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THE GCD SEQUENCES OF THE ALTERED LUCAS SEQUENCES

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Abstract. In this study, we give two sequences $\{L_n^+\}_{n\geq1}$ and $\{L_n^-\}_{n\geq1}$ derived by altering the Lucas numbers with $\{\pm1, \pm3\}$, terms of which are called as altered Lucas numbers. We give relations connected with the Fibonacci F_n and Lucas L_n numbers, and construct recurrence relations and Binet's like formulas of the L_n^+ and L_n^- numbers. It is seen that the altered Lucas numbers have two distinct factors from the Fibonacci and Lucas sequences. Thus, we work out the greatest common divisor (GCD) of r-consecutive altered Lucas numbers. We obtain r-consecutive GCD sequences according to the altered Lucas numbers, and show that their GCD sequences are unbounded or periodic in terms of values r.

1. Introduction

Let F_n and L_n denote nth Fibonacci and Lucas numbers, respectively. The numbers F_n and L_n, are entries of sequences $\{F_n\}_{n\geq0}$ and $\{L_n\}_{n\geq0}$, are given by the linear recurrence relations,

$$F_{n+2} = F_{n+1} + F_n, \quad L_{n+2} = L_{n+1} + L_n, \quad n \geq 0 \tag{1.1}$$

with the initial values $F_0 = 0$, $F_1 = 1$, $L_0 = 2$, $L_1 = 1$ (see [6]).

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A quick look at the greatest common divisor (GCD) properties of the numbers F_n and L_n shows that the GCD of two Fibonacci numbers is always a Fibonacci number, $(F_m, F_n) = F_{(m,n)}$. Thus, the successive Fibonacci and Lucas numbers are relatively prime, $(F_n, F_{n+1}) = (F_n, F_{n+2}) = 1$ and $(L_n, L_{n+1}) = (L_n, L_{n+2}) = 1$. In addition to these properties, there exist a number of divisibility and GCD properties for these numbers such as

$$L_m \,|F_n \quad \Leftrightarrow \quad 2m\,|n\,, \quad m \geq 2,$$

$$L_m \,|L_n \quad \Leftrightarrow \quad n = (2k-1)\,m, \quad m \geq 2,$$

$$(F_n, L_n) = \begin{cases} 2, & n \equiv 0 \pmod 3\,, \\ 1, & \text{otherwise}, \end{cases}$$

$$(L_m, L_n) = L_d \quad \text{if} \ \frac{m}{d} \ \text{ and } \ \frac{n}{d} \ \text{ is odd.}$$

Several authors investigate the above numbers finding many values of a, $b \in \mathbb{Z}$ for the Fibonacci $\{F_n \pm a\}_{n\geq 0}$ and Lucas $\{L_n \pm b\}_{n\geq 0}$ sequences. For example, in [2], two sequences are defined with $\{G_n\}_{n\geq 0} = \{F_n + (-1)^n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0} = \{F_n - (-1)^n\}_{n\geq 0}$, which are called as the altered Fibonacci numbers. It is shown that the sequences $\{G_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ are multiplication of Fibonacci and Lucas subsequences according to their indices n ([1], [2], [6]). And also, in [2], the authors investigate some GCD cases for successive terms of the $\{G_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$. It is noted that (G_{4n+k}, G_{4n+k+1}) and (H_{4n+k}, H_{4n+k+1}), $(k = 0, 2)$ are not relatively prime. In addition to the sequences $\{G_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$, in [1], K. Chen defines a sequence $\{F_n + a\}_{n\geq 0}$, $a \in \mathbb{Z}$, called as a shifted Fibonacci sequence. And also, the author establishes a sequence $\{f_n(a)\}_{n\geq 0} = \{\gcd(F_n + a, F_{n+1} + a)\}_{n\geq 0}$, which is called as a GCD sequence of the shifted Fibonacci sequence. He shows that some successive terms of the altered and shifted sequences have a different behavior such as

$$(G_{4n}, G_{4n+1}) = L_{2n+1} = (G_{4n+1}, G_{4n+3})\,, \quad (G_{4n+2}, G_{4n+3}) = F_{2n+2},$$

$$(H_{4n}, H_{4n+1}) = F_{2n+1} = (H_{4n+1}, H_{4n+3})\,, \quad (H_{4n+2}, H_{4n+3}) = L_{2n+2},$$

$$f_{4n-1}(1) = F_{2n-1}, \quad f_{4n+1}(1) = L_{2n},$$

$$f_{4n-1}(-1) = L_{2n-1}, \quad f_{4n+1}(-1) = F_{2n}.$$

In [1], the author shows that $\{f_n(a)\}_{n\geq 0}$ is bounded from above if $a \neq \pm 1$. In addition to the properties of $\{f_n(a)\}_{n\geq 0}$ given in [1], we can give Spilker's result about $f_n(a)$ as follows (see [8]): let n and a be integers. If $m := a^4 - 1$ is not 0 and $f_n\,(a)$ divides $a^2 + (-1)^n$, then $f_n(a)$ is simply periodic such that a period p is defined by $F_p \equiv 0 \pmod{m}$, $F_{p+1} \equiv 0 \pmod{m}$. Also, the author produces explicit formulas for the number $f_n(a)$ and generalizes it to a wider class of recursive second order sequences.

In [7], the authors establish a sequence $\{f_n(\pm 3)\}_{n\geq 0}$, and show that their results correspond with bounds and periods given in [1] and [8].

In [4], the authors study cases of $(F_m + b, F_n + a)$, for a, $b \in \mathbb{Z}$ by varying positive integers m and n. For example, they show that there exists a constant c such that $\gcd(F_m + a, F_n + a) > e^{cm}$ holds for infinitely many pairs of positive integers $m > n$.

In [5], the author studies two shifted sequences $U_a \pm k$ of the Lucas sequences of the first kind, where $U_a = \{u_n\}_{n\geq 0}$, $a \in \mathbb{Z}$, $u_n = au_{n-1} + u_{n-2}$ for $n \geq 2$, $u_0 = 0$, $u_1 = 1$, and shows that there exist infinitely many integers k such that two sequences are prime free. This result extends previous works for the shifted Fibonacci sequences, when $a = 1$ and $k = 1$.

In [2], the authors mention that the sequences $\{L_n + (-1)^n\}_{n\geq 0}$ and $\{L_n - (-1)^n\}_{n\geq 0}$ are not considered as altered Lucas sequences. Fortunately, in [1], the author also derives GCD sequences $(L_{4n+k-1} + 1, L_{4n+k} + 1)$, $k = 0, 1, 2, 3$, and mentions that if $n \equiv l (\mathrm{mod}\, m)$, $m = 3, 6$ and $l \in \{0, 1, 2, 3, 4, 5\}$, then the sequences $\gcd(L_{4n+k-1} + 1, L_{4n+k} + 1)$, $k = 0, 1, 2, 3$ are constant.

In this study, our goal is to define two altered Lucas sequences, $\{L_n \pm k_1\}_{n\geq 0}$ and $\{L_n \mp k_2\}_{n\geq 0}$, for specific integers k_1 and k_2. Since it is seen that theirs terms have two distinct factors such as the Fibonacci and Lucas numbers, we work out GCD sequences for r-consecutive terms of the altered Lucas sequences. And also, we determine relations between GCD sequences and the Fibonacci or Lucas sequences. In the last part, we establish some r-consecutive GCD shifted sequences from two altered Lucas sequences, and give some properties of them.

2. The altered Lucas sequences

In this section, we define two altered Lucas sequences $\{L_n^+\}_{n\geq 1}$ and $\{L_n^-\}_{n\geq 1}$ by

$$L_n^+ = \begin{cases} L_n - 1, & \text{if } n \text{ is odd}, \\ L_n + 3, & \text{otherwise}, \end{cases} \tag{2.1}$$

$$L_n^- = \begin{cases} L_n + 1, & \text{if } n \text{ is odd}, \\ L_n - 3, & \text{otherwise}. \end{cases} \tag{2.2}$$

Based on the definitions given in (2.1) and (2.2), we can give the first 12 terms of the $\{L_n^+\}_{n\geq 1}$ and $\{L_n^-\}_{n\geq 1}$ in the following table:

(2.3)

n	1	2	3	4	5	6	7	8	9	10	11	12
L_n^+	0	6	3	10	10	21	28	50	75	126	198	325
L_n^-	2	0	5	4	12	15	30	44	77	120	200	319

We see that some interesting observations can be made for L_n^+ and L_n^- given in (2.3) according to both divisibility properties and recurrence relation. For example, the numbers $L_{3n}^{\pm}$ (i.e., L_{3n}^+ and L_{3n}^-) have odd parity, and the numbers $L_{3n+1}^{\pm}$ and $L_{3n+2}^{\pm}$ have even parity. In addition, recurrence relations of $\{L_n^+\}_{n\geq 1}$ and $\{L_n^-\}_{n\geq 1}$ are shown by using $L_{n+1}^{\pm} + L_n^{\pm} = L_{n+2} \pm 2$, namely, the Lucas type recurrence relations are given as

$$L_n^{\pm} + L_{n+1}^{\pm} = \begin{cases} L_{n+2}^{\pm} \pm 3, & \text{if } n \text{ is odd,} \\ L_{n+2}^{\pm} \mp 1, & \text{otherwise,} \end{cases}$$

$$L_{n+1}^{\pm} - L_n^{\pm} = \begin{cases} L_{n-1}^{\pm} \pm 1, & \text{if } n \text{ is odd,} \\ L_{n-1}^{\pm} \mp 3, & \text{otherwise.} \end{cases}$$

Let us take a look at differences $L_{2n+1}^{\pm} - L_{2n-1}^{\pm}$ and $L_{2n+2}^{\pm} - L_{2n}^{\pm}$. It is seen they are the Lucas numbers: $L_{2n+1}^{\pm} - L_{2n-1}^{\pm} = L_{2n}$, $L_{2n+2}^{\pm} - L_{2n}^{\pm} = L_{2n+1}$.

The following equations, which are the relations for the difference and sum of indices of the Lucas numbers given in [6],

$$L_{m+n} + L_{m-n} = \begin{cases} L_m L_n, & \text{if } n \text{ is even,} \\ 5F_m F_n, & \text{otherwise,} \end{cases} \tag{2.4}$$

$$L_{m+n} - L_{m-n} = \begin{cases} 5F_m F_n, & \text{if } n \text{ is even,} \\ L_m L_n, & \text{otherwise,} \end{cases} \tag{2.5}$$

will enable us to determine a number of properties for the altered Lucas sequences.

THEOREM 2.1. *Let L_n^+ and L_n^- be the nth altered Lucas numbers given in* (2.1) *and* (2.2)*, respectively. The following equations are valid:*

$$\begin{aligned} L_{4k}^+ &= 5F_{2k+1}F_{2k-1}, & L_{4k}^- &= L_{2k+1}L_{2k-1}, \\ L_{4k+1}^+ &= 5F_{2k+1}F_{2k}, & L_{4k+1}^- &= L_{2k+1}L_{2k}, \\ L_{4k+2}^+ &= L_{2k+2}L_{2k}, & L_{4k+2}^- &= 5F_{2k+2}F_{2k}, \\ L_{4k+3}^+ &= L_{2k+2}L_{2k+1}, & L_{4k+3}^- &= 5F_{2k+2}F_{2k+1}. \end{aligned}$$

PROOF. By substituting $2k+1$ and $2k-1$ for m and n given in (2.4), $2k+1$ and $2k$ for m and n given in (2.5), respectively, we rewrite equalities into the forms

$$L_{(2k+1)+(2k-1)}+3=5F_{2k+1}F_{2k-1},$$
$$L_{(2k+1)+2k}-1=5F_{2k+1}F_{2k}.$$

Also, the desired results can be given with similar applications taking suitable values for m and n. □

In the rest of this study, similar proofs of all results are generally omitted for the sake of brevity.

Now, we show that the altered Lucas numbers L_n^+ and L_n^- satisfy interrelationships with the Fibonacci and Lucas numbers.

THEOREM 2.2. *If L_n^+ and L_n^- are the nth altered Lucas numbers, then*

$$L_{2n}^+ + L_{2n+1}^+ = \begin{cases} L_{n+1}^2, & \text{if } n \text{ is odd},\\ 5F_{n+1}^2, & \text{otherwise}, \end{cases}$$

$$L_{2n+1}^+ + L_{2n+2}^+ = \begin{cases} L_nL_{n+3}+6, & \text{if } n \text{ is odd},\\ 5F_nF_{n+3}+6, & \text{otherwise}, \end{cases}$$

$$L_{2n}^- + L_{2n+1}^- = \begin{cases} L_{n+1}^2, & \text{if } n \text{ is even},\\ 5F_{n+1}^2, & \text{otherwise}, \end{cases}$$

$$L_{2n+1}^- + L_{2n+2}^- = \begin{cases} L_nL_{n+3}+2, & \text{if } n \text{ is odd},\\ 5F_nF_{n+3}+2, & \text{otherwise}. \end{cases}$$

PROOF. By using the definitions given in (2.1) and (2.2), and all results of Theorem 2.1, we obtain

$$L_{2n}^+ + L_{2n+1}^+ = \begin{cases} L_{n+1}\left(L_n+L_{n-1}\right), & \text{if } n \text{ is odd},\\ 5F_{n+1}\left(F_n+F_{n-1}\right), & \text{otherwise}, \end{cases}$$

$$L_{2n+1}^+ + L_{2n+2}^+ = \begin{cases} L_n\left(L_{n+2}+L_{n+1}\right)+6, & \text{if } n \text{ is odd},\\ 5F_n\left(F_{n+2}+F_{n+1}\right)+6, & \text{otherwise}. \end{cases}$$

□

As an alternative method to the definitions given in (2.1), (2.2) and all results of Theorem 2.1, we investigate a Binet's like formula, which is commonly used in the proof of the properties of the integer sequences. Then, the altered Lucas numbers can be expressed in terms of α and $\beta = -\alpha^{-1}$, where $\alpha = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

Theorem 2.3. *The Binet's like formulas of the numbers L_n^+ and L_n^- are given, respectively, by*

$$L_n^+ = \left(\alpha^{\left\lfloor \frac{n}{2}+1 \right\rfloor} - (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} \beta^{\left\lfloor \frac{n}{2}+1 \right\rfloor}\right)\left(\alpha^{\left\lceil \frac{n}{2}-1 \right\rceil} - (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} \beta^{\left\lceil \frac{n}{2}-1 \right\rceil}\right),$$

$$L_n^- = \left(\alpha^{\left\lfloor \frac{n}{2}+1 \right\rfloor} + (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} \beta^{\left\lfloor \frac{n}{2}+1 \right\rfloor}\right)\left(\alpha^{\left\lceil \frac{n}{2}-1 \right\rceil} + (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} \beta^{\left\lceil \frac{n}{2}-1 \right\rceil}\right),$$

where $\lfloor x \rfloor$ and $\lceil x \rceil$ denote the floor and ceiling integer functions.

Proof. By using the Binet's formulas of the Fibonacci and Lucas numbers, we achieve the desired results. □

3. Properties of the GCD sequences of the altered Lucas sequences

In this section, we consider two greatest common divisor (GCD) sequences, $\{L_{n,r}^+\}_{n\geq 1}$ and $\{L_{n,r}^-\}_{n\geq 1}$, which are called as r-consecutive GCD sequences,

$$L_{n,r}^+ = \gcd\left(L_n^+, L_{n+r}^+\right), \tag{3.1}$$

$$L_{n,r}^- = \gcd\left(L_n^-, L_{n+r}^-\right). \tag{3.2}$$

It is known that the Lucas sequence has some GCD properties such as $(L_m, L_n) \neq L_{(m,n)}$ for $n, m \in \mathbb{Z}^+$, and if $\frac{m}{d}$ and $\frac{n}{d}$ are odd, $(L_m, L_n) = L_d$ and $(F_n, L_n) = 1$ or 2.

Firstly, our aim is to investigate the 1-*consecutive* GCD sequences, $\{L_{n,1}^+\}_{n\geq 0} = \{\gcd(L_n^+, L_{n+1}^+)\}_{n\geq 1}$ and $\{L_{n,1}^-\}_{n\geq 1} = \{\gcd(L_n^-, L_{n+1}^-)\}_{n\geq 1}$, and also to study some properties of them.

The first 14 terms of the sequence $\{L_{n,1}^+\}_{n\geq 0}$ are given with

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$L_{n,1}^+$	$5F_1$	6	L_2	1	$5F_3$	1	L_4	2	$5F_5$	3	L_6	1	$5F_7$	2	L_8

The sequence $\{L^+_{n,1}\}_{n\geq 1}$ is neither constant nor decreasing, or increasing. But, there are some subsequences of the sequence $\{L^+_{n,1}\}_{n\geq 1}$, which are either periodic or increasing. It is seen that the sequence $\{L^+_{2k,1}\}_{k\geq 0}$ includes $\{L_{k+1}\}$ for $k = 1, 3, 5, ...$ and $\{5F_{k+1}\}$ for $k = 0, 2, 4, 6, \ldots$. Also, the sequence $\{L^+_{2k+1,1}\}_{k\geq 0}$ is $\{6, 1, 1, 2, 3, 1, 2, 1, 3, 2, 1, 1\}$ for $k = 0, 1, 2, 3, \ldots, 11$, which is periodic according to $k \equiv 0 - 11 (\bmod 12)$.

Now, according to observations made for the numbers $L^+_{n,1}$, the numbers $L^-_{n,1}$ are given with

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$L^-_{n,1}$	L_1	2	$5F_2$	1	L_3	3	$5F_4$	2	L_5	1	$5F_6$	1	L_7	6	$5F_8$

It is seen that $L^-_{2k,1} = 5F_{k+1}$ for $k = 1, 3, 5, \ldots$, and $L^-_{2k,1} = L_{k+1}$ for $k = 0, 2, 4, 6, \ldots$. Also, the sequence $\{L^-_{2k+1,1}\} = \{2, 1, 3, 2, 1, 1, 6, 1, 1, 2, 3, 1\}$, $k \equiv 0 - 11 \,(\bmod 12)$ is periodic.

Lemma 3.1. *For any integers m and n,*

$$(L_n - L_m - F_{m-1}, L_{n+1} + L_{m-1} + F_{m-2}) = (L_{n-2} - L_{m+2} - F_{m+1}, L_{n-1} + L_{m+1} + F_m)\,. \tag{3.3}$$

Proof. By applying property $(x, y) = (x, y - x)$ for the left hand side of (3.3), we have

$$\begin{aligned}
&\big(L_n - L_m - F_{m-1}, L_{n+1} + L_{m-1} + F_{m-2}\big)\\
&\quad= (L_n - L_m - F_{m-1}, L_{n+1} - L_n + L_{m-1} + L_m + F_{m-2} + F_{m-1})\\
&\quad= (L_n - L_{n-1} - L_m - L_{m+1} - F_{m-1} - F_m, L_{n-1} + L_{m+1} + F_m)\\
&\quad= (L_{n-2} - L_{m+2} - F_{m+1}, L_{n-1} + L_{m+1} + F_m)
\end{aligned}$$

by using $F_{n+1} - F_n = F_{n-1}$ and $L_{n+1} - L_n = L_{n-1}$ given in (1.1). □

Lemma 3.2. *For any integers m and n,*

$$\big(L_n - 1, L_{n+1} + 3\big) = (L_{n-2m} - L_{2m+1} - F_{2m}, L_{n-2m+1} + L_{2m} + F_{2m-1})\,, \tag{3.4}$$

$$\big(L_n + 1, L_{n+1} - 3\big) = (L_{n-2m} + L_{2m+1} + F_{2m}, L_{n-2m+1} - L_{2m} - F_{2m-1})\,. \tag{3.5}$$

PROOF. Note that $F_{-1} = F_1 = 1$ and $F_0 = 0$. Thus, by applying property $(x, y) = (x, y - x)$ for the left hand side of (3.4), we get

$$\begin{aligned}(L_n - 1, L_{n+1} + 3) &= (L_n - F_1 L_1 - F_0 L_2, L_{n+1} + F_0 L_1 + F_{-1} L_2)\\ &= (L_n - F_1 L_1 - F_0 L_2, L_{n-1} + F_2 L_1 + F_1 L_2)\\ &= (L_{n-2} - F_3 - 3F_2, L_{n-1} + F_2 + 3F_1)\,.\end{aligned}$$

By using $L_n = F_{n-1} + F_{n+1}$, we obtain

$$\begin{aligned}(L_{n-2} - F_3 - 3F_2, L_{n-1} + F_2 + 3F_1) &= (L_{n-2} - F_4 - 2F_2, L_{n-1} + F_3 + 2F_1)\\ &= (L_{n-2} - L_3 - F_2, L_{n-1} + L_2 + F_1)\,.\end{aligned} \tag{3.6}$$

The equation in (3.6) is a special case for $m = 1$ of equation given in (3.3). Thus, by applying property $(x, y) = (x, y - x)$, $m-1$ times to (3.6), we achieve the desired result. $\square$

THEOREM 3.3. *Let $L^+_{2k,1}$ and $L^-_{2k,1}$ be the 1-consecutive GCD numbers given in (3.1) and (3.2) with $r = 1$, respectively. Then*

$$L^+_{2k,1} = \begin{cases} L_{k+1}, & \text{for odd } k,\\ 5F_{k+1}, & \text{for even } k,\end{cases} \qquad L^-_{2k,1} = \begin{cases} 5F_{k+1}, & \text{for odd } k,\\ L_{k+1}, & \text{for even } k.\end{cases}$$

PROOF. Since $L^+_{2k,1} = (L^+_{2k}, L^+_{2k+1})$, by applying $k+1$ for m, and $k-1$ and k for n in equations given (2.4) and (2.5), respectively, we can rewrite the values L^+_{2k} and L^+_{2k+1} as

$$L_{(k+1)+(k-1)} + L_{(k+1)-(k-1)} = \begin{cases} L_{k+1}L_{k-1}, & \text{if } k \text{ is odd},\\ 5F_{k-1}F_{k+1}, & \text{otherwise},\end{cases}$$

$$L_{(k+1)+k} - L_{(k+1)-k} = \begin{cases} 5F_kF_{k+1}, & \text{if } k \text{ is even},\\ L_{k+1}L_k, & \text{otherwise}.\end{cases}$$

Since $(L_k, L_{k-1}) = 1$ and $(F_k, F_{k-1}) = 1$, (L^+_{2k}, L^+_{2k+1}) is L_{k+1} or $5F_{k+1}$. The other equation is shown with a similar way. $\square$

THEOREM 3.4. *If $L^{+}_{2k-1,1}$ and $L^{-}_{2k-1,1}$ are the $(2k-1)$th entries of the 1-consecutive GCD sequences, respectively, then $L^{+}_{2k-1,1}$ and $L^{-}_{2k-1,1}$ are periodic such as*

$$L^{+}_{2k-1,1}=\begin{cases}1, & k\equiv 0,2,3,6,8,11 \pmod{12},\\ 2, & k\equiv 4,7,10 \pmod{12},\\ 3, & k\equiv 5,9 \pmod{12},\\ 6, & k\equiv 1 \pmod{12},\end{cases}$$

$$L^{-}_{2k-1,1}=\begin{cases}1, & k\equiv 0,2,5,6,8,9 \pmod{12},\\ 2, & k\equiv 1,4,10 \pmod{12},\\ 3, & k\equiv 3,11 \pmod{12},\\ 6, & k\equiv 7 \pmod{12}.\end{cases}$$

PROOF. Since $L^{+}_{2k-1,1}=(L_{2k-1}-1,L_{2k}+3)$, firstly, for an even k, we can write (3.4) with $n=2k-1$ and $m=\frac{k}{2}$ as

$$\begin{aligned}(L_{2k-1}-1,L_{2k}+3)&=(L_{k-1}-L_{k+1}-F_k,2L_k+F_{k-1})\\ &=(-L_k-F_k,2L_k+F_{k-1}).\end{aligned}$$

By using properties $L_k=F_{k+1}+F_{k-1}$ and $(x,y)=(x,y+zx)$, we have

$$\begin{aligned}L^{+}_{2k-1,1}&=(-2F_{k+1},2F_{k+1}+3F_{k-1})\\ &=(-2F_{k+1},3F_{k-1}).\end{aligned}$$

Since $(F_{k+1},3)=1$ for even k, it is valid $(-2F_{k+1},3F_{k-1})=(2,F_{k-1})$. Thus, $L^{+}_{2k-1,1}$ is 1 or 2.

Secondly, for an odd k, we can write (3.4) with $n=2k-1$ and $m=\frac{k-1}{2}$ as

$$(L_{2k-1}-1,L_{2k}+3)=(-F_{k-1},L_{k+1}+L_{k-1}+F_{k-2}).$$

By using properties $5F_k=L_{k+1}+L_{k-1}$ and $(x,y)=(x,y+zx)$, we have

$$\begin{aligned}L^{+}_{2k-1,1}&=(-F_{k-1},5F_{k-1}+6F_{k-2})\\ &=(-F_{k-1},6F_{k-2}).\end{aligned}$$

It follows $(-F_{k-1}, 6F_{k-2}) = (F_{k-1}, 6)$, so $L^+_{2k-1,1}$ is one of the entries of $\{1, 2, 3, 6\}$ for odd k. In both cases, the following properties are valid

$$(2, F_k) = 2 \text{ if and only if } k \equiv 0 \,(\mathrm{mod}\, 3)\,,$$

$$(3, F_k) = 3 \text{ if and only if } k \equiv 0 \,(\mathrm{mod}\, 4)\,,$$

$$(6, F_k) = 6 \text{ if and only if } k \equiv 0 \,(\mathrm{mod}\, 12)\,.$$

Thus, in case $(F_{k-1}, 6) = 6$, for $k \equiv 1(\mathrm{mod}\, 12)$, it is clear that $(L^+_{2k-1}, L^+_{2k}) = 6$. If $(F_{k-1}, 6) = 3$, $k \neq 1$, for $k \equiv 1(\mathrm{mod}\, 4)$ for odd k, that is $k = 4l + 1$, for $k \equiv 5, 9(\mathrm{mod}\, 12)$, then $(L^+_{2k-1}, L^+_{2k}) = 3$. Now, assume $(F_{k-1}, 6) = 2$, for $k \equiv 1(\mathrm{mod}\, 3)$ for odd k, that is $k = 3m + 1$, for $k \equiv 7(\mathrm{mod}\, 12)$, then $(L^+_{2k-1}, L^+_{2k}) = 2$. Finally, in the cases $k \equiv 3, 11(\mathrm{mod}\, 12)$, we have $(F_{k-1}, 6) = 1$. Suppose that $(2, F_{k-1}) = 2$, $k \equiv 1(\mathrm{mod}\, 3)$ for even k, that is $k = 3s + 1$, for $k \equiv 4, 10(\mathrm{mod}\, 12)$, it is clear that $(L^+_{2k-1}, L^+_{2k}) = 2$. Otherwise, in cases $k \equiv 0(\mathrm{mod}\, 3)$ and $k \equiv 2(\mathrm{mod}\, 3)$, it is $(2, F_{k-1}) = 1$, for $k \equiv 0, 6, (\mathrm{mod}\, 12)$ and $k \equiv 2, 8, (\mathrm{mod}\, 12)$, respectively. All results complete the proof for all cases of $L^+_{2k-1,1} = (L^+_{2k-1}, L^+_{2k})$.

Now, since $L^-_{2k-1,1} = (L^-_{2k-1}, L^-_{2k})$, we suppose for even k, $n = 2k - 1$ and $m = \frac{k}{2}$ given in (3.5):

$$(L_{k-1} + L_{k+1} + F_k, -F_{k-1}) = (2F_k, -F_{k-1})\,.$$

And also, we assume for odd k, $n = 2k - 1$ and $m = \frac{k+1}{2}$ given in (3.5):

$$\begin{aligned}(L_{k-2} + L_{k+2} + F_{k+1}, L_{k-1} - L_{k+1} - F_k) &= (3L_k + F_{k+1}, -L_k - F_k)\\ &= (3F_{k-1} + 4F_{k+1}, -2F_{k+1})\\ &= (3F_{k-1}, -2F_{k+1})\,.\end{aligned}$$

Depending on whether k is odd or even, the calculations of expressions $L^-_{2k-1,1} = (2F_k, -F_{k-1})$ and $L^-_{2k-1,1} = (3F_{k-1}, -2F_{k+1})$ can be made with similar methods. □

As a brief summary of the mentioned above, the sequence $\{L^+_{4k-2,1}\}_{k\geq 1} = \{\gcd(L^+_{4k-2}, L^+_{4k-1})\}_{k\geq 1}$ is $\{L_{2k}\}_{k\geq 1}$, and the sequence $\{L^+_{4k,1}\}_{k\geq 1}$ is $\{5F_{2k+1}\}_{k\geq 1}$. And also, $\{L^-_{4k-2,1}\}_{k\geq 1} = \{5F_{2k}\}_{k\geq 1}$ and $\{L^-_{4k,1}\}_{k\geq 1} = \{L_{2k+1}\}_{k\geq 1}$. These results given in the following lemma are consequences of Theorem 3.3.

Lemma 3.5. *Let $L^+_{n,1}$ and $L^-_{n,1}$ be the nth numbers of 1-consecutive GCD sequences. Then*

$$\begin{aligned} L^+_{4k,1} &= 5F_{2k+1}, \quad L^-_{4k,1} = L_{2k+1}, \\ L^+_{4k+2,1} &= L_{2k+2}, \quad L^-_{4k+2,1} = 5F_{2k+2}. \end{aligned}$$

In addition, the $\{L^+_{4k+1,1}\}_{k\geq 1} = \{6, 1, 3, 2, 3, 1\}$, $k \in Z_6$ is periodic; that is $L^+_{4k+1,1} = 6$ iff $k \equiv 0 \pmod 6$, $L^+_{4k+1,1} = 1$ iff $k \equiv 1 \pmod 6$ and so on, respectively. The sequence $\{L^+_{4k-1,1}\}_{k\geq 1} = \{1, 2, 1, 1, 2, 1\}$, $k \in Z_6$ is periodic. The sequence $\{L^-_{4k+1,1}\}_{k\geq 1} = \{2, 3, 1, 6, 1, 3\}$, $k \in Z_6$ is periodic. In addition, the $\{L^-_{4k-1,1}\}_{k\geq 1} = \{1, 2, 1\}$, $k \in Z_3$ is also periodic. These results given in the following lemma are consequences of Theorem 3.4.

Lemma 3.6. *Let $L^+_{n,1}$ and $L^-_{n,1}$ be the nth numbers of 1-consecutive GCD sequences, $L^\pm_{n,1}$ denotes both the numbers $L^+_{n,1}$ and $L^-_{n,1}$. Then*

$$L^+_{4k+1,1} = \begin{cases} 6, & k = 0 \pmod 6, \\ 3, & k = 2, 4 \pmod 6, \\ 2, & k = 3 \pmod 6, \\ 1, & k = 1, 5 \pmod 6, \end{cases}$$

$$L^-_{4k+1,1} = \begin{cases} 6, & k = 4 \pmod 6, \\ 3, & k = 0, 2 \pmod 6, \\ 2, & k = 1 \pmod 6, \\ 1, & k = 3, 5 \pmod 6, \end{cases}$$

and

$$L^\pm_{4k+3,1} = \begin{cases} 2, & k = 1 \pmod 3, \\ 1, & \textit{otherwise.} \end{cases}$$

It is well known that $(F_n, F_{n+2}) = 1$ and $(L_n, L_{n+2}) = 1$. Similarly, sequences $\{L^+_{n,2}\}_{k\geq 1}$ and $\{L^-_{n,2}\}_{k\geq 1}$ are obtained as the periodic constant sequences.

Theorem 3.7. *Let $L^+_{n,2}$ and $L^-_{n,2}$ be the nth 2-consecutive GCD numbers. Then*

$$L^+_{4k,2} = L^+_{4k+3,2} = L^-_{4k+3,2} = \begin{cases} 2, & k \equiv 2 \pmod 3, \\ 1, & \textit{otherwise,} \end{cases}$$

$$L^+_{4k+2,2} = \begin{cases} 2, & k \equiv 0 \,(\mathrm{mod}\, 3)\,, \\ 1, & \textit{otherwise}, \end{cases} \qquad L^-_{4k,2} = \begin{cases} 4, & k \equiv 2 \,(\mathrm{mod}\, 3)\,, \\ 1, & \textit{otherwise}, \end{cases}$$

$$L^-_{4k+2,2} = \begin{cases} 4, & k \equiv 0 \,(\mathrm{mod}\, 3)\,, \\ 1, & \textit{otherwise}, \end{cases}$$

$$L^-_{4k+1,2} = \begin{cases} 1, & k \equiv 0, 2 \,(\mathrm{mod}\, 6)\,, \\ 2, & k \equiv 4 \,(\mathrm{mod}\, 6)\,, \\ 3, & k \equiv 3, 5 \,(\mathrm{mod}\, 6)\,, \\ 6, & k \equiv 1 \,(\mathrm{mod}\, 6)\,, \end{cases} \qquad L^+_{4k+1,2} = \begin{cases} 1, & k \equiv 3, 5 \,(\mathrm{mod}\, 6)\,, \\ 2, & k \equiv 1 \,(\mathrm{mod}\, 6)\,, \\ 3, & k \equiv 0, 2 \,(\mathrm{mod}\, 6)\,, \\ 6, & k \equiv 4 \,(\mathrm{mod}\, 6)\,. \end{cases}$$

Proof. From $L^+_{4k,2} = (L^+_{4k}, L^+_{4k+2})$ and $L^-_{4k+2,2} = (L^-_{4k+2}, L^-_{4k+4})$, we get

$$\begin{aligned} \big(L_{4k} + 3, L_{4k+2} + 3\big) &= (5F_{2k+1}F_{2k-1}, L_{2k+2}L_{2k}) \\ &= (5F_{2k+1}, L_{2k+2})\,(F_{2k-1}, L_{2k})\,(5F_{2k+1}, L_{2k})\,(F_{2k-1}, L_{2k+2}) \\ &= (F_{2k-1}, F_{2k+3} + F_{2k+1}) = (F_{2k-1}, 4F_{2k}) \end{aligned}$$

and

$$\begin{aligned} \big(L_{4k+2} - 3, L_{4k+4} - 3\big) &= (5F_{2k+2}F_{2k}, L_{2k+3}L_{2k+1}) \\ &= (5F_{2k+2}, L_{2k+3})\,(F_{2k}, L_{2k+1})\,(5F_{2k+2}, L_{2k+1})\,(F_{2k}, L_{2k+3}) \\ &= (F_{2k}, F_{2k+4} + F_{2k+2}) = (F_{2k}, 4F_{2k+1})\,. \end{aligned}$$

By using the properties $(2, F_k) = 2$ if and only if $k \equiv 0 \,(\mathrm{mod}\, 3)$ and $(4, F_k) = 4$ if and only if $k \equiv 0 \,(\mathrm{mod}\, 6)$, we obtain $L^+_{4k,2} = 2$ iff $k \equiv 2 \,(\mathrm{mod}\, 3)$ and $L^-_{4k+2,2} = 4$ iff $k \equiv 0 \,(\mathrm{mod}\, 3)$, then the desired results are found. The other properties are obtained in a similar way by using $(3, F_k) = 3$ if and only if $k \equiv 0 \,(\mathrm{mod}\, 4)$. □

It is well known that $(F_n, F_{n+3}) = 2$ and $(L_n, L_{n+3}) = 2$ iff $n \equiv 0 \,(\mathrm{mod}\, 3)$, otherwise $(F_n, F_{n+3}) = (L_n, L_{n+3}) = 1$. And, sequences $\{L^+_{n,3}\}_{k\geq 1}$ and $\{L^-_{n,3}\}_{k\geq 1}$ are established by Theorem 3.8.

Theorem 3.8. *Let $L^+_{n,3}$ and $L^-_{n,3}$ be the nth 3-consecutive GCD numbers. Then*

$$L^+_{4k+1,3} = \begin{cases} 10F_{2k+1}, & k \equiv 0 \,(\mathrm{mod}\, 3)\,, \\ 5F_{2k+1}, & \textit{otherwise}, \end{cases}$$

$$L^{+}_{4k+3,3}=\begin{cases}2L_{2k+2}, & k\equiv 1\,(\mathrm{mod}\,3)\,,\\ L_{2k+2}, & \textit{otherwise},\end{cases}$$

$$L^{-}_{4k+1,3}=\begin{cases}2L_{2k+1}, & k\equiv 0\,(\mathrm{mod}\,3)\,,\\ L_{2k+1}, & \textit{otherwise},\end{cases}$$

$$L^{-}_{4k+3,3}=\begin{cases}10F_{2k+2}, & k\equiv 1\,(\mathrm{mod}\,3)\,,\\ 5F_{2k+2}, & \textit{otherwise}.\end{cases}$$

Proof. From $L^{+}_{4k+1,3}=\left(L^{+}_{4k+1},L^{+}_{4k+4}\right)$ and $L^{-}_{4k+1,3}=\left(L^{-}_{4k+1},L^{-}_{4k+4}\right)$, we get

$$\left(L_{4k+1}-1,L_{4k+4}+3\right)=5F_{2k+1}\left(F_{2k},F_{2k+3}\right),$$

$$\left(L_{4k+1}+1,L_{4k+4}-3\right)=L_{2k+1}\left(L_{2k},L_{2k+3}\right).$$

Thus, the properties $(F_n,F_{n+3})=2$ and $(L_n,L_{n+3})=2$ iff $n\equiv 0\,(\mathrm{mod}\,3)$ complete the proof. □

Theorem 3.9. *Let $L^{+}_{n,3}$ and $L^{-}_{n,3}$ be the nth 3-consecutive GCD numbers. Then*

$$L^{+}_{4k,3}=\begin{cases}1, & k\equiv 0,3\,(\mathrm{mod}\,6)\,,\\ 2, & k\equiv 1,2,4,5\,(\mathrm{mod}\,6)\,,\end{cases}\qquad L^{-}_{4k,3}=\begin{cases}1, & k\equiv 0,3\,(\mathrm{mod}\,6)\,,\\ 2, & k\equiv 1,4\,(\mathrm{mod}\,6)\,,\\ 4, & k\equiv 2,5\,(\mathrm{mod}\,6)\,,\end{cases}$$

$$L^{+}_{4k+2,3}=\begin{cases}1, & k\equiv 4\,(\mathrm{mod}\,6)\,,\\ 2, & k\equiv 0,2\,(\mathrm{mod}\,6)\,,\\ 3, & k\equiv 1\,(\mathrm{mod}\,6)\,,\\ 6, & k\equiv 3,5\,(\mathrm{mod}\,6)\,,\end{cases}\qquad L^{-}_{4k+2,3}=\begin{cases}1, & k\equiv 1\,(\mathrm{mod}\,6)\,,\\ 2, & k\equiv 5\,(\mathrm{mod}\,6)\,,\\ 3, & k\equiv 4\,(\mathrm{mod}\,6)\,,\\ 4, & k\equiv 3\,(\mathrm{mod}\,6)\,,\\ 6, & k\equiv 2,\,(\mathrm{mod}\,6)\,,\\ 12, & k\equiv 0\,(\mathrm{mod}\,6)\,.\end{cases}$$

Proof. Since $L^{+}_{4k+2,3}=(L^{+}_{4k+2},L^{+}_{4k+5})$ and $L^{-}_{4k,3}=(L^{-}_{4k},L^{-}_{4k+3})$, by applying appropriate values of equations given in (3.4) and (3.5), we obtain proofs of all results. □

Since $(F_n,F_{n+4})=(F_n,3F_{n+1})$ and $(L_n,L_{n+4})=(L_n,3L_{n+1})$, it is seen that $(F_n,F_{n+4})=3$ iff $n\equiv 0\,(\mathrm{mod}\,4)$ and $(L_n,L_{n+4})=3$ iff $n\equiv 2\,(\mathrm{mod}\,4)$, otherwise it equals to 1. Now, we give the form of the sequences $\{L^{+}_{n,4}\}_{n\geq 1}$ and $\{L^{-}_{n,4}\}_{n\geq 1}$.

THEOREM 3.10. *Let $L^{+}_{n,4}$ and $L^{-}_{n,4}$ be the nth 4-consecutive GCD numbers. Then*

$$L^{+}_{4k,4} = 5F_{2k+1}, \quad L^{-}_{4k,4} = L_{2k+1},$$

$$L^{+}_{4k+2,4} = \begin{cases} L_{2k+2}, & k \equiv 0 \pmod 2, \\ 3L_{2k+2}, & \textit{otherwise}, \end{cases}$$

$$L^{-}_{4k+2,4} = \begin{cases} 15F_{2k+2}, & k \equiv 0 \pmod 2, \\ 5F_{2k+2}, & \textit{otherwise}. \end{cases}$$

PROOF. From $L^{-}_{4k+2,4} = (L^{-}_{4k+2}, L^{-}_{4(k+1)+2})$, it follows that $L^{-}_{4k+2,4} = (L_{4k+2}-3, L_{4(k+1)+2}-3)$ and $L^{-}_{4k+2,4} = 5F_{2k+2}(F_{2k}, 3F_{2k+1})$. Since $(F_{2k}, 3) = 3$ iff $k \equiv 0 \pmod 2$, we achieve desired results. □

LEMMA 3.11. *Let $L^{+}_{n,4}$ and $L^{-}_{n,4}$ be the nth 4-consecutive GCD numbers. Then*

$$L^{+}_{4k+1,4} = 5L^{-}_{4k+1,4} = \begin{cases} 5, & k \equiv 1, 2 \pmod 3, \\ 10, & k \equiv 0 \pmod 3, \end{cases}$$

$$5L^{+}_{4k+3,4} = L^{-}_{4k+3,4} = \begin{cases} 5, & k \equiv 0, 2 \pmod 3, \\ 10, & k \equiv 1 \pmod 3. \end{cases}$$

PROOF. Since $L^{+}_{4k+1,4} = (L^{+}_{4k+1}, L^{+}_{4k+5})$ and $L^{-}_{4k+3,4} = (L^{-}_{4k+3}, L^{-}_{4(k+1)+3})$, by applying Theorem 2.1, we get the desired results. □

Now, in addition to the sequences $L^{+}_{n,r}$ and $L^{-}_{n,r}$ defined in (3.1) and (3.2), by selecting r-consecutive elements as mixed from the numbers L^{+}_{n} and L^{-}_{n} defined in (2.1) and (2.2), we establish two different *GCD* sequences of the altered Lucas sequences such as

$$\left\{L^{+\,-}_{n,r}\right\}_{n\geq 1} = \left\{\gcd\left(L^{+}_{n}, L^{-}_{n+r}\right)\right\}_{n\geq 1},$$

$$\left\{L^{-\,+}_{n,r}\right\}_{n\geq 1} = \left\{\gcd\left(L^{-}_{n}, L^{+}_{n+r}\right)\right\}_{n\geq 1}.$$

It is well known that $(F_n, L_n) = 2$ if and only if $3|n$, otherwise $(F_n, L_n) = 1$. Similarly, the sequence $\{L^{+\,-}_{n,0}\}_{n\geq 1}$ (or $\{L^{-\,+}_{n,0}\}_{n\geq 1}$), i.e. 0-consecutive *GCD* sequence, is a constant periodic sequence.

LEMMA 3.12. *Let* $L_{n,0}^{+\,-}=L_{n,0}^{-\,+}=\gcd(L_n^+,L_n^-)$ *be the nth* 0*-consecutive GCD numbers. Then*

$$L_{4k,0}^{+\,-}=\begin{cases}1, & k\equiv 0\,(\mathrm{mod}\,3)\,,\\ 2, & \textit{otherwise},\end{cases}\qquad L_{4k+1,0}^{+\,-}=\begin{cases}1, & k\equiv 2\,(\mathrm{mod}\,3)\,,\\ 2, & \textit{otherwise},\end{cases}$$

$$L_{4k+2,0}^{+\,-}=\begin{cases}3, & k\equiv 1\,(\mathrm{mod}\,3)\,,\\ 6, & \textit{otherwise},\end{cases}\qquad L_{4k+3,0}^{+\,-}=\begin{cases}1, & k\equiv 0\,(\mathrm{mod}\,3)\,,\\ 2, & \textit{otherwise}.\end{cases}$$

PROOF. Since $L_{4k,0}^{+\,-}=(L_{4k}^+,L_{4k}^-)$ and $L_{4k+1,0}^{+\,-}=(L_{4k+1}^+,L_{4k+1}^-)$, by applying Theorem 2.1 with appropriate values, we can write

$$\begin{aligned}L_{4k,0}^{+\,-}&=(5F_{2k+1}F_{2k-1},L_{2k+1}L_{2k-1})\\&=(5F_{2k+1},L_{2k+1})\,(F_{2k-1},L_{2k-1})\,.\end{aligned}$$

By using $(L_{2k+1},2)=2$ if and only if $k\equiv 1(\mathrm{mod}\,3)$, and $(F_{2k-1},2)=2$ if and only if $k\equiv 2(\mathrm{mod}\,3)$, others cases are 1, we achieve desired result. The other results are produced with similar ways. □

Firstly, we have not encountered in the literature with (F_n,L_{n+1}) and (F_{n+1},L_n), but, we can write $(F_n,F_n+F_{n+2})=1$ and $(F_{n+1},F_{n-1}+F_{n+1})=1$, respectively. Therefore, we study on 1-consecutive GCD sequences.

THEOREM 3.13. *Let* $L_{n,1}^{+\,-}$ *and* $L_{n,1}^{-\,+}$ *be the nth numbers of* 1*-consecutive GCD sequences. Then*

$$L_{4k+1,1}^{+\,-}=5F_{2k},\quad L_{4k+1,1}^{-\,+}=L_{2k},$$

$$L_{4k+3,1}^{+\,-}=L_{2k+1},\quad L_{4k+3,1}^{-\,+}=5F_{2k+1}.$$

PROOF. From the definitions given in (2.1), (2.2) and Theorem 2.1, we have

$$\begin{aligned}\left(L_{4k+1}^+,L_{4k+2}^-\right)&=(L_{4k+1}-1,L_{4k+2}-3)=5F_{2k}\,(F_{2k+1},F_{2k+2})\,,\\ \left(L_{4k+1}^-,L_{4k+2}^+\right)&=(L_{4k+1}+1,L_{4k+2}+3)=L_{2k}\,(L_{2k+1},L_{2k+2})\,.\end{aligned}$$

Thus, all results are obtained, since $(F_{2k+1},F_{2k+2})=1=(L_{2k+1},L_{2k+2})$. □

Lemma 3.14. *If $L^{+\ -}_{n,1}$ and $L^{-\ +}_{n,1}$ are the nth 1-consecutive GCD numbers, then*

$$L^{\pm\ \mp}_{4k,1} = \begin{cases} 2, & k \equiv 1 \pmod 3, \\ 1, & \textit{otherwise}, \end{cases}$$

$$L^{+\ -}_{4k+2,1} = \begin{cases} 1, & k \equiv 0, 4 \pmod 6, \\ 2, & k \equiv 2 \pmod 6, \\ 3, & k \equiv 1, 3 \pmod 6, \\ 6, & k \equiv 5 \pmod 6, \end{cases} \qquad L^{-\ +}_{4k+2,1} = \begin{cases} 1, & k \equiv 1, 3 \pmod 6, \\ 2, & k \equiv 5 \pmod 6, \\ 3, & k \equiv 0, 4 \pmod 6, \\ 6, & k \equiv 2 \pmod 6. \end{cases}$$

Secondly, $(F_n, L_{n+2}) = (F_n, 3F_{n+1})$ and $(F_{n+2}, L_n) = (F_{n-2}, 3F_{n-1})$ give $(F_n, L_{n+2}) = 3$ iff $n \equiv 0 \pmod 4$ and $(F_{n+2}, L_n) = 3$ iff $n \equiv 2 \pmod 4$, otherwise, it equals to 1. Now, we study on numbers $L^{+\ -}_{n,2}$ and $L^{-\ +}_{n,2}$.

Theorem 3.15. *Let $L^{+\ -}_{n,2}$ and $L^{-\ +}_{n,2}$ be the nth numbers of 2-consecutive GCD sequences. Then*

$$L^{+\ -}_{4k+1,2} = 5F_{2k+1}, \quad L^{-\ +}_{4k+1,2} = L_{2k+1},$$

$$L^{+\ -}_{4k+3,2} = L_{2k+2}, \quad L^{-\ +}_{4k+1,2} = 5F_{2k+2}.$$

Proof. From the definitions given in (2.1), (2.2) and Theorem 2.1, we have

$$\left(L^{+}_{4k+1}, L^{-}_{4k+3}\right) = 5F_{2k+1} \left(F_{2k}, F_{2k+1} + F_{2k}\right).$$

Since $(F_{2k}, F_{2k+1}) = 1$ and $(x, y) = (x, y - x)$, we have the proof of the first of our equalities. The proofs of the remaining properties are similar. □

Lemma 3.16. *If $L^{+\ -}_{n,2}$ and $L^{-\ +}_{n,2}$ are the nth 2-consecutive GCD numbers, then*

$$L^{+\ -}_{4k,2} = 5L^{-\ +}_{4k,2} = \begin{cases} 10, & k \equiv 2 \pmod 3, \\ 5, & \textit{otherwise}, \end{cases}$$

$$L^{-\ +}_{4k+2,2} = 5L^{+\ -}_{4k+2,2} = \begin{cases} 10, & k \equiv 0 \pmod 3, \\ 5, & \textit{otherwise}. \end{cases}$$

As the third, since $(F_n, L_{n+3}) = (F_n, 4F_{n+1})$ and $(F_{n+3}, L_n) = (2F_n, L_n)$, it is seen that $(F_n, L_{n+3}) = 4$ iff $n \equiv 0 \pmod 6$ and (F_{n+3}, L_n) is 4 iff $n \equiv 3 \pmod 6$ or 2 iff $n \equiv 0 \pmod 6$, otherwise, $(F_{n+3}, L_n) = (F_n, L_{n+3}) = 1$. So, we derive numbers $L_{n,3}^{+\,-}$ and $L_{n,3}^{-\,+}$.

Theorem 3.17. *Let $L_{n,3}^{+\,-}$ and $L_{n,3}^{-\,+}$ be the nth numbers of 3-consecutive GCD sequences. Then*

$$L_{4k,3}^{+\,-} = \begin{cases} 10F_{2k+1}, & k \equiv 2 \pmod 3, \\ 5F_{2k+1}, & \textit{otherwise}, \end{cases} \qquad L_{4k+2,3}^{+\,-} = \begin{cases} 2L_{2k+2}, & k \equiv 0 \pmod 3, \\ L_{2k+2}, & \textit{otherwise}, \end{cases}$$

$$L_{4k,3}^{-\,+} = \begin{cases} 2L_{2k+1}, & k \equiv 2 \pmod 3, \\ L_{2k+1}, & \textit{otherwise}, \end{cases} \qquad L_{4k+2,3}^{-\,+} = \begin{cases} 10F_{2k+2}, & k \equiv 0 \pmod 3, \\ 5F_{2k+2}, & \textit{otherwise}. \end{cases}$$

Proof. Since $(L_{4k}^{+}, L_{4k+3}^{-}) = (L_{4k} + 3, L_{4k+3} + 1)$, by using Theorem 2.1, we have

$$\begin{aligned} (L_{4k} + 3, L_{4k+3} + 1) &= (5F_{2k+1}F_{2k-1}, 5F_{2k+2}F_{2k+1}) \\ &= 5F_{2k+1}\,(F_{2k-1}, F_3) \\ &= 5F_{2k+1}F_{(2k-1,3)}. \end{aligned}$$

Since $\left(L_{4k}^{-}, L_{4k+3}^{+}\right) = (L_{4k} - 3, L_{4k+3} - 1)$, we get

$$\begin{aligned} (L_{4k} - 3, L_{4k+3} - 1) &= (L_{2k+1}L_{2k-1}, L_{2k+2}L_{2k+1}) \\ &= L_{2k+1}\,(L_{2k-1}, 2L_{2k}) \\ &= L_{2k+1}\,(L_{2k-1}, 2)\,. \end{aligned}$$

Because the other proofs are similar, we omit them. □

Lemma 3.18. *If $L_{n,3}^{+\,-}$ and $L_{n,3}^{-\,+}$ are the nth 3-consecutive GCD numbers, then*

$$L_{4k+1,3}^{+\,-} = \begin{cases} 1, & k \equiv 2 \pmod 3, \\ 2, & k \equiv 1 \pmod 3, \\ 4, & k \equiv 0 \pmod 3, \end{cases} \qquad L_{4k+1,3}^{-\,+} = \begin{cases} 1, & k \equiv 2 \pmod 3, \\ 2, & \textit{otherwise}, \end{cases}$$

$$L_{4k+3,3}^{+\,-}=\begin{cases}1, & k\equiv 3\,(\mathrm{mod}\,6)\,,\\ 2, & k\equiv 5\,(\mathrm{mod}\,6)\,,\\ 3, & k\equiv 0\,(\mathrm{mod}\,6)\,,\\ 4, & k\equiv 1\,(\mathrm{mod}\,6)\,,\\ 6, & k\equiv 2\,(\mathrm{mod}\,6)\,,\\ 12, & k\equiv 4\,(\mathrm{mod}\,6)\,,\end{cases}\qquad L_{4k+3,3}^{-\,+}=\begin{cases}1, & k\equiv 0\,(\mathrm{mod}\,6)\,,\\ 2, & k\equiv 2,4\,(\mathrm{mod}\,6)\,,\\ 3, & k\equiv 3\,(\mathrm{mod}\,6)\,,\\ 6, & k\equiv 1,5\,(\mathrm{mod}\,6)\,.\end{cases}$$

Finally, we establish 4-consecutive GCD sequences $\{L_{n,4}^{+\,-}\}_{n\geq 1}$ and $\{L_{n,4}^{-\,+}\}_{n\geq 1}$.

Lemma 3.19. *Let $L_{n,4}^{+\,-}$ and $L_{n,4}^{-\,+}$ be the nth 4-consecutive GCD numbers. Then*

$$L_{4k,4}^{\pm\,\mp}=\begin{cases}2, & k\equiv 1\,(\mathrm{mod}\,3)\,,\\ 1, & otherwise,\end{cases}$$

$$L_{4k+1,4}^{+\,-}=\begin{cases}1, & k\equiv 1,5\,(\mathrm{mod}\,6)\,,\\ 3, & k\equiv 2,4\,(\mathrm{mod}\,6)\,,\\ 4, & k\equiv 3\,(\mathrm{mod}\,6)\,,\\ 12, & k\equiv 0\,(\mathrm{mod}\,6)\,,\end{cases}\qquad L_{4k+3,4}^{+\,-}=\begin{cases}1, & k\equiv 3,5\,(\mathrm{mod}\,6)\,,\\ 3, & k\equiv 0,2\,(\mathrm{mod}\,6)\,,\\ 4, & k\equiv 1\,(\mathrm{mod}\,6)\,,\\ 12, & k\equiv 4\,(\mathrm{mod}\,6)\,,\end{cases}$$

$$L_{4k+1,4}^{-\,+}=\begin{cases}1, & k\equiv 2,4\,(\mathrm{mod}\,6)\,,\\ 2, & k\equiv 0\,(\mathrm{mod}\,6)\,,\\ 3, & k\equiv 1,5\,(\mathrm{mod}\,6)\,,\\ 6, & k\equiv 3\,(\mathrm{mod}\,6)\,,\end{cases}\qquad L_{4k+3,4}^{-\,+}=\begin{cases}1, & k\equiv 0,2\,(\mathrm{mod}\,6)\,,\\ 2, & k\equiv 4\,(\mathrm{mod}\,6)\,,\\ 3, & k\equiv 3,5\,(\mathrm{mod}\,6)\,,\\ 6, & k\equiv 1\,(\mathrm{mod}\,6)\,,\end{cases}$$

$$L_{4k+2,4}^{+\,-}=\begin{cases}3, & k\equiv 0,1,3,4,7,9\,(\mathrm{mod}\,12)\,,\\ 6, & k\equiv 5,8,11\,(\mathrm{mod}\,12)\,,\\ 21, & k\equiv 6,10\,(\mathrm{mod}\,12)\,,\\ 42, & k\equiv 2\,(\mathrm{mod}\,12)\,,\end{cases}$$

$$L_{4k+2,4}^{-\,+}=\begin{cases}3, & k\equiv 1,3,6,7,9,10\,(\mathrm{mod}\,12)\,,\\ 6, & k\equiv 2,5,11\,(\mathrm{mod}\,12)\,,\\ 21, & k\equiv 0,4\,(\mathrm{mod}\,12)\,,\\ 42, & k\equiv 8\,(\mathrm{mod}\,12)\,.\end{cases}$$

4. Conclusion

In this study, two altered Lucas sequences $\{L_n^+\}_{n\geq 1}$ and $\{L_n^-\}_{n\geq 1}$ are derived by altering the Lucas numbers with $\{\pm 1, \pm 3\}$. Thus, the $\{L_n^+\}_{n\geq 1}$ and $\{L_n^-\}_{n\geq 1}$ sequences are separated from the shifted and altered sequences in the literature. But, the L_n^+ and L_n^- are related to the Fibonacci and Lucas numbers. It is seen that they have two different Fibonacci and Lucas factors. Therefore, we study several different type $r-$consecutive GCD sequences, $\{L_{n,r}^+\}_{n\geq 1}$ and $\{L_{n,r}^-\}_{n\geq 1}$ for the altered Lucas sequences $\{L_n^+\}_{n\geq 1}$ and $\{L_n^-\}_{n\geq 1}$, respectively. According to values r, it is seen that these sequences are periodic or unbounded. But for now, we leave other properties of the altered Lucas and r-consecutive GCD sequences for researches in the future.

References

[1] K.-W. Chen, *Greatest common divisors in shifted Fibonacci sequences*, J. Integer. Seq. **14** (2011), no. 4, Article 11.4.7, 8 pp.
[2] U. Dudley and B. Tucker, *Greatest common divisors in altered Fibonacci sequences*, Fibonacci Quart. **9** (1971), no. 1, 89–91.
[3] L. Hajdu and M. Szikszai, *On the GCD-s of k consecutive terms of Lucas sequences*, J. Number Theory **132** (2012), no. 12, 3056–3069.
[4] S. Hernández and F. Luca, *Common factors of shifted Fibonacci numbers*, Period. Math. Hungar. **47** (2003), no. 1-2, 95–110.
[5] L. Jones, *Primefree shifted Lucas sequences*, Acta Arith. **170** (2015), no. 3, 287–298.
[6] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley-Interscience, New York, 2001.
[7] A. Rahn and M. Kreh, *Greatest common divisors of shifted Fibonacci sequences revisited*, J. Integer. Seq. **21** (2018), no. 6, Art. 18.6.7, 12 pp.
[8] J. Spilker, *The GCD of the shifted Fibonacci sequence*, in: J. Sander et al. (eds.), *From Arithmetic to Zeta-Functions*, Springer, Cham, 2016, pp. 473–483.

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