LEFT DERIVABLE MAPS AT NON-TRIVIAL IDEMPOTENTS ON NEST ALGEBRAS

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Abstract. Let \( \text{Alg}\,N \) be a nest algebra associated with the nest \( N \) on a (real or complex) Banach space \( X \). Suppose that there exists a non-trivial idempotent \( P \in \text{Alg}\,N \) with range \( P(X) \in N \), and \( \delta: \text{Alg}\,N \to \text{Alg}\,N \) is a continuous linear mapping (generalized) left derivable at \( P \), i.e. \( \delta(ab) = a\delta(b) + b\delta(a) \) \( (\delta(ab) = a\delta(b) + b\delta(a) - ba\delta(I)) \) for any \( a, b \in \text{Alg}\,N \) with \( ab = P \), where \( I \) is the identity element of \( \text{Alg}\,N \). We show that \( \delta \) is a (generalized) Jordan left derivation. Moreover, in a strongly operator topology we characterize continuous linear maps \( \delta \) on some nest algebras \( \text{Alg}\,N \) with the property that \( \delta(P) = 2P\delta(P) \) or \( \delta(P) = 2P\delta(P) - P\delta(I) \) for every idempotent \( P \) in \( \text{Alg}\,N \).

1. Introduction

Throughout this paper, all algebras and vector spaces will be over \( \mathbb{F} \), where \( \mathbb{F} \) is either the real field \( \mathbb{R} \) or the complex field \( \mathbb{C} \). Let \( A \) be an algebra with unity \( 1 \), \( M \) be a left \( A \)-module and \( \delta: A \to M \) be a linear mapping. The mapping \( \delta \) is said to be a left derivation (or a generalized left derivation) if \( \delta(ab) = a\delta(b) + b\delta(a) \) (or \( \delta(ab) = a\delta(b) + b\delta(a) - ba\delta(1) \)) for all \( a, b \in A \). It is called a Jordan left derivation (or a generalized Jordan left derivation) if \( \delta(a^2) = 2a\delta(a) \) (or \( \delta(a^2) = 2a\delta(a) - a^2\delta(1) \)) for any \( a \in A \). Obviously, any (generalized) left derivation is a (generalized) Jordan left derivation, but in general the converse is not true (see [15] Example 1.1). The concepts of left derivation and Jordan left derivation were introduced by Brešar and Vukman.

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in [4]. For results concerning left derivations and Jordan left derivations we refer the readers to [10] and the references therein.

In recent years, several authors studied the linear (additive) maps that behave like homomorphisms, derivations or left derivations when acting on special products (for instance, see [3, 7, 9, 6, 11, 12, 16] and the references therein). In this article we study the linear maps on nest algebras behaving like left derivations at idempotent-product elements.

Let $A$ be an algebra with unity $1$, $M$ be a left $A$-module and $\delta : A \rightarrow M$ be a linear mapping. We say that $\delta$ is left derivable (or generalized left derivable) at a given point $z \in A$ if $\delta(ab) = a\delta(b) + b\delta(a)$ (or $\delta(ab) = a\delta(b) + b\delta(a) - ba\delta(1)$) for any $a, b \in A$ with $ab = z$. In this paper, we characterize the continuous linear maps on nest algebras which are (generalized) left derivable at a non-trivial idempotent operator $P$. Moreover, in a strongly operator topology we describe continuous linear maps $\delta$ on some nest algebra $\text{Alg} N$ with the property that $\delta(P) = 2P\delta(P)$ or $\delta(P) = 2P\delta(P) - P\delta(I)$ for every idempotent $P$ in $\text{Alg} N$, where $I$ is the identity element of $\text{Alg} N$.

The following are the notations and terminologies which are used throughout this article.

Let $X$ be a Banach space. We denote by $\mathcal{B}(X)$ the algebra of all bounded linear operators on $X$, and $\mathcal{F}(X)$ denotes the algebra of all finite rank operators in $\mathcal{B}(X)$. A subspace lattice $\mathcal{L}$ on a Banach space $X$ is a collection of closed (under norm topology) subspaces of $X$ which is closed under the formation of arbitrary intersection and closed linear span (denoted by $\vee$), and which includes $\{0\}$ and $X$. For a subspace lattice $\mathcal{L}$, we define $\text{Alg} \mathcal{L}$ by

$$\text{Alg} \mathcal{L} = \{ T \in \mathcal{B}(X) \mid T(N) \subseteq N \text{ for all } N \in \mathcal{L} \}.$$ 

A totally ordered subspace lattice $\mathcal{N}$ on $X$ is called a nest and $\text{Alg} \mathcal{N}$ is called a nest algebra. When $\mathcal{N} \neq \{\{0\}, X\}$, we say that $\mathcal{N}$ is non-trivial. It is clear that if $\mathcal{N}$ is trivial, then $\text{Alg} \mathcal{N} = \mathcal{B}(X)$. Denote $\text{Alg}_F \mathcal{N} := \text{Alg} \mathcal{N} \cap \mathcal{F}(X)$, the set of all finite rank operators in $\text{Alg} \mathcal{N}$ and for $N \in \mathcal{N}$, let $N_- = \vee\{ M \in \mathcal{N} \mid M \subset N \}$. The identity element of a nest algebra will be denoted by $I$. An element $P$ in a nest algebra is called a non-trivial idempotent if $P \neq 0, I$ and $P^2 = P$.

Let $N$ be a non-trivial nest on a Banach space $X$. If there exists a non-trivial idempotent $P \in \text{Alg} \mathcal{N}$ with range $P(X) \in \mathcal{N}$, then we have $(I - P)(\text{Alg} \mathcal{N})P = \{0\}$ and hence

$$\text{Alg} \mathcal{N} = P(\text{Alg} \mathcal{N})P \perp P(\text{Alg} \mathcal{N})(I - P) \perp (I - P)(\text{Alg} \mathcal{N})(I - P)$$

as sum of linear spaces. This is so-called Peirce decomposition of $\text{Alg} \mathcal{N}$. The sets $P(\text{Alg} \mathcal{N})P$, $P(\text{Alg} \mathcal{N})(I - P)$ and $(I - P)(\text{Alg} \mathcal{N})(I - P)$ are closed.
in \( \text{Alg} \mathcal{N} \). In fact, \( P(\text{Alg} \mathcal{N})P \) and \( (I - P)(\text{Alg} \mathcal{N})(I - P) \) are Banach sub-algebras of \( \text{Alg} \mathcal{N} \) whose unit elements are \( P \) and \( I - P \), respectively and \( P(\text{Alg} \mathcal{N})(I - P) \) is a Banach \( (P(\text{Alg} \mathcal{N})P, (I - P)(\text{Alg} \mathcal{N})(I - P)) \)-bimodule. Also \( P(\text{Alg} \mathcal{N})(I - P) \) is faithful as a left \( P(\text{Alg} \mathcal{N})P \)-module as well as a right \( (I - P)(\text{Alg} \mathcal{N})(I - P) \)-module. For more information on nest algebras, we refer to \([\text{II}]\).

A subspace lattice \( \mathcal{L} \) on a Hilbert space \( \mathbb{H} \) is called a \textit{commutative subspace lattice}, or a \textit{CSL}, if the projections of \( \mathbb{H} \) onto the subspaces of \( \mathcal{L} \) commute with each other. If \( \mathcal{L} \) is a CSL, then \( \text{Alg} \mathcal{L} \) is called a \textit{CSL algebra}. Each nest algebra on a Hilbert space is a CSL-algebra.

2. Main results

In order to prove our results we need the following result.

\textbf{Theorem 2.1} (\[\text{III}\]). Let \( \mathbb{A} \) be a Banach algebra with unity \( 1 \), \( \mathbb{X} \) be a Banach space and let \( \phi : \mathbb{A} \times \mathbb{A} \to \mathbb{X} \) be a continuous bilinear map with the property that

\[ a, b \in \mathbb{A}, \ ab = 1 \Rightarrow \phi(a, b) = \phi(1, 1). \]

Then

\[ \phi(a, a) = \phi(a^2, 1) \]

for all \( a \in \mathbb{A} \).

\textbf{Proposition 2.2}. Let \( \mathbb{A} \) be a Banach algebra with unity \( 1 \), and \( \mathbb{M} \) be a unital Banach left \( \mathbb{A} \)-module. Let \( \delta : \mathbb{A} \to \mathbb{M} \) be a continuous linear map. If \( \delta \) is left derivable at \( 1 \), then \( \delta \) is a Jordan left derivation.

\textbf{Proof}. Since \( 1 \cdot 1 = 1 \), it follows that \( \delta(1) = 2\delta(1) \) and therefore \( \delta(1) = 0 \). Define a continuous bilinear map \( \phi : \mathbb{A} \times \mathbb{A} \to \mathbb{M} \) by \( \phi(a, b) = a\delta(b) + b\delta(a) \). Then \( \phi(a, b) = \phi(1, 1) \) for all \( a, b \in \mathbb{A} \) with \( ab = 1 \), since \( \delta \) is left derivable at \( 1 \). By applying Theorem \[\text{2.1}\] we obtain \( \phi(a, a) = \phi(a^2, 1) \) for all \( a \in \mathbb{A} \). So,

\[ \delta(a^2) = 2a\delta(a) \quad (a \in \mathbb{A}). \]
Corollary 2.3. Let $\mathbb{A}$ be a Banach algebra with unity 1, and $\mathbb{M}$ be a unital Banach left $\mathbb{A}$-module. Let $x, y \in \mathbb{A}$ with $x + y = 1$ and let $\delta : \mathbb{A} \to \mathbb{M}$ be a continuous linear map. If $\delta$ is left derivable at $x$ and $y$, then $\delta$ is a Jordan left derivation.

Proof. For $a, b \in \mathbb{A}$ with $ab = 1$, we have $abx = x$ and $aby = y$. Since $\delta$ is left derivable at $x$ and $y$, it follows that

$$\delta(x) = \delta(abx) = a\delta(bx) + bx\delta(a)$$

and

$$\delta(y) = \delta(aby) = a\delta(by) + by\delta(a).$$

Combining the two above equations, we get that

$$\delta(1) = \delta(x + y) = a\delta(bx) + bx\delta(a) + a\delta(by) + by\delta(a) = a\delta(b) + b\delta(a),$$

i.e. $\delta$ is left derivable at 1. It follows from Proposition 2.2 that $\delta$ is a Jordan left derivation. $\square$

Remark 2.4. If $\mathbb{A}$ is a CSL-algebra or a unital semisimple Banach algebra, then by [12] and [14] every continuous Jordan left derivation on $\mathbb{A}$ is zero. Hence it follows from Proposition 2.2 that every continuous linear map $\delta : \mathbb{A} \to \mathbb{A}$ which is left derivable at 1 is zero.

The following is our main result.

Theorem 2.5. Let $\mathcal{N}$ be a nest on a Banach space $\mathbb{X}$ such that there exists non-trivial idempotent $P \in \text{Alg}\mathcal{N}$ with range $P(\mathbb{X}) \in \mathcal{N}$. If $\delta : \text{Alg}\mathcal{N} \to \text{Alg}\mathcal{N}$ is a continuous left derivable map at $P$, then $\delta$ is a Jordan left derivation.

Proof. For a notational convenience, we denote $\mathbb{A} = \text{Alg}\mathcal{N}$, $\mathbb{A}_{11} = P\mathbb{A}P$, $\mathbb{A}_{12} = P\mathbb{A}(I - P)$ and $\mathbb{A}_{22} = (I - P)\mathbb{A}(I - P)$. As mentioned in the introduction $\mathbb{A} = \mathbb{A}_{11} + \mathbb{A}_{12} + \mathbb{A}_{22}$. Throughout the proof, $a_{ij}$ and $b_{ij}$ will denote arbitrary elements in $\mathbb{A}_{ij}$ for $1 \leq i, j \leq 2$.

First we show that $\delta(P) = 0$. Since $P^2 = P$, we have $2P\delta(P) = \delta(P)$. It follows from equation $2P\delta(P) = \delta(P)$ that $P\delta(P) = 0$ and it implies that $\delta(P) = 0$.

We complete the proof by verifying the following steps.
Step 1. \( P\delta(a_{11}^2)P = 2a_{11}P\delta(a_{11})P \) and \( P\delta(a_{11}^2)(I-P) = 2a_{11}P\delta(a_{11})(I-P) \).

For any \( a_{11}, b_{11} \) with \( a_{11}b_{11} = P \), we have

\[
a_{11}\delta(b_{11}) + b_{11}\delta(a_{11}) = \delta(P).
\]

Multiplying this identity by \( P \) both from the left and from the right, we find

\[
a_{11}P\delta(b_{11})P + b_{11}P\delta(a_{11})P = P\delta(P)P \quad (a_{11}b_{11} = P).
\]

Define a continuous linear map \( d : \mathbb{A}_{11} \rightarrow \mathbb{A}_{11} \) by \( d(a_{11}) = P\delta(a_{11})P \). By above identity \( d \) is left derivable at \( P \). Hence by Proposition 2.2 \( d \) is a Jordan left derivation, which implies

\[
P\delta(a_{11}^2)P = 2a_{11}P\delta(a_{11})P \quad (a_{11} \in \mathbb{A}_{11}).
\]

By multiplying (2.1) by \( P \) from the left and by \( (I - P) \) from the right, we arrive at

\[
a_{11}P\delta(b_{11})(I - P) + b_{11}P\delta(a_{11})(I - P) = P\delta(P)(I - P) \quad (a_{11}b_{11} = P).
\]

Define a continuous linear map \( D : \mathbb{A}_{11} \rightarrow \mathbb{A}_{12} \) by \( D(a_{11}) = P\delta(a_{11})(I - P) \). It is easy to see that \( D \) is a left derivable at \( P \). It follows from Proposition 2.2 that \( D \) is a Jordan left derivation. Thus,

\[
P\delta(a_{11}^2)(I - P) = 2a_{11}P\delta(a_{11})(I - P) \quad (a_{11} \in \mathbb{A}_{11}).
\]

Step 2. \( P\delta(a_{22}) = 0 \).

Since \((P + a_{22})P = P\), we have

\[
(P + a_{22})\delta(P) + P\delta(P + a_{22}) = \delta(P).
\]

From \( \delta(P) = 0 \) we get

\[
P\delta(a_{22}) = 0.
\]

Step 3. \( P\delta(a_{12}) = 0 \).

Applying \( \delta \) to \((P + a_{12})P = P\), we get

\[
(P + a_{12})\delta(P) + P\delta(P + a_{12}) = \delta(P).
\]

Since \( \delta(P) = 0 \), it follows that

\[
P\delta(a_{12}) = 0.
\]
Step 4. \((I - P)\delta(a_{11})(I - P) = 0\).

For any \(a_{11}, b_{11}\) with \(b_{11}a_{11} = P\), we have \((I - P + b_{11})a_{11} = P\) and hence
\[
(I - P + b_{11})\delta(a_{11}) + a_{11}\delta(I - P + b_{11}) = \delta(P).
\]
Multiplying this identity by \(I - P\) both from the left and from the right we arrive at
\[
(I - P)\delta(a_{11})(I - P) = 0.
\]
Since any element in a Banach algebra with unit element is a sum of its invertible elements (\([\Pi]\)), by the linearity of \(\delta\) and above identity we have
\[
(I - P)\delta(a_{11})(I - P) = 0
\]
for all \(a_{11} \in \mathbb{A}_{11}\).

Step 5. \((I - P)\delta(a_{12})(I - P) = 0\).

Since \((P - a_{12})(I + a_{12}) = P\), it follows that
\[
(P - a_{12})\delta(I + a_{12}) + (I + a_{12})\delta(P - a_{12}) = \delta(P).
\]
Multiplying this identity by \(I - P\) both from the left and from the right and using the fact that \(\delta(P) = 0\), we find
\[
(I - P)\delta(a_{12})(I - P) = 0.
\]

Step 6. \((I - P)\delta(a_{22})(I - P) = 0\).

Applying \(\delta\) to \((P + a_{12})(P - a_{12}a_{22} + a_{22}) = P\), we see that
\[
(P + a_{12})\delta(P - a_{12}a_{22} + a_{22}) + (P - a_{12}a_{22} + a_{22})\delta(P + a_{12}) = \delta(P).
\]
Now, multiplying this identity from the left by \(P\), from the right by \(I - P\) and by Steps 2,3 and 5 and the fact that \(\delta(P) = 0\), we get
\[
a_{12}((I - P)\delta(a_{22})(I - P)) = \{0\}. \tag{1}
\]
From the fact that \(\mathbb{A}_{12}\) is faithful as right \(\mathbb{A}_{22}\)-module, we arrive at
\[
(I - P)\delta(a_{22})(I - P) = 0.
\]
Since \(ab = PaPbP + PaPb(I - P) + P\alpha(I - P)b(I - P) + (I - P)a(I - P)b(I - P)\), for any \(a, b \in \mathbb{A}\), by Steps 1–6, it follows that \(\delta\) is a Jordan left derivation. \(\Box\)
Our next result characterizes the linear mappings on $\text{Alg} \mathcal{N}$ which are generalized left derivable at $P$.

**Theorem 2.6.** Let $\mathcal{N}$ be a nest on a Banach space $X$ such that there exists a non-trivial idempotent $P \in \text{Alg} \mathcal{N}$ with range $P(X) \in \mathcal{N}$. If $\delta: \text{Alg} \mathcal{N} \to \text{Alg} \mathcal{N}$ is a continuous generalized left derivable map at $P$, then $\delta$ is a generalized Jordan left derivation.

**Proof.** Define $\Delta: \text{Alg} \mathcal{N} \to \text{Alg} \mathcal{N}$ by $\Delta(a) = \delta(a) - a\delta(1)$. It is easy to see that $\Delta$ is a continuous left derivable map at $P$. By Theorem 2.5, $\Delta$ is a Jordan left derivation. Therefore

\[
\delta(a^2) = \Delta(a^2) + a^2\delta(1) \\
= 2a\Delta(a) + a^2\delta(1) \\
= 2a(\delta(a) - a\delta(1)) + a^2\delta(1) \\
= 2a\delta(a) - a^2\delta(1)
\]

for all $a \in \text{Alg} \mathcal{N}$. So $\delta$ is a generalized Jordan left derivation. $\square$

Since every continuous Jordan left derivation on a CSL algebra is zero (12), we have the following result.

**Corollary 2.7.** Let $\mathcal{N}$ be a non-trivial nest on a Hilbert space $\mathbb{H}$. Let $P$ be a non-trivial idempotent in $\text{Alg} \mathcal{N}$ with range $P(\mathbb{H}) \in \mathcal{N}$ and $\delta: \text{Alg} \mathcal{N} \to \text{Alg} \mathcal{N}$ be a continuous linear map.

(i) If $\delta$ is left derivable at $P$, then $\delta$ is zero.
(ii) If $\delta$ is generalized left derivable at $P$, then $\delta(a) = a\delta(1)$ for all $a \in \text{Alg} \mathcal{N}$.

**Proof.** (i) Since every continuous Jordan left derivation on a CSL algebra is zero (12), by Theorem 2.5, $\delta$ is zero.

(ii) By Theorem 2.6, $\delta$ is a generalized Jordan left derivation, so the mapping $\Delta: \text{Alg} \mathcal{N} \to \text{Alg} \mathcal{N}$ defined by $\Delta(a) = \delta(a) - a\delta(1)$ is a continuous Jordan left derivation. Therefore $\Delta = 0$ and hence $\delta(a) = a\delta(1)$ for all $a \in \text{Alg} \mathcal{N}$. $\square$

Now, we characterize (generalized) left Jordan derivations which are continuous in the strongly operator topology, but in order to prove our result we must assume an additional (mild) condition concerning the nest $\mathcal{N}$. At present we have no counter-example, so it remains an open problem if this additional condition can be omitted.

The idea of the proof of Proposition 2.8 (i) comes from [2].
Proposition 2.8. Let $N$ be a nest on a Banach space $X$, with each $N \in N$ complemented in $X$ whenever $N_\perp = N$. Let $\delta : \operatorname{Alg} N \to \operatorname{Alg} N$ be a continuous linear map in a strong operator topology.

(i) If $\delta(P) = 2P\delta(P)$ for every idempotent $P$ in $\operatorname{Alg} N$, then $\delta = 0$.

(ii) If $\delta(P) = 2P\delta(P) - P\delta(I)$ for every idempotent $P$ in $\operatorname{Alg} N$, then $\delta(a) = a\delta(I)$ for all $a \in \operatorname{Alg} N$.

Proof. (i) For arbitrary idempotent operator $P \in \operatorname{Alg} N$, by hypothesis we have $\delta(P) = 2P\delta(P)$. It follows from equation $2P\delta(P) = \delta(P)$ that $P\delta(P) = 0$ and it implies that $\delta(P) = 0$.

Notice that $\operatorname{Alg} F N$ is contained in the linear span of the idempotents in $\operatorname{Alg} N$ (see [11]), which implies that $\delta(F) = 0$ for all finite rank operator $F$ in $\operatorname{Alg} N$. Since $\delta$ is continuous under the strong operator topology and $\operatorname{Alg} F N_{\text{SOT}} = \operatorname{Alg} N$ (see [13]), we find that $\delta(a) = 0$ for all $a \in \operatorname{Alg} N$.

(ii) Define $\Delta : \operatorname{Alg} N \to \operatorname{Alg} N$ by $\Delta(a) = \delta(a) - a\delta(I)$. It is easy to see that $\Delta$ is a continuous left map satisfying $\Delta(P) = 2P\Delta(P)$ for every idempotent $P$ in $\operatorname{Alg} N$. So by (i) we have $\Delta = 0$ and hence $\delta(a) = a\delta(I)$ for all $a \in \operatorname{Alg} N$. □

It is obvious that the nests on Hilbert spaces, finite nests and the nests having order-type $\omega + 1$ or $1 + \omega^*$, where $\omega$ is the order-type of the natural numbers, satisfy the condition in Proposition 2.8 automatically.

References

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