}_{yk} = E_\eta \tilde{y}_k m_z = E_\eta (\tilde{o}_k + \tilde{\eta}_k) m_z = \tilde{o}_k m_z .$$

Fig. 1. Spectra of a model series representing sum of 7 monochromatic oscillations ($N=2048$): original signal (black), phase modulated signal ($\Phi = \pi/2$, blue), amplitude modulated signal ($A = 1.0$, green), amplitude and phase modulated signal ($\Phi = \pi/2$, $A = 1.0$, brown).

Furthermore, simple calculation shows that

$$E_z |\tilde{r}_{yk} - E_\eta E_z \tilde{r}_{yk}|^2 = E_z |\tilde{r}_{yk} - E_z \tilde{r}_{yk} + E_z \tilde{r}_{yk} - E_\eta E_z \tilde{r}_{yk}|^2 = E_z |\tilde{r}_{yk} - E_z \tilde{r}_{yk}|^2 + |E_z \tilde{r}_{yk} - E_\eta E_z \tilde{r}_{yk}|^2$$

and analogously as in (8) we obtain

$$E_z |\tilde{r}_{yk} - E_\eta E_z \tilde{r}_{yk}|^2 = \frac{\sigma_z^2}{N^2} \sum_{t=0}^{N-1} |y_t|^2 + |m_z (\tilde{y}_k - \tilde{o}_k)|^2,$$

which by (11) together with the equalities $E_\eta \eta_t = 0$, $E_\eta \eta_t \bar{\eta}_s = \delta_{st}$, $s, t = 0, 1, \dots, N-1$, and lemma 1 finally gives

$$\begin{aligned} E_\eta E_z |\tilde{r}_{yk} - E_\eta E_z \tilde{r}_{yk}|^2 &= \frac{\sigma_z^2}{N^2} \sum_{t=0}^{N-1} E_\eta |y_t|^2 + E_\eta |m_z \tilde{\eta}_k|^2 = \frac{\sigma_z^2}{N^2} \sum_{t=0}^{N-1} [o_t^2 + E_\eta |\eta_t|^2] + \frac{|m_z|^2 \sigma_\eta^2}{N} \\ &= \frac{\sigma_z^2}{N^2} \sum_{t=0}^{N-1} |o_t|^2 + \frac{(\sigma_z^2 + |m_z|^2) \sigma_\eta^2}{N} \end{aligned} \quad (14)$$

for $k = 0, 1, \dots, N-1$, as well as

$$\begin{aligned} |E_\eta E_z (\tilde{r}_{yk} - E_\eta E_z \tilde{r}_{yk}) (\bar{\tilde{r}}_{yl} - E_\eta E_z \bar{\tilde{r}}_{yl})| &\leq E_\eta E_z |(\tilde{r}_{yk} - E_\eta E_z \tilde{r}_{yk}) (\bar{\tilde{r}}_{yl} - E_\eta E_z \bar{\tilde{r}}_{yl})| \\ &\leq \sqrt{E_\eta E_z |\tilde{r}_{yk} - E_\eta E_z \tilde{r}_{yk}|^2} \sqrt{E_\eta E_z |\bar{\tilde{r}}_{yl} - E_\eta E_z \bar{\tilde{r}}_{yl}|^2} \leq \frac{\sigma_z^2}{N^2} \sum_{t=0}^{N-1} |o_t|^2 + \frac{(\sigma_z^2 + |m_z|^2) \sigma_\eta^2}{N} \end{aligned} \quad (15)$$

for $k, l = 0, 1, \dots, N-1$.

Hence, we see that the presence of errors corrupting the original signal values does not change the character of the DFT spectrum of the modulated series $r_{yt} = z_t y_t$, $t = 0, 1, \dots, N-1$. Indeed, the formulae (13), (14), (15) are analogous to (7), (8), (9), respectively, except for addition of the term $\sigma_\eta^2(\sigma_z^2 + |m_z|^2)/N = \sigma_\eta^2 E_z |z_t|^2 / N$ which now occurs because of non-zero second moment of the errors.

For bounded signals $|o_t| \leq B$, $t = 0, 1, \dots, N-1$, we can easily obtain the inequalities analogous to (10), namely,

$$E_\eta E_z |\tilde{r}_{yk} - E_\eta E_z \tilde{r}_{yk}|^2 \leq \frac{\sigma_z^2 B^2 + (\sigma_z^2 + |m_z|^2) \sigma_\eta^2}{N}, \quad (16)$$

$$|E_\eta E_z (\tilde{r}_{yk} - E_\eta E_z \tilde{r}_{yk})(\bar{\tilde{r}}_{yl} - E_\eta E_z \bar{\tilde{r}}_{yl})| \leq \frac{\sigma_z^2 B^2 + (\sigma_z^2 + |m_z|^2) \sigma_\eta^2}{N}$$

for $k, l = 0, 1, \dots, N-1$.

Thus, our earlier assertions concerning the character of the modulated signal spectrum hold also in the case of a bounded signal corrupted by uncorrelated random errors which have zero mean and finite second moment. This means that only in the case of amplitude modulation the DFT spectrum of the modulated corrupted signal can resemble the one of the corrupted signal itself.

4. SUMMARY

The properties of the DFT spectrum examined in this work are helpful for understanding the possible changes such a spectrum undergoes in the case of random amplitude and phase modulation of an original signal. Our modulation model includes additive distortions of stochastic nature in the amplitudes and phases of original signal values at observation moments. For bounded signals of deterministic character (like a sum of monochromatic oscillations with constant amplitudes and phases) it is proved that occurrence of random phase modulation of the signal can completely change the character of its DFT spectrum. Namely, the phase modulated signal spectrum can show purely stochastic character without any frequencies distinguished, like the spectrum of white noise. In the case of random amplitude modulation of such a signal it is shown that the modulated signal spectrum can still resemble the spectrum of the original signal, although small peaks can be significantly smoothed. Similar conclusions can be deduced also in the case of a deterministic signal which is corrupted at the moments of observation by uncorrelated random errors with zero mean and finite second moment.

It is worth remarking that our conclusions agree with observations of Ni and Huo (2007), concerning the importance of phase and amplitude information in signal and image reconstruction. The concept of phase randomization used for obtaining multivariate surrogate time series (Mammen and Nandi 2008) with distribution similar to observed series is also related to the subject considered here.

Hinich (2003) used similar approach to amplitude modulation modeling, as the one applied in this work, to derive statistics for detecting randomly modulated pulses in noise. The Singular Spectrum Analysis (SSA) method has an important property, first noted by Vautard and Ghil (1989), that it may be used directly to identify modulated oscillations in the presence of noise. Allen and Robertson (1996) proposed a generalization of the ‘‘Monte Carlo SSA’’ algorithm which allows for objective testing for the presence of modulated oscillations at low

signal-to-noise ratios in multivariate data. They demonstrated the application of the test to the analysis of interannual variability in tropical Pacific sea-surface temperatures.

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