TOWARDS VALIDATION OF SATELLITE GRADIOMETRIC DATA USING MODIFIED VERSION OF 2nd ORDER PARTIAL DERIVATIVES OF EXTENDED STOKES' FORMULA

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ABSTRACT. The satellite gradiometric data should be validated prior to being used. One way of such a validation process is to use some integral estimators which are the second-order partial derivatives of the extended Stokes formula to regenerate the data from the gravity anomaly at the topographic surface. In this paper, we present how least-squares modification methods are used to modify such integral estimators. Our concentration will be on validation of the vertical-horizontal and horizontal-horizontal elements of the gravitational tensor at satellite level. The paper will formulate the elements of the system of equations from which the modification parameters are derived based on all types of least-squares modification. The truncation and Paul's coefficients will also be modelled.

Keywords: Combination of gradients, gradiometry, gravity gradient, isotropic kernel, integral gradient estimator, Paul's coefficients, signal and error spectra, truncation coefficients, validation

1 INTRODUCTION

Satellite gradiometry is a technique to measure second-order derivatives of the Earth's gravity field from space. It is expected to determine the geopotential coefficients to higher degrees and orders than those are obtained from other satellite techniques. The gravity field and steady-state ocean circulation explorer (GOCE) (ESA 1999, Albertella et al. 2002, Balmino et al. 1998 and 2001) is the first satellite mission which uses the satellite gradiometry technique. It is orbiting the Earth at an altitude of around 250 km above sea level. It is expected to determine a geopotential model to degree and order 300 using the GOCE gradiometric data. Quality of the GOCE data is important in gravity field modelling and geophysical studies. Any systematic error in the data defects the quality of the recovered gravity field. Therefore, the quality of GOCE data should be controlled prior to processing, or in other words, the data should be validated.

Different methods of validating satellite gradiometric data have been proposed. A simple way could be the direct comparison of the real satellite gradiometric data with the synthesized gravity gradients using an existing Earth's gravity model. Another idea is to use regional gravity data to generate the gradients at satellite level. Haagmans et al. (2002) and Kern and Haagmans (2004) used the extended Stokes formula and extended Hotine formula to generate the gravity gradients using terrestrial gravity data. Denker (2002) used the least-squares spectral combination technique to generate and validate the gradients. Bouman et al. (2003) has set up a calibration model based on instrument (gradiometer) characteristics to validate the measurements. Mueller et al. (2004) used the terrestrial gravity anomalies to generate the gravity gradients, and after that Wolf (2007) investigated the deterministic approaches to modify the integrals and validate the satellite gradiometric data. In fact, the spectral weighting scheme (Sjöberg 1980 and 1981 and Wenzel 1981) was used by Wolf (2007). Stochastic methods of modifying Stokes' formula, or in other words least-squares modification can be used for the extended Stokes formula as well; see Sjöberg (1984a), (1984b), (1991) and (2003). Least-squares collocation can be used for validation purposes. Tscherning et al. (2006) considered this method and concluded that the gradients can be predicted with an error of 2-3 mE in the case of an optimal size of the collection area and optimal resolution of data. Zielinski and Petrovskaya (2003) proposed a balloon-borne gradiometer to fly at 20-40 km altitude simultaneously with satellite mission and proposed downward continuation of satellite data and comparing them with balloon-borne data. Bouman and Koop (2003) presented an along-track interpolation method to detect the outliers. Their idea is to compare the along-track interpolated gradients with measured gradients. If the interpolation error is small enough, the differences should be predicted reasonably by an error model. Pail (2003) proposed a combined adjustment method supporting high quality gravity field information within the well-surveyed test area for continuation of local gravity field upward and validating the satellite gradiometric data. Bouman et al. (2004) stated that there were some limitations in generating the gravity gradients using terrestrial gravimetry data and the Earth gravity models. When such a model is used, high degrees and orders of geopotential coefficients should be taken into account and the recent gravity models seem to be able to remove the greater part of the systematic errors. In their regional approach they concluded that the bias of the gradients can accurately be recovered using least-squares collocation. Also, they concluded that the method of validation using high-low satellite-to-satellite tracking data fails unless a higher resolution the Earth gravity model is available. Kern and Haagmans (2004) and Kern et al. (2005) presented an algorithm for detecting the outliers in the satellite gradiometric data in the time domain.

Kern and Haagmans (2004), Mueller et al. (2004) and Wolf (2007) used the second-order partial derivatives of extended Stokes formulas for generating the gravity gradients at satellite level. Kern and Haagmans (2004) did not consider any modification for these formulas. However, Mueller et al. (2004) and Wolf (2007) used a deterministic approach of modification to modify these integral formulas. The stochastic modification used by Eshagh (2010a) to modify the second-order radial derivative of the extended Stokes formula. He proposed two methods of a) modification prior to derivative and b) derivative prior to modification. The former

method is similar to the work done by Mueller et al. (2004) and Wolf (2007) but not in deterministic way of modification. Eshagh (2010b) also modified the second-order radial derivative of the Abel-Poisson formula in a least-squares sense to generate the second-order radial gradient at satellite level using an existing geoid model. The least-squares modification of the vertical-horizontal and horizontal-horizontal derivatives of the extended Stokes formula is a new subject in the scope of satellite gradiometry which is theoretically presented in this paper.

Modifications of integrals have frequently been discussed in geodetic literature. The main goal – reduction of truncation errors – was reached by re-weighting the contribution of data based on their distance from the computation point. These methods were often used in connection with the residual gravity field modelling since local data cannot supply low frequencies of the recovered function. This paper presents the way of using least-squares modification in validation of satellite gradiometric data. However, there are some discrepancies between this theoretical study and the real case of satellite gradiometry. First, the gradients are generated on a regular grid at satellite level, whereas we already know that the gradiometric data are measured on the orbit. Second, we use the local north-oriented reference frame, while in reality the gradients are measured in the gradiometer reference frame. Having these shortcomings in the mind, we start the discussion with the extended Stokes formula.

The disturbing potential can be expressed by an integral which is well-known as the extended Stokes formula. This integral formula is (Heiskanen and Moritz 1967, p. 319, Eq. 8-88):

$$T(P) = \frac{R}{4\pi} \iint_{\sigma} S(r_p, \psi) \Delta g(Q) d\sigma, \quad (1a)$$

where R is the radius of the reference sphere, r_p is the geocentric distance at computation point P , ψ is the geocentric angle between the computation point P and the integration point Q with the following expression:

$$\cos \psi = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\lambda' - \lambda) \quad (1b)$$

and θ and λ are the co-latitude and longitude of P and θ' and λ' are of the integration point Q . σ is the unit sphere, $\Delta g(Q)$ is the gravity anomaly at a sphere of radius R approximating sea level and

$$S(r_p, \psi) = \sum_{n=2}^{\infty} \frac{2n+1}{2} \Omega_n(r_p) P_n(\cos \psi), \quad (1c)$$

is the spectral form of the extended Stokes function with the spectrum:

$$\Omega_n(r_p) = \frac{2}{n-1} \left(\frac{R}{r_p} \right)^{n+1} \quad (1d)$$

and $P_n(\cos\psi)$ stands for the Legendre polynomial.

Equation (1a) shows that the integration should be performed globally, which means that $\Delta g(Q)$ with a global coverage is required. Therefore we should look for an approach to modify the integral in such a way that the contribution of the far zone data is minimized. Different methods for modifying Stokes' formula have been presented, but here our concentration is on the stochastic approaches of Sjöberg (1984a) and (1984b). In fact, the theory behind this part of the study was presented by him, but just on Stokes' integral for geoid determination. However, we are going to show the feasibility of using these stochastic approaches in modifying the integral estimators and validating vertical-horizontal and horizontal-horizontal gradients. The advantage of the stochastic modifications is to minimize the errors of terrestrial, geopotential coefficients and the truncation of the integral formula simultaneously. In fact, it considers the quality of the existing data in its modification process as well.

2 SPECTRAL FORMS OF VERTICAL-HORIZONTAL GRADIENT ESTIMATORS

Here we assume the local north oriented reference frame as the frame in which the gradients are expressed. Such a frame is defined as the one whose z -axis is pointing upwards in geocentric radial direction, x -axis towards the north and the frame is right-handed, implying that y -axis is directed to the west. The spherical harmonic expressions of the satellite gradiometric data in this frame are shown e.g. in Petrovskaya and Vershkov (2006), Eshagh (2009a) and (2009b). We name the gradients $T_{xz}(P)$ and $T_{yz}(P)$ vertical-horizontal gradients. These gradients can be expressed by the vertical-horizontal derivatives of the extended Stokes integral formula:

$$\begin{bmatrix} T_{xz}(P) \\ T_{yz}(P) \end{bmatrix} = \frac{R}{4\pi} \iint_{\sigma} V(r_p, \psi) \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \Delta g(Q) d\sigma, \quad (2a)$$

where

$$V(r_p, \psi) = \frac{1}{r_p^2} S_{\psi}(r_p, \psi) - \frac{1}{r_p} S_{r_p \psi}(r_p, \psi), \quad (2b)$$

is the isotropic part of the integral kernel and α is the azimuth between the points P and Q . Here we have (Heiskanen and Moritz 1967, Eq. 6-64b, p. 235):

$$\frac{S_{\psi}(r_p, \psi)}{r_p^2} = \frac{-t^2 \sin \psi}{r_p^2} \left(\frac{2}{D^3} + \frac{6}{D} - 8 - 3 \frac{1-t \cos \psi - D}{D \sin^2 \psi} - 3 \ln \frac{1-t \cos \psi + D}{2} \right), \quad (2c)$$

and (Reed 1973, Eq. 5.36, p. 72):

$$\frac{S_{r_p\psi}(r_p, \psi)}{r_p} = \frac{t^3 \sin \psi}{Rr_p} \left(\frac{3(1-t^2)}{D^5} + \frac{4}{D^3} + \frac{6}{D} - 13 - 6 \ln \frac{1-t \cos \psi + D}{2} + 3 \frac{t \cos \psi (D+1)}{D(1-t \cos \psi + D)} \right), \quad (2d)$$

where $t = R/r_p$ and $D = \sqrt{1-2t \cos \psi + t^2}$. The spectral form of Eq. (2b) is:

$$V(r_p, \psi) = - \sum_{n=2}^{\infty} \frac{2n+1}{2} \Omega_n^{\text{vH}}(r_p) P_{n1}(\cos \psi), \quad (2e)$$

where $P_{n1}(\cos \psi)$ is the associated Legendre function of degree n and order 1 and

$$\Omega_n^{\text{vH}}(r_p) = \frac{\Omega_n(r_p)}{r_p^2} - \frac{\Omega'_n(r_p)}{r_p} \quad (2f)$$

and

$$\Omega'_n(r_p) = - \frac{2}{r_p} \frac{n+1}{n-1} \left(\frac{R}{r_p} \right)^{n+1}. \quad (2g)$$

2.1 TRUNCATION ERROR FORMULAS

The truncation error formulation for vertical-horizontal gradients is done by defining the following isotropic kernel for the integral formula:

$$\bar{V}(r_p, \psi) = \begin{cases} 0 & 0 \leq \psi < \psi_0 \\ \frac{1}{2} V(r_p, \psi) & \psi_0 \leq \psi \leq \pi \end{cases}, \quad (3a)$$

where ψ_0 is the truncation radius. The kernel has the following spectral form for this kernel (see Hagiwara 1972):

$$\bar{V}(r_p, \psi) = \sum_{n=2}^{\infty} \frac{2n+1}{2n(n+1)} Q_n^V(r_p, \psi_0) P_{n1}(\cos \psi), \quad (3b)$$

The truncation error coefficients are derived by:

$$Q_n^V(r_p, \psi_0) = \frac{1}{2} \int_{\psi_0}^{\pi} V(r_p, \psi) P_{n1}(\cos \psi) \sin \psi d\psi. \quad (3c)$$

We can write the truncation error of the vertical-horizontal gradients by the following integral formula:

$$\begin{bmatrix} \delta T_{xz}^1(P) \\ \delta T_{yz}^1(P) \end{bmatrix} = \frac{R}{4\pi} \iint_{\sigma} \bar{V}(r_P, \psi) \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \Delta g(Q) d\sigma. \quad (4)$$

Hagiwara (1972) presented spectral forms for the integral estimators of deflection of vertical. Here, we take advantage his results to find a spectral form for the truncation error of the vertical-horizontal gradients. It is presented in the following theorem.

Theorem 1 (Hagiwara 1972):

$$\iint_{\sigma} \Delta g(Q) P_{n1}(\cos \psi) \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} d\sigma = \frac{4\pi}{2n+1} \begin{pmatrix} \frac{\partial \Delta g_n(P)}{\partial \theta} \\ \frac{\partial \Delta g_n(P)}{\sin \theta \partial \lambda} \end{pmatrix}.$$

Now, by substituting Eq. (3b) into Eq(4) we have:

$$\begin{bmatrix} \delta T_{xz}^1(P) \\ \delta T_{yz}^1(P) \end{bmatrix} = \frac{R}{4\pi} \sum_{n=2}^{\infty} \frac{2n+1}{2n(n+1)} Q_n^V(r_P, \psi_0) \iint_{\sigma} \Delta g(Q) P_{n1}(\cos \psi) \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} d\sigma. \quad (5a)$$

According to Theorem 1 we can simplify the integral part of Eq. (5a) to:

$$\begin{bmatrix} \delta T_{xz}^1(P) \\ \delta T_{yz}^1(P) \end{bmatrix} = \frac{R}{2} \sum_{n=2}^{\infty} Q_n^{VH}(r_P, \psi_0) \begin{pmatrix} \frac{\partial \Delta g_n(P)}{\partial \theta} \\ \frac{\partial \Delta g_n(P)}{\sin \theta \partial \lambda} \end{pmatrix}, \quad (5b)$$

where

$$Q_n^{VH}(r_P, \psi_0) = \frac{Q_n^V(r_P, \psi_0)}{n(n+1)}. \quad (5c)$$

Equation (5c) is the spectral form of the truncation error of the vertical-horizontal derivative of the extended Stokes formula.

2.2 TERRESTRIAL DATA ERROR FORMULA

The error of the terrestrial data can be expressed by the following integral:

$$\begin{bmatrix} \delta T_{xz}^0(P) \\ \delta T_{yz}^0(P) \end{bmatrix} = \frac{R}{4\pi} \iint_{\sigma} [V(r_P, \psi) - \bar{V}(r_P, \psi)] \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \varepsilon(Q) d\sigma. \quad (6a)$$

where ε stands for error of the terrestrial data. Inserting the spectral forms of $V(r_p, \psi)$ and $\bar{V}(r_p, \psi)$ into Eq. (6a) we have

$$\begin{bmatrix} \delta T_{xz}^0(P) \\ \delta T_{yz}^0(P) \end{bmatrix} = \frac{R}{4\pi} \sum_{n=2}^{\infty} \frac{2n+1}{2} [\Omega_n^{\text{VH}}(r_p) - Q_n^{\text{VH}}(r_p, \psi_0)] \iint_{\sigma} \varepsilon(Q) P_{n1}(\cos \psi) \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} d\sigma \quad (6b)$$

According to Theorem 1 we can write

$$\begin{bmatrix} \delta T_{xz}(P) \\ \delta T_{yz}(P) \end{bmatrix} = \frac{R}{2} \sum_{n=2}^{\infty} [\Omega_n^{\text{VH}}(r_p) - Q_n^{\text{VH}}(r_p, \psi_0)] \begin{pmatrix} \frac{\partial \varepsilon_n^T(P)}{\partial \theta} \\ \frac{\partial \varepsilon_n^T(P)}{\sin \theta \partial \lambda} \end{pmatrix}. \quad (6c)$$

Equation (6c) shows that the spectral form of the terrestrial error is derived from the spectrum of the kernel function minus the truncation error coefficients. We will take this advantage in the least-squares modification of vertical-horizontal gradient estimators.

3 LEAST-SQUARES MODIFICATION OF VERTICAL-HORIZONTAL GRADIENT ESTIMATORS

Let the following general integral estimators for the vertical-horizontal gradients:

$$\begin{bmatrix} \tilde{T}_{xz}(P) \\ \tilde{T}_{yz}(P) \end{bmatrix} = \frac{R}{4\pi} \iint_{\sigma_0} V^L(r_p, \psi) \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \Delta g(Q) d\sigma + \frac{R}{2} \sum_{n=2}^L b_n^{\text{VH}}(r_p) \begin{bmatrix} \frac{\partial \Delta g_n(P)}{\partial \theta} \\ \frac{\partial \Delta g_n(P)}{\sin \theta \partial \lambda} \end{bmatrix} \quad (7a)$$

where

$$V^L(r_p, \psi) = V(r_p, \psi) + \sum_{n=1}^L \frac{2n+1}{2} v_n(r_p) P_{n1}(\cos \psi). \quad (7b)$$

We are going to find the modification parameters $v_n(r_p)$ in a least-squares sense. As we can see in Eq. (2a), the integral kernel is not isotropic because it depends on the azimuth α between the computation point P and the integration point Q . This problem is very similar with the modification of Vening-Meinesz's integral formula. The truncation error of the formula was given by Hagiwara (1972). The formula was also modified by the least-squares approach for the first time by Sjöberg (1984a) considering the error of the spherical harmonic coefficients and truncation error. Later on Sjöberg (1984b) developed the formulation to consider the error of terrestrial data as well. Neyman et al. (1996) worked further on the modification of this formula, but they used the Molodensky modification (Molodensky et al. 1962) method in their study. Tziavos and Andritsanos (1998) also used this method and Hwang (1998) presented inverse formulas for Vening-Meinesz's integral to recover

gravity anomaly from satellite altimetry data. However, in this section we use the strategy of Sjöberg (1984b) to modify the integral formulas of the vertical-horizontal gradients. Let us present the general form of the system of equations from which the modification parameters are derived:

$$\sum_{r=2}^L a_{kr} s_r(r_p) = h_k(r_p), \quad (8)$$

where the coefficients a_{kr} and $h_k(r_p)$ are derived based on the method of least-squares modification and $s_r(r_p)$ is the representation of the modification parameters.

Proposition 1 *The biased type least-squares modification parameters are derived by setting $b_n^{\text{VH}}(r_p) = v_n(r_p)$ for the vertical-horizontal gradient estimator by solving Eq. (8) with following elements:*

$$\begin{aligned} s_n(r_p) &= v_n(r_p) \\ a_{rk} &= a_{kr} = k(k+1)(\sigma_k^2 + dc_k)\delta_{rk} - r(r+1)E_{rk}^1(\psi_0)\sigma_k^2 - \\ &\quad - k(k+1)E_{kr}^1(\psi_0)\sigma_r^2 + \sum_{n=2}^{\infty} E_{nr}^1(\psi_0)E_{nk}^1(\psi_0)n(n+1)(\sigma_n^2 + c_n), \\ h_k(r_p) &= k(k+1)\{\Omega_k^{\text{VH}}(r_p) - Q_k^{\text{VH}}(r_p, \psi_0)\}\sigma_k^2 - \sum_{n=2}^{\infty} n(n+1)E_{nk}^1(\psi_0) \times \\ &\quad \times \{Q_n^{\text{VH}}(r_p, \psi_0)(\sigma_n^2 + c_n) - \Omega_n^{\text{VH}}(r_p)\sigma_n^2\}. \end{aligned}$$

where

$$E_{nk}^1(\psi_0) = \frac{2k+1}{2} e_{nk}^1(\psi_0), \quad (9a)$$

$$e_{nk}^1(\psi_0) = - \int_{\psi_0}^{\pi} P_{n1}(\cos\psi) P_{k1}(\cos\psi) \sin\psi d\psi, \quad (9b)$$

which we call $e_{nk}^1(\psi_0)$ the first-order Paul coefficients in this study.

σ_k^2 is the error spectrum of the terrestrial gravity anomalies; see Ågren (2004), Ellmann (2004 and 2005) for the details of its computations. dc_k is the error spectrum of the Earth's gravity model which is used. c_n is the signal degree variance of gravity anomaly. According to Ågren (2004) and Ellmann (2004 and 2005), the Tscherning-Rapp (1974) model of gravity anomaly degree variance is the best among other existing models. Eshagh (2009c) showed some improvement in the least-squares modification of Stokes formula using the recent Earth's gravity model EGM08 (Pavlis and Holmes 2006) for generating c_n . However, it was revealed that

the improvement is due to small error spectra of the model and not to the generated c_n from EGM08.

Proof. We suppose that the global mean square error of the estimator, Eq. (7a), contains three terms:

$$\|\delta T_{\text{VH}}\|^2 = \|\delta T_{\text{VH}}^0\|^2 + \|\delta T_{\text{VH}}^1\|^2 + \|\delta T_{\text{VH}}^2\|^2, \quad (9c)$$

where $\|\delta T_{\text{VH}}^0\|^2$ is the term related with terrestrial data error, $\|\delta T_{\text{VH}}^1\|^2$ is related to the truncation error and $\|\delta T_{\text{VH}}^2\|^2$ to the error of the geopotential coefficients. Now, we attempt to obtain these terms one by one in the following.

Consider the spectral form of the modified Stokes formulas which is restricted to the cap size in which we have data. Considering the relations between the gradients in geocentric and local frame and after some manipulation we obtain:

$$\begin{bmatrix} \tilde{T}_{xz}(P) \\ \tilde{T}_{yz}(P) \end{bmatrix} = \frac{R}{2} \sum_{n=2}^{\infty} \left\{ \Omega_n^{\text{VH}}(r_P) + v_n^*(r_P) - [Q_n^{\text{VH}}(r_P, \psi_0)]^L \right\} \begin{bmatrix} \frac{\partial \varepsilon_n(P)}{\partial \theta} \\ \frac{\partial \varepsilon_n(P)}{\sin \theta \partial \lambda} \end{bmatrix}, \quad (10a)$$

where $\varepsilon_n(P)$ is the error spectrum of gravity anomaly and

$$v_n^*(r_P) = \begin{cases} v_n(r_P) & n \leq L \\ 0 & n > 0 \end{cases}. \quad (10b)$$

Squaring Eq. (10a) reads:

$$\begin{aligned} \begin{bmatrix} \delta \tilde{T}_{xz}^2(P) \\ \delta \tilde{T}_{yz}^2(P) \end{bmatrix} &= \frac{R^2}{4} \sum_{n=2}^{\infty} \sum_{n'=2}^{\infty} \left\{ v_{n'}^*(r_P) - \Omega_{n'}^{\text{VH}}(r_P) - [Q_{n'}^{\text{VH}}(r_P, \psi_0)]^L \right\} \times \\ &\times \left\{ v_n^*(r_P) - \Omega_n^{\text{VH}}(r_P) - [Q_n^{\text{VH}}(r_P, \psi_0)]^L \right\} \begin{bmatrix} \frac{\partial \varepsilon_n(P)}{\partial \theta} \frac{\partial \varepsilon_{n'}(P)}{\partial \theta} \\ \frac{\partial \varepsilon_n(P)}{\sin \theta \partial \lambda} \frac{\partial \varepsilon_{n'}(P)}{\sin \theta \partial \lambda} \end{bmatrix}, \quad (10c) \end{aligned}$$

where

$$[Q_n^{\text{VH}}(r_P, \psi_0)]^L = Q_n^{\text{VH}}(r_P, \psi_0) - \sum_{k=2}^L v_k(r_P) E_{nk}^1(\psi_0). \quad (10d)$$

According to Heiskanen and Mortiz (1967, Eq. 7-37, p. 262) we can write

$$\begin{aligned}\|\delta T_{\text{VH}}^0\|^2 &= \frac{R^2}{4} \sum_{n=2}^{\infty} \left\{ v_n^*(r_P) - \Omega_n^{\text{VH}}(r_P) - [Q_n^{\text{VH}}(r_P, \psi_0)]^L \right\}^2 \left[\left(\frac{\partial \varepsilon_n(P)}{\partial \theta} \right)^2 + \left(\frac{\partial \varepsilon_n(P)}{\sin \theta \partial \lambda} \right)^2 \right] \\ &= \frac{R^2}{4} \sum_{n=2}^{\infty} \left\{ v_n^*(r_P) - \Omega_n^{\text{VH}}(r_P) - [Q_n^{\text{VH}}(r_P, \psi_0)]^L \right\}^2 n(n+1) \sigma_n^2.\end{aligned}\quad (10e)$$

In a very similar way can obtain:

$$\|\delta T_{\text{VH}}^1\|^2 = \frac{R^2}{4} \sum_{n=2}^{\infty} \left\{ [Q_n^{\text{VH}}(r_P, \psi_0)]^L \right\}^2 n(n+1) c_n, \quad (10f)$$

and

$$\|\delta T_{\text{VH}}^2\|^2 = \frac{R^2}{4} \sum_{n=2}^L v_n^2(r_P) n(n+1) dc_n. \quad (10g)$$

Substituting Eqs. (10e)-(10g) into Eq. (9c) and considering

$$\frac{\partial \|\delta T_{\text{VH}}\|^2}{\partial v_k(r_P)} = 0, \quad (10h)$$

and after relatively long derivations the proposition is proved.

Proposition 2 *The unbiased type least-squares modification parameters are derived by selecting $b_n^{\text{VH}}(r_P) = [Q_n^L(r_P, \psi)]^{\text{VH}} + v_n^{\text{VH}}(r_P)$ for the vertical-horizontal gradient estimator and solving Eq. (8) with following elements:*

$$\begin{aligned}s_n(r_P) &= v_n(r_P) \\ a_{rk} &= a_{kr} = k(k+1)d_k \delta_{rk} - r(r+1)E_{rk}^1(\psi_0)d_r - \\ &\quad - k(k+1)E_{kr}^1(\psi_0)d_k + \sum_{n=2}^{\infty} E_{nr}^1(\psi_0)E_{nk}^1(\psi_0)n(n+1)d_n, \\ h_k(r_P) &= k(k+1) \left\{ \Omega_k^{\text{VH}}(r_P) \sigma_k^2 - [Q_k(r_P, \psi_0)]^{\text{VH}} d_k \right\} + \sum_{n=2}^{\infty} n(n+1)E_{nk}^1(\psi_0) \times \\ &\quad \times \left\{ [Q_n(r_P, \psi_0)]^{\text{VH}} d_n - \Omega_n^{\text{VH}}(r_P) \sigma_n^2 \right\},\end{aligned}$$

where

$$d_k = \sigma_k^2 + dc_n^*$$

and

$$dc_n^* = \begin{cases} dc_n & 2 \leq n \leq L \\ c_n & n > L \end{cases}.$$

Proof. Sjöberg (1991) showed that if $b_n^{\text{VH}}(r_p) = [Q_n^{\text{VH}}(r_p, \psi_0)]^L + v_n(r_p)$, then the estimator will be unbiased. In this case, we can consider Eq. (9c) for the error of the estimator. The first term which is related to the terrestrial data error is the same with the biased type, but we can merge the truncation and geopotential coefficients errors together. Therefore we can write

$$\|\delta T_{\text{VH}}^1\|^2 + \|\delta T_{\text{VH}}^2\|^2 = \frac{R^2}{4} \sum_{n=2}^{\infty} \left\{ [Q_n^{\text{VH}}(r_p, \psi_0)]^L + v_n^*(r_p) \right\}^2 n(n+1) dc_n^*, \quad (11a)$$

After summing up Eq. (10e) and the above equation and taking derivative with respect to $v_k^*(r_p)$ and equating the results to zero and some simplification the proposition is proved. The global mean square error of the estimator can be obtained by using Eq. (9c).

Proposition 3 *The optimum type least-squares modification parameters are derived by setting $b_n^{\text{VH}}(r_p) = \left\{ [Q_n^{\text{VH}}(r_p, \psi_0)]^L + v_n(r_p) \right\} c_n / (c_n + dc_n)$ for the vertical-horizontal gradient estimator and solving Eq. (8) and with the following elements:*

$$s_n(r_p) = v_n(r_p)$$

$$a_{kr} = a_{rk} = k(k+1)C_r \delta_{kr} - r(r+1)E_{rk}^1(\psi_0)C_r - k(k+1)E_{kr}^1(\psi_0)C_k + \sum_{n=2}^{\infty} E_{nr}^1(\psi_0)E_{nk}^1(\psi_0)n(n+1)C_n, \quad k, r = 2, 3, \dots, M$$

$$h_k(r_p) = k(k+1) \left\{ \Omega_k^{\text{VH}}(r_p) \sigma_k^2 - [Q_k(r_p, \psi_0)]^{\text{VH}} C_k \right\} + \sum_{n=2}^{\infty} \left\{ [Q_n(r_p, \psi_0)]^{\text{VH}} C_n - \Omega_n^{\text{VH}}(r_p) \sigma_n^2 \right\} n(n+1) E_{nk}^1(\psi_0)$$

where

$$C_k = \sigma_k^2 + \begin{cases} c_k dc_k / (c_k + dc_k) & 2 \leq k \leq M \\ c_k & k > M \end{cases}, \quad (12a)$$

Proof. The global mean square error of the estimator is will be:

$$\|\delta T_{\text{VH}}\|^2 = \frac{R^2}{4} \sum_{n=2}^{\infty} \left\{ [b_n^*(r_p) - v_n^*(r_p) - [Q_n^{\text{VH}}(r_p, \psi_0)]^L] c_n + \right.$$

$$+\left[\Omega_n^{\text{VH}}(r_p)-v_n^*(r_p)-[Q_n^{\text{VH}}(r_p,\psi_0)]^L\right]^2\sigma_n^2\}+\frac{R^2}{4}\sum_{n=2}^M(b_n^{\text{VH}}(r_p))^2dc_n, \quad (12b)$$

where

$$b_n^*(r_p)=\begin{cases} b_n^{\text{VH}}(r_p) & n \leq L \\ 0 & n > 0 \end{cases}, \quad (12c)$$

taking derivative with respect to $v_k(r_p)$ and equating the results to zero and further simplifications the proposition is proved.

4 SPECTRAL FORMS OF HORIZONTAL-HORIZONTAL GRADIENT ESTIMATORS

We name the gradients $T_{xx}(P)$, $T_{yy}(P)$ and $T_{xy}(P)$, horizontal horizontal gradients. Analogous to the previous section, we consider the following integral formula:

$$\left[\frac{T_{xx}(P)-T_{yy}(P)}{2T_{xy}(P)}\right]=\frac{R}{4\pi}\iint_{\sigma}H(r_p,\psi)\begin{pmatrix}\cos 2\alpha \\ \sin 2\alpha\end{pmatrix}\Delta g(Q)d\sigma, \quad (13a)$$

where (Reed 1973, Eq. 5.37)

$$H(r_p,\psi)=S_{\psi\psi}(r_p,\psi)-\cot\psi S_{\psi}(r_p,\psi), \quad (13b)$$

and

$$\begin{aligned} S_{\psi\psi}(r_p,\psi)= & -t^2\cos\psi\left(\frac{2}{D^3}+\frac{6}{D}-8-3\frac{1-t\cos\psi-D}{D\sin^2\psi}-\right. \\ & \left.-3\ln\frac{1-t\cos\psi+D}{2}\right)+t^3\sin^2\psi\left[\left(\frac{6}{D^5}+\frac{6}{D^3}+3\frac{D-1}{D^2\sin^2\psi}+\right.\right. \\ & \left.\left.+3\frac{D+1}{D(1-t\cos\psi+D)}-3\frac{1-t\cos\psi-D}{D\sin^2\psi}\left(\frac{2\cos\psi}{t\sin^2\psi}+\frac{1}{D^2}\right)\right]\right], \quad (13c) \end{aligned}$$

the spectral form of Eq. (13b) is:

$$H(r_p,\psi)=\sum_{n=2}^{\infty}\frac{2n+1}{2}\Omega_n(r_p)P_{n2}(\cos\psi), \quad (13d)$$

where $\Omega_n(r_p)$ is same as that of Eq. (1d).

4.1 TRUNCATION ERROR FORMULAS

Let the following kernel for the horizontal-horizontal gradients:

$$\bar{H}(r_p, \psi) = \begin{cases} 0 & 0 \leq \psi < \psi_0 \\ \frac{1}{2} H(r_p, \psi) & \psi_0 \leq \psi \leq \pi \end{cases}, \quad (14a)$$

with the following spectral form

$$\bar{H}(r_p, \psi) = \sum_{n=2}^{\infty} \frac{2n+1}{2(n+2)(n+1)n(n-1)} Q_n^H(r_p, \psi_0) P_{n2}(\cos \psi), \quad (14b)$$

where $P_{n2}(\cos \psi)$ is the associated Legendre function of degree n and order 2, and truncation error coefficients

$$Q_n^H(r_p, \psi_0) = \frac{1}{2} \int_{\psi_0}^{\pi} H(r_p, \psi) P_{n2}(\cos \psi) \sin \psi d\psi. \quad (14c)$$

Now, we want to formulate a truncation error formula for the horizontal-horizontal gradients. By using the following theorem the least-squares modification of the horizontal-horizontal derivative of the extended Stokes formula will be performed.

Theorem 2:

$$\begin{aligned} \iint_{\sigma} \Delta g(Q) P_{n2}(\cos \psi) \begin{pmatrix} \cos 2\alpha \\ \sin 2\alpha \end{pmatrix} d\sigma = \\ = \frac{4\pi}{2n+1} \left[\begin{aligned} & \frac{\partial^2 \Delta g_n(P)}{\partial \theta^2} - \cot \theta \frac{\partial \Delta g_n(P)}{\partial \theta} - \frac{1}{\sin^2 \theta} \frac{\partial^2 \Delta g_n(P)}{\partial \lambda^2} \\ & \frac{1}{\sin \theta} \frac{\partial^2 \Delta g_n(P)}{\partial \lambda \partial \theta} - \cot \theta \frac{1}{\sin \theta} \frac{\partial \Delta g_n(P)}{\partial \lambda} \end{aligned} \right]. \end{aligned}$$

Proof. Let us first consider that part of the theorem containing $\cos 2\alpha$, this term can be written as:

$$\begin{aligned} \iint_{\sigma} \Delta g(Q) P_{n2}(\cos \psi) \cos 2\alpha d\sigma = \\ = \iint_{\sigma} \Delta g(Q) \left[\frac{\partial^2 P_n(\cos \psi)}{\partial \theta^2} - \cot \theta \frac{\partial P_n(\cos \psi)}{\partial \theta} - \frac{1}{\sin^2 \theta} \frac{\partial^2 P_n(\cos \psi)}{\partial \lambda^2} \right] d\sigma, \end{aligned} \quad (15a)$$

Equation (15a) consists of three terms, and, here we simplified each terms in the following. First, we consider the first term. By using the addition theorem of spherical harmonics (Heiskanen and Moritz 1967, p. 33, Eq. 1-82’):

$$(2n+1)P_n(\cos\psi) = \sum_{m=-n}^n Y_{nm}(Q)Y_{nm}(P), \quad (15b)$$

we have

$$\begin{aligned} \iint_{\sigma} \Delta g(Q) \frac{\partial^2 P_n(\cos\psi)}{\partial \theta^2} d\sigma &= \frac{1}{2n+1} \sum_{m=-n}^n \frac{\partial^2 Y_{nm}(P)}{\partial \theta^2} \iint_{\sigma} \Delta g(Q) Y_{nm}(Q) d\sigma + \\ &= \frac{4\pi}{2n+1} \sum_{m=-n}^n \Delta g_{nm} \frac{\partial^2 Y_{nm}(P)}{\partial \theta^2} = \frac{4\pi}{2n+1} \frac{\partial^2 \Delta g_n(P)}{\partial \theta^2}, \end{aligned} \quad (16a)$$

the second term

$$\begin{aligned} \iint_{\sigma} \Delta g(Q) \cot \theta \frac{\partial P_n(\cos\psi)}{\partial \theta} d\sigma &= \frac{\cot \theta}{2n+1} \frac{\partial Y_{nm}(P)}{\partial \theta} \iint_{\sigma} \Delta g(Q) Y_{nm}(Q) d\sigma + \\ &= \frac{4\pi \cot \theta}{2n+1} \sum_{m=-n}^n \Delta g_{nm} \frac{\partial Y_{nm}(P)}{\partial \theta} = \frac{4\pi \cot \theta}{2n+1} \frac{\partial \Delta g_n(P)}{\partial \theta}, \end{aligned} \quad (16b)$$

and the last term

$$\begin{aligned} \iint_{\sigma} \Delta g(Q) \frac{1}{\sin^2 \theta} \frac{\partial^2 P_n(\cos\psi)}{\partial \lambda^2} d\sigma &= \frac{1}{2n+1} \frac{1}{\sin^2 \theta} \sum_{m=-n}^n \frac{\partial^2 Y_{nm}(P)}{\partial \lambda^2} \iint_{\sigma} \Delta g(Q) Y_{nm}(Q) d\sigma \\ &= \frac{4\pi}{2n+1} \frac{1}{\sin^2 \theta} \sum_{m=-n}^n \Delta g_{nm} \frac{\partial^2 Y_{nm}(P)}{\partial \lambda^2} = \frac{4\pi}{2n+1} \frac{1}{\sin^2 \theta} \frac{\partial^2 \Delta g_n(P)}{\partial \lambda^2}. \end{aligned} \quad (16c)$$

Now by substituting Eqs. (16a)-(16c) into Eq. (15a) we obtain:

$$\begin{aligned} \iint_{\sigma} \Delta g(Q) P_{n_2}(\cos\psi) \cos 2\alpha d\sigma &= \frac{4\pi}{2n+1} \left(\frac{\partial^2 \Delta g_n(P)}{\partial \theta^2} - \right. \\ &\quad \left. - \cot \theta \frac{\partial \Delta g_n(P)}{\partial \theta} - \frac{1}{\sin^2 \theta} \frac{\partial^2 \Delta g_n(P)}{\partial \lambda^2} \right). \end{aligned} \quad (17a)$$

In a very similar way we can prove that

$$\iint_{\sigma} \Delta g(Q) P_{n_2}(\cos\psi) \sin 2\alpha d\sigma = \frac{4\pi}{2n+1} \left(\frac{\partial^2 \Delta g_n(P)}{\sin \theta \partial \lambda \partial \theta} - \cot \theta \frac{\partial \Delta g_n(P)}{\sin \theta \partial \lambda} \right). \text{ Q.E.D. } (17b)$$

By considering Theorem 2, the spectral form of the truncation error formula will not be difficult to obtain. According to Eqs. (17a) and (17b) and after further simplifications we have:

$$\begin{bmatrix} \delta T_{xx}^1(P) - \delta T_{yy}^1(P) \\ 2\delta T_{xy}^1(P) \end{bmatrix} = \frac{R}{2} \sum_{n=2}^{\infty} \frac{Q_n^H(r_P, \psi_0)}{(n+2)(n+1)n(n-1)} \times \\ \times \iint_{\sigma} \Delta g(Q) P_{n2}(\cos \psi) \begin{pmatrix} \cos 2\alpha \\ \sin 2\alpha \end{pmatrix} d\sigma, \quad (18)$$

According to Theorem 2 the integral part of Eq. (18) is simplified and finally we can write the spectral form of the truncation error formula as:

$$\begin{bmatrix} \delta T_{xx}^1(P) - \delta T_{yy}^1(P) \\ 2\delta T_{xy}^1(P) \end{bmatrix} = \sum_{n=2}^{\infty} Q_n^{HH}(r_P, \psi_0) \times \\ \times \begin{bmatrix} \frac{\partial^2 \Delta g_n(P)}{\partial \theta^2} - \cot \theta \frac{\partial \Delta g_n(P)}{\partial \theta} - \frac{1}{\sin^2 \theta} \frac{\partial^2 \Delta g_n(P)}{\partial \lambda^2} \\ \frac{1}{\sin \theta} \frac{\partial^2 \Delta g_n(P)}{\partial \lambda \partial \theta} - \cot \theta \frac{1}{\sin \theta} \frac{\partial \Delta g_n(P)}{\partial \lambda} \end{bmatrix}, \quad (19a)$$

where

$$Q_n^{HH}(r_P, \psi_0) = \frac{Q_n^H(r_P, \psi_0)}{(n+2)(n+1)n(n-1)}. \quad (19b)$$

4.2 TERRESTRIAL DATA ERROR FORMULA

Let the following integral formula for the error of terrestrial data

$$\begin{bmatrix} \delta T_{xx}^0(P) - \delta T_{yy}^0(P) \\ 2\delta T_{xy}^0(P) \end{bmatrix} = \frac{R}{4\pi} \iint_{\sigma} [H(r_P, \psi) - \bar{H}(r_P, \psi)] \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \varepsilon(Q) d\sigma. \quad (20)$$

Considering the spectral forms of $H(r_P, \psi)$ and $\bar{H}(r_P, \psi)$, and inserting them into Eq. (20) we have

$$\begin{bmatrix} \delta T_{xx}^0(P) - \delta T_{yy}^0(P) \\ 2\delta T_{xy}^0(P) \end{bmatrix} = \frac{R}{4\pi} \sum_{n=2}^{\infty} \frac{2n+1}{2} [\Omega_n(r_P) - Q_n^H(r_P, \psi_0)] \\ \iint_{\sigma} \varepsilon(Q) P_{n2}(\cos \psi) \begin{pmatrix} \cos 2\alpha \\ \sin 2\alpha \end{pmatrix} d\sigma. \quad (21a)$$

According to Theorem 2 we obtain

$$\begin{bmatrix} \delta T_{xx}^0(P) - \delta T_{yy}^0(P) \\ 2\delta T_{xy}^0(P) \end{bmatrix} = \frac{R}{2} \sum_{n=2}^{\infty} [\Omega_n(r) - Q_n^H(r, \psi_0)] \times$$

$$\times \begin{bmatrix} \frac{\partial^2 \varepsilon_n(P)}{\partial \theta^2} - \cot \theta \frac{\partial \varepsilon_n(P)}{\partial \theta} - \frac{1}{\sin^2 \theta} \frac{\partial^2 \varepsilon_n(P)}{\partial \lambda^2} \\ \frac{1}{\sin \theta} \frac{\partial^2 \varepsilon_n(P)}{\partial \lambda \partial \theta} - \cot \theta \frac{1}{\sin \theta} \frac{\partial \varepsilon_n(P)}{\partial \lambda} \end{bmatrix}. \quad (21b)$$

5 LEAST-SQUARES MODIFICATION OF HORIZONTAL-HORIZONTAL GRADIENT ESTIMATORS

Let us consider the following estimator for the horizontal-horizontal gradients:

$$\begin{bmatrix} \tilde{T}_{xx}(P) - \tilde{T}_{yy}(P) \\ 2\tilde{T}_{xy}(P) \end{bmatrix} = \frac{R}{4\pi} \iint_{\sigma_0} H^L(r_p, \psi) \begin{pmatrix} \cos 2\alpha \\ \sin 2\alpha \end{pmatrix} \Delta g(Q) d\sigma + \frac{R}{2} \sum_{n=2}^L b_n^{\text{HH}}(r_p) \times \\ \times \begin{bmatrix} \frac{\partial^2 \Delta g_n(P)}{\partial \theta^2} - \cot \theta \frac{\partial \Delta g_n(P)}{\partial \theta} - \frac{1}{\sin^2 \theta} \frac{\partial^2 \Delta g_n(P)}{\partial \lambda^2} \\ \frac{1}{\sin \theta} \frac{\partial^2 \Delta g_n(P)}{\partial \theta \partial \lambda} + \frac{\cos \theta}{\sin^2 \theta} \frac{\partial \Delta g_n(P)}{\partial \lambda} \end{bmatrix}, \quad (22a)$$

where

$$H^L(r_p, \psi) = H(r_p, \psi) - \sum_{n=1}^L \frac{2n+1}{2} h'_n(r_p) P_{n2}(\cos \psi). \quad (22b)$$

Similar to the previous derivation, the problem is to determine the modification parameters $h'_n(r_p)$, so as to minimize the global mean squares error of the estimators in a least-squares sense. In the following, the mathematical derivations are summarized in three propositions as before.

Proposition 4 *The biased type least-squares modification parameters $h'_n(r_p)$ are derived by setting $b_n^{\text{HH}}(r_p) = h'_n(r_p)$ for the horizontal-horizontal gradient estimator solving Eq. (8) with following elements:*

$$\begin{aligned} s_n(r_p) &= h'_n(r_p) \\ a_{rk} &= a_{kr} = (k-1)k(k+1)(k+2)(\sigma_k^2 + dc_k) \delta_{rk} - \\ &\quad - (r-1)r(r+1)(r+2)E_{rk}^2(\psi_0)\sigma_k^2 - \\ &\quad - (k-1)k(k+1)(k+2)E_{kr}^2(\psi_0)\sigma_r^2 + \\ &\quad + \sum_{n=2}^{\infty} E_{nr}^2(\psi_0)E_{nk}^2(\psi_0)(n-1)n(n+1)(n+2)(\sigma_n^2 + c_n), \\ h_k(r_p) &= (k-1)k(k+1)(k+2) \{ \Omega_k(r_p) - \mathcal{Q}_k^{\text{HH}}(r_p, \psi_0) \} \sigma_k^2 + \\ &\quad + \sum_{n=2}^{\infty} (n-1)n(n+1)(n+2)E_{nk}^2(\psi_0) \{ \mathcal{Q}_n^{\text{HH}}(r_p, \psi_0)(\sigma_n^2 + c_n) - \Omega_n(r_p)\sigma_n^2 \}, \end{aligned}$$

where

$$E_{nk}^2(\psi_0) = \frac{2k+1}{2} e_{nk}^2(\psi_0), \quad (23a)$$

$$e_{nk}^2(\psi_0) = \int_{\psi_0}^{\pi} P_{n2}(\cos\psi) P_{k2}(\cos\psi) \sin\psi d\psi, \quad (23b)$$

which we call $e_{nk}^2(\psi_0)$ the second-order Paul coefficients.

Proof. Again the global mean square error of the estimator can be written as:

$$\|\delta T_{HH}\|^2 = \|\delta T_{HH}^0\|^2 + \|\delta T_{HH}^1\|^2 + \|\delta T_{HH}^2\|^2. \quad (24)$$

The first term is related to the terrestrial data error; the second and the third are related to the truncation and geopotential coefficients errors with following expressions

$$\|\delta T_{HH}^0\|^2 = \frac{R^2}{4r_p^2} \sum_{n=2}^{\infty} \left\{ h_n'^*(r_p) - \Omega_n(r_p) - Q_n^{HH}(r_p, \psi_0) \right\}^2 (n-1)n(n+1)(n+2) \sigma_n^2, \quad (25a)$$

$$\|\delta T_{HH}^1\|^2 = \frac{R^2}{4r_p^2} \sum_{n=2}^{\infty} \left[Q_n^{HH}(r_p, \psi_0) \right]^2 (n-1)n(n+1)(n+2) c_n, \quad (25b)$$

$$\|\delta T_{HH}^2\|^2 = \frac{R^2}{4r_p^2} \sum_{n=2}^{\infty} \left\{ h_n'^*(r_p) \right\}^2 (n-1)n(n+1)(n+2) d c_n, \quad (25c)$$

where

$$h_n'^*(r_p) = \begin{cases} h_n'(r_p) & n \leq L \\ 0 & n > L \end{cases}, \quad (26a)$$

$$\left[Q_n^{HH}(r_p, \psi_0) \right]^L = Q_n^{HH}(r_p, \psi_0) - \sum_{k=2}^L h_k'(r_p) E_{nk}^2(\psi_0). \quad (26b)$$

According to

$$\frac{\partial \|\delta T_{HH}\|^2}{\partial h_k'(r_p)} = 0, \quad (26c)$$

and simplifying the results the proposition is proved.

Proposition 5 *The unbiased type least-squares modification parameters $h'_n(r_p)$ of the horizontal-horizontal gradient estimator is derived by setting $b_n^{\text{HH}}(r_p) = [Q_n^{\text{HH}}(r_p, \psi_0)]^L + h'_n(r_p)$ and solving Eq. (8) with the following elements*

$$\begin{aligned}
 s_n(r_p) &= h'_n(r_p) \\
 a_{rk} &= a_{kr} = (k-1)k(k+1)(k+2)d_k\delta_{rk} - \\
 &\quad -(r-1)r(r+1)(r+2)E_{rk}^2(\psi_0)d_r - \\
 &\quad -(k-1)k(k+1)(k+2)E_{kr}^2(\psi_0)d_k + \\
 &\quad + \sum_{n=2}^{\infty} E_{nr}^2(\psi_0)E_{nk}^2(\psi_0)(n-1)n(n+1)(n+2)d_n \\
 h_k(r_p) &= (k-1)k(k+1)(k+2)\{\Omega_k(r_p)\sigma_k^2 - Q_k^{\text{HH}}(r_p, \psi_0)d_k\} + \\
 &\quad + \sum_{n=2}^{\infty} (n-1)n(n+1)(n+2)E_{nk}^2(\psi_0)\{Q_n^{\text{HH}}(r_p, \psi_0)d_n - \Omega_n(r_p)\sigma_n^2\}.
 \end{aligned}$$

Proof. If we consider Eq. (21b) as the first term of global mean squares error which is related to the terrestrial data error and

$$\|\delta T_{\text{HH}}^1\|^2 + \|\delta T_{\text{HH}}^2\|^2 = \frac{R^2}{4r_p^2} \sum_{n=2}^{\infty} \{Q_n^{\text{HH}}(r_p, \psi_0) + h_n'^*(r_p)\}^2 (n-1)n(n+1)(n+2)dc_n^*,$$

for truncation and geopotential coefficients error. The proposition is proved in a similar way of Proposition 3.

Proposition 6 *The optimum type least-squares modification parameters of the horizontal-horizontal gradient estimator are derived by setting $b_n^{\text{HH}}(r_p) = \{[Q_n^{\text{HH}}(r_p, \psi_0)]^L + h'_n(r_p)\}c_n / (c_n + dc_n)$ solving Eq. (8) with the following elements*

$$\begin{aligned}
 s_n(r_p) &= h'_n(r_p) \\
 a_{rk} &= a_{kr} = (k-1)k(k+1)(k+2)C_k\delta_{rk} - \\
 &\quad -(r-1)r(r+1)(r+2)E_{rk}^2(\psi_0)C_r - \\
 &\quad -(k-1)k(k+1)(k+2)E_{kr}^2(\psi_0)C_k + \\
 &\quad + \sum_{n=2}^{\infty} E_{nr}^2(\psi_0)E_{nk}^2(\psi_0)(n-1)n(n+1)(n+2)C_n \\
 h_k(r_p) &= (k-1)k(k+1)(k+2)\{\Omega_k(r_p)\sigma_k^2 - Q_k^{\text{HH}}(r_p, \psi_0)C_k\} + \\
 &\quad + \sum_{n=2}^{\infty} (n-1)n(n+1)(n+2)E_{nk}^2(\psi_0)\{Q_n^{\text{HH}}(r_p, \psi_0)C_n - \Omega_n(r_p)\sigma_n^2\}.
 \end{aligned}$$

Proof. The proof is similar to that presented for Proposition 3.

6 A DISCUSSION ON VALIDATION OF SATELLITE GRADIOMETRIC DATA

Isotropy of the kernel is important in modifying the extended Stokes formula, but such a property is destroyed by taking its horizontal derivative. In this case, modification of the formula will not be as easy as it is for the ordinary extended Stokes formula or its second-order radial derivative (see Eshagh 2010a). An idea to generate the vertical-horizontal and horizontal-horizontal gradients is to modify the extended Stokes formula prior to taking the partial derivatives. However, as Eshagh (2010a) showed, this method is just suitable when a deterministic modification method is used. He showed that modification of the extended Stokes formula prior to taking derivative and taking derivative prior to modification of the formula yields similar results if the modification technique is deterministic and not stochastic. Validation of the second-order radial derivative of satellite gradiometric data was also discussed by Eshagh (2010a). According to the estimator presented in Eq. (22a) it is easy to validate the diagonal elements of the gravitational tensor. Here, we assume that $T_{zz}(P)$ has been already validated and we know that $T_{zz}(P) = -(T_{xx}(P) + T_{yy}(P))$, by estimating the $T_{zz}(P)$ the summation of the $T_{xx}(P) + T_{yy}(P)$ is validated as well. However, Eq. (22a) is the estimator of $T_{xx}(P) - T_{yy}(P)$. Therefore $T_{xx}(P)$ and $T_{yy}(P)$ can be validated by using estimates of $T_{zz}(P) = -(T_{xx}(P) + T_{yy}(P))$ and $T_{xx}(P) - T_{yy}(P)$. No problem is seen for evaluation of the vertical-horizontal gradients according to Eq. (7a) as they can be estimated separately. $T_{xy}(P)$ can also be validated using Eq. (22a) as well. Let the following estimator for $T_{zz}(P)$ (Eshagh 2010a):

$$\tilde{T}_{zz}(P) = \frac{R}{4\pi} \iint_{\sigma_0} S_{zz}^L(r_p, \psi) \Delta g(Q) d\sigma + \frac{R}{2} \sum_{n=2}^L b_n^{zz}(r_p) \Delta g_n(P). \quad (27)$$

According to Eqs. (27) and (22a) it will not be difficult to obtain the following estimator for $T_{xx}(P)$ and $T_{yy}(P)$:

$$\begin{aligned} \begin{pmatrix} \tilde{T}_{xx}(P) \\ \tilde{T}_{yy}(P) \end{pmatrix} &= \pm \frac{R}{8\pi} \iint_{\sigma_0} (H^L(r_p, \psi) \cos 2\alpha \mp S_{zz}^L(r_p, \psi)) \Delta g(Q) d\sigma \pm \\ &\pm \frac{R}{2} \sum_{n=2}^L [b_n^{HH}(r_p) \left(\frac{\partial^2 \Delta g_n(P)}{\partial \theta^2} - \cot \theta \frac{\partial \Delta g_n(P)}{\partial \theta} - \frac{1}{\sin^2 \theta} \frac{\partial^2 \Delta g_n(P)}{\partial \lambda^2} \right) \\ &\mp b_n^{zz}(r_p) \Delta g_n(P)]. \end{aligned} \quad (28)$$

An important issue that should be stated here is that in mathematical derivation of Eq. (28) it was assumed that the cap size of integration for both Eqs. (27) and (22a) are the same. Otherwise Eq. (28) will not be valid. In practice it is also possible to

have different cap sizes for Eqs. (22a) and (27), but in this case Eq. (28) should not be used and one has to work directly on the estimates of $T_{zz}(P) = -(T_{xx}(P) + T_{yy}(P))$ and $T_{xx}(P) - T_{yy}(P)$.

Equations (7a), (22a), (27) and (28) are suited for the case where the gravity anomaly at sea level is available. The surface anomalies should be continued downward to sea level; or the surface anomalies can be used directly to estimate the approximate gradients at satellite level and after that the effect of disregarded downward continuation effect can be taken into account on the approximate gradients. The effect of topographic and atmospheric masses should also be accounted for in this case as well; see e.g. Eshagh (2009a, 2009d, 2009e, 2009f and 2010c), Eshagh and Sjöberg (2008, 2009a and 2009b).

Another important issue is that the satellite gradiometric data are not measured on a regular grid. One simple idea is to use some interpolators for transferring the satellite data to such a grid. Also the downward/upward continuation method presented by Tóth et al. (2004, 2006 and 2007) can also be used in this respect. As our estimators show, the gradients are expressed in the local north-oriented frame and not in the gradiometer reference frame. However, the gradients can be transferred to the local frame if the transformation parameters between the frames are available. Mueller (2003) mentioned that the error of the parameters will influence the transferred gradients, and it is better to keep the frame of computations as close as possible to the gradiometer reference frame. Due to the orbital perturbations of the satellites they will not fly on the mean orbital sphere, and in this case the modification parameters should be computed for each position of satellite, which is very laborious work and impractical. Therefore it will be more reasonable to find the minimum and maximum altitude of the satellite over the desired region and generate the sets of approximate gradients at different altitudes between these two levels. One can also increase the resolution of the estimated gradients to create a dense three-dimensional network between these two levels, or in the shell in which the satellite flies. Since the gradients are rather smooth at satellite altitude, it will be straightforward to interpolate the satellite data using some interpolators and the position of the satellite. In such a case, one can look for a corrective surface to model the systematic errors of the satellite gradiometric data.

7 THE TRUNCATION COEFFICIENTS $Q_n^{VH}(r_p, \psi_0)$ AND $Q_n^{HH}(r_p, \psi_0)$

The mathematical formula of the truncation coefficient $Q_n^{VH}(r_p, \psi_0)$ was presented in Eq. (5c). The important parameter of this formula is related to $Q_n^V(r_p, \psi_0)$ which plays the major role in generation of $Q_n^{VH}(r_p, \psi_0)$. According to Eshagh (2010a) numerical integration of such integral is not a suitable approach as the numerical integration error will influence the estimated modification parameters. He suggested the use of spectral form of the truncation error for the second-order radial derivative of the extended Stokes formula. Adopting his conclusion we obtain a similar spectral form for $Q_n^{VH}(r_p, \psi_0)$. Substituting Eq. (2e) into Eq. (3c) and by comparing the results to Eq. (9b) it will be straightforward to obtain:

$$Q_n^{\text{VH}}(r_P, \psi_0) = \frac{1}{2n(n+1)} \sum_{n'=2}^{\infty} \frac{2n'+1}{2} \Omega_{n'}^{\text{VH}}(r_P) e_{nn'}^1(\psi_0). \quad (29)$$

Analogously, by substituting Eq. (13d) into (14c) considering Eq. (23b) and inserting the result in Eq. (19b) we obtain:

$$Q_n^{\text{HH}}(r_P, \psi_0) = \frac{1}{2(n-1)n(n+1)(n+2)} \sum_{n'=2}^{\infty} \frac{2n'+1}{2} \Omega_{n'}(r_P) e_{nn'}^2(\psi_0). \quad (30)$$

Equations (29) and (30) can be used for generating the truncation coefficients $Q_n^{\text{VH}}(r_P, \psi_0)$ and $Q_n^{\text{HH}}(r_P, \psi_0)$, but the practical problem is to perform the summation to infinity. However, Eshagh (2010a) stated that for the case of second-order radial derivative these series can be truncated to a certain number; say to 5000. As we observe Eqs. (29) and (30) are dependent on the first- and second-order Paul coefficients. We discuss these coefficients in the next section.

8 DERIVATION OF $e_{nk}^1(\psi_0)$ AND $e_{nk}^2(\psi_0)$

In the least-squares modification of the vertical-vertical, vertical-horizontal and horizontal-horizontal gradients we are involved with $e_{nk}^0(\psi_0)$, $e_{nk}^1(\psi_0)$ and $e_{nk}^2(\psi_0)$, respectively. We already named these coefficients, zero-, first- and second-order Paul's coefficients. Therefore we have to derive some formulas to practically evaluate these coefficients. The first-order Paul coefficients $e_{nk}^0(\psi_0)$ is the integral of product of two Legendre polynomials and Paul (1973) proposed a recursive formula to generate these coefficients. The first-order Paul coefficients $e_{nk}^1(\psi_0)$ are the integral product of two associated Legendre functions of order 1. A general recursive formula for the integral product of the two associated Legendre functions was obtained by Mainville (1986). After that Hwang (1995) worked further on this matter and developed a formula to convert this integral product to a simple integral of the Legendre functions. Since the recursive formula for the integral was derived by Paul (1973), Hwang (1995) idea can be used to compute the integral product of two Legendre functions. However, we already know that higher order terms than 2 is not needed for the least-squared modification goal. It would be preferable to develop some relations between $e_{nk}^1(\psi_0)$ and $e_{nk}^0(\psi_0)$. The computation of $e_{nk}^1(\psi_0)$ which is integral product of two first-order associated Legendre functions was also presented by those geodesists working on Vening-Meinesz's formulas and it can easily be shown that $e_{nk}^1(\psi_0)$ is related to $e_{nk}^0(\psi_0)$ by the following relation (Eshagh 2009a):

$$e_{nk}^1(\psi_0) = n(n+1)e_{nk}^0(\psi_0) - n P_k(\cos \psi_0) [P_{n-1}(\cos \psi_0) - \cos \psi_0 P_n(\cos \psi_0)]. \quad (31)$$

For more details and other forms for this relation the reader is referred e.g. to Chen (1982), Hsu (1984) and Neyman et al. (1996). According to this relation it is not difficult to compute $e_{nk}^1(\psi_0)$. The problem of deriving the second-order Paul coefficient $e_{nk}^2(\psi_0)$ was not considered until now, because this problem is mostly related to gradiometry. In the following we obtain a relation between this function and the first-order Paul coefficient $e_{nk}^1(\psi_0)$.

Proposition 7 *The relation between $e_{nk}^2(\psi_0)$ and $e_{nk}^1(\psi_0)$ is:*

$$e_{nk}^2(\psi_0) = -\frac{\cos\psi_0}{2} [n(n+1)+1] [k(k+1)+1] P_{n1}(\cos\psi_0) P_{k1}(\cos\psi_0) + e_{nk}^1(\psi_0) [n(n+1)-2] + n(n+1)k [P_{k-1}(\cos\psi_0) - \cos\psi_0 P_k(\cos\psi_0)].$$

Proof. Let the well-known Legendre equation (Heiskanen and Mortitz 1967, p. 22):

$$(1-x^2) \frac{d^2 P_n(x)}{dx^2} - 2x \frac{dP_n(x)}{dx} = -n(n+1)P_n(x), \quad -1 \leq x \leq 1 \quad (32a)$$

we can also write

$$\begin{aligned} (1-x^2)^2 \frac{d^2 P_n(x)}{dx^2} \frac{d^2 P_k(x)}{dx^2} - 2x(1-x^2) \frac{dP_n(x)}{dx} \frac{d^2 P_k(x)}{dx^2} = \\ = -n(n+1)(1-x^2)P_n(x) \frac{d^2 P_k(x)}{dx^2}. \end{aligned} \quad (32b)$$

After integrating Eq. (32b), the first term in the left hand side will be nothing else but $e_{nk}^2(\psi_0)$, namely:

$$e_{nk}^2(x_0) = \int_{-1}^{x_0} (1-x^2)^2 \frac{d^2 P_n(x)}{dx^2} \frac{d^2 P_k(x)}{dx^2} dx = \int_{-1}^{x_0} P_{n2}(x) P_{k2}(x) dx. \quad (32c)$$

Therefore Eq. (32b) can be re-written as:

$$e_{nk}^2(x_0) = 2 \int_{-1}^{x_0} x(1-x^2) \frac{dP_n(x)}{dx} \frac{d^2 P_k(x)}{dx^2} dx - n(n+1) \int_{-1}^{x_0} (1-x^2) P_n(x) \frac{d^2 P_k(x)}{dx^2} dx \quad (33d)$$

The first and second integrals in the right hand side of Eq. (33d) can be solved partially

$$e_{nk}^2(x_0) = 2x(1-x^2) \frac{dP_n(x)}{dx} \frac{d^2 P_k(x)}{dx^2} \Big|_{-1}^{x_0} - 2C -$$

$$-n(n+1)\left[\left(1-x^2\right)P_n(x)\frac{d^2P_k(x)}{dx^2}\Big|_{-1}^{x_0}-D\right], \quad (34a)$$

where

$$C = \int_{-1}^{x_0} (1-x^2) \frac{dP_n(x)}{dx} \frac{dP_k(x)}{dx} dx + \int_{-1}^{x_0} x \frac{d}{dx} \left[(1-x^2) \frac{dP_n(x)}{dx} \right] \frac{dP_k(x)}{dx} dx, \quad (34b)$$

$$e_{nk}^1(x_0) = \int_{-1}^{x_0} (1-x^2) \frac{dP_n(x)}{dx} \frac{dP_k(x)}{dx} dx = \int_{-1}^{x_0} P_{n1}(x) P_{n1}(x) dx, \quad (34c)$$

$$\int_{-1}^{x_0} x \frac{d}{dx} \left[(1-x^2) \frac{dP_n(x)}{dx} \right] \frac{dP_k(x)}{dx} dx = -n(n+1) \int_{-1}^{x_0} x P_n(x) \frac{dP_k(x)}{dx} dx, \quad (34d)$$

and after further simplifications

$$C = e_{nk}^1(x_0) + n(n+1) \int_{-1}^{x_0} x P_n(x) \frac{dP_k(x)}{dx} dx, \quad (34e)$$

$$\begin{aligned} D &= \int_{-1}^{x_0} \frac{d}{dx} \left[(1-x^2) P_n(x) \right] \frac{dP_k(x)}{dx} dx = \int_{-1}^{x_0} \left[-2x P_n(x) + (1-x^2) \frac{dP_n(x)}{dx} \right] \frac{dP_k(x)}{dx} dx = \\ &= -2 \int_{-1}^{x_0} x P_n(x) \frac{dP_k(x)}{dx} dx + \int_{-1}^{x_0} (1-x^2) \frac{dP_n(x)}{dx} \frac{dP_k(x)}{dx} dx \\ &= -2 \int_{-1}^{x_0} x P_n(x) \frac{dP_k(x)}{dx} dx + e_{nk}^1(x_0). \end{aligned} \quad (34f)$$

Substituting Eqs. (34e) and (34f) into (34a) we obtain:

$$\begin{aligned} e_{nk}^2(x_0) &= 2x(1-x^2) \frac{dP_n(x)}{dx} \frac{dP_k(x)}{dx} \Big|_{-1}^{x_0} + [n(n+1)-2] e_{nk}^1(x_0) - \\ &\quad -n(n+1) (1-x^2) P_n(x) \frac{dP_k(x)}{dx} \Big|_{-1}^{x_0}, \end{aligned} \quad (35)$$

and after changing the variable x to $\cos \psi$ we have:

$$\begin{aligned} e_{nk}^2(\psi_0) &= 2 \cos \psi \sin^2 \psi \frac{dP_n(\cos \psi)}{d \cos \psi} \frac{dP_k(\cos \psi)}{d \cos \psi} \Big|_{\psi_0}^{\pi} + [n(n+1)-2] e_{nk}^1(\psi_0) - \\ &\quad -n(n+1) \sin^2 \psi P_n(\cos \psi) \frac{dP_k(\cos \psi)}{d \cos \psi} \Big|_{\psi_0}^{\pi}. \end{aligned} \quad (36)$$

Substitution of the limits and further simplifications prove the proposition.

9 SUMMARY

In this paper we obtained the spectral forms of the integral gradient estimators for the vertical-horizontal and horizontal-horizontal gradients. These spectral forms were used to formulate the elements of the system of equations from its solution the modification parameters are derived. These elements were derived based on biased, unbiased and optimum least-squares modification methods for modifying the vertical-vertical and horizontal-horizontal gradients integral estimator. We formulated the truncation error formulas of $Q_n^{VH}(r_p, \psi_0)$ or $Q_n^{HH}(r_p, \psi_0)$ and first- and second-order coefficients of Paul, which were required for modifying the integral formulas. The general gradient estimators (7a) and (22a) can easily be modified in a least-squares sense by solving Eq. (8) considering each one of Propositions 1-6. The modified integral gradient estimators can be used for validation of all the elements of the gradiometric tensor at satellite level.

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