LEFT DERIVABLE MAPS AT NON-TRIVIAL IDEMPOTENTS ON NEST ALGEBRAS

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Abstract. Let $\text{Alg} \mathcal{N}$ be a nest algebra associated with the nest $\mathcal{N}$ on a (real or complex) Banach space $X$. Suppose that there exists a non-trivial idempotent $P \in \text{Alg} \mathcal{N}$ with range $P(X) \in \mathcal{N}$, and $\delta : \text{Alg} \mathcal{N} \to \text{Alg} \mathcal{N}$ is a continuous linear mapping (generalized) left derivable at $P$, i.e. $\delta(ab) = a\delta(b) + b\delta(a)$ $(\delta(ab) = a\delta(b) + b\delta(a) - ba\delta(I))$ for any $a,b \in \text{Alg} \mathcal{N}$ with $ab = P$, where $I$ is the identity element of $\text{Alg} \mathcal{N}$. We show that $\delta$ is a (generalized) Jordan left derivation. Moreover, in a strongly operator topology we characterize continuous linear maps $\delta$ on some nest algebras $\text{Alg} \mathcal{N}$ with the property that $\delta(P) = 2P\delta(P)$ or $\delta(P) = 2P\delta(P) - P\delta(I)$ for every idempotent $P$ in $\text{Alg} \mathcal{N}$.

1. Introduction

Throughout this paper, all algebras and vector spaces will be over $\mathbb{F}$, where $\mathbb{F}$ is either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. Let $A$ be an algebra with unity $1$, $M$ be a left $A$-module and $\delta : A \to M$ be a linear mapping. The mapping $\delta$ is said to be a left derivation (or a generalized left derivation) if $\delta(ab) = a\delta(b) + b\delta(a)$ (or $\delta(ab) = a\delta(b) + b\delta(a) - ba\delta(1)$) for all $a,b \in A$. It is called a Jordan left derivation (or a generalized Jordan left derivation) if $\delta(a^2) = 2a\delta(a)$ (or $\delta(a^2) = 2a\delta(a) - a^2\delta(1)$) for any $a \in A$. Obviously, any (generalized) left derivation is a (generalized) Jordan left derivation, but in general the converse is not true (see [15] Example 1.1). The concepts of left derivation and Jordan left derivation were introduced by Brešar and Vukman.
in [4]. For results concerning left derivations and Jordan left derivations we refer the readers to [10] and the references therein.

In recent years, several authors studied the linear (additive) maps that behave like homomorphisms, derivations or left derivations when acting on special products (for instance, see [3, 7, 9, 6, 11, 12, 16] and the references therein). In this article we study the linear maps on nest algebras behaving like left derivations at idempotent-product elements.

Let \( \mathbb{A} \) be an algebra with unity \( 1 \), \( \mathbb{M} \) be a left \( \mathbb{A} \)-module and \( \delta: \mathbb{A} \to \mathbb{M} \) be a linear mapping. We say that \( \delta \) is left derivable (or generalized left derivable) at a given point \( z \in \mathbb{A} \) if

\[
\delta(ab) = a\delta(b) + b\delta(a) \quad \text{or} \quad \delta(ab) = a\delta(b) + b\delta(a) - ba\delta(1)
\]

for any \( a, b \in \mathbb{A} \) with \( ab = z \). In this paper, we characterize the continuous linear maps on nest algebras which are (generalized) left derivable at a non-trivial idempotent operator \( P \). Moreover, in a strongly operator topology we describe continuous linear maps \( \delta \) on some nest algebra \( \text{Alg} \mathcal{N} \) with the property that

\[
\delta(P) = 2P\delta(P) \quad \text{or} \quad \delta(P) = 2P\delta(P) - P\delta(I)
\]

for every idempotent \( P \) in \( \text{Alg} \mathcal{N} \), where \( I \) is the identity element of \( \text{Alg} \mathcal{N} \).

The following are the notations and terminologies which are used throughout this article.

Let \( \mathbb{X} \) be a Banach space. We denote by \( \mathcal{B}(\mathbb{X}) \) the algebra of all bounded linear operators on \( \mathbb{X} \), and \( \mathcal{F}(\mathbb{X}) \) denotes the algebra of all finite rank operators in \( \mathcal{B}(\mathbb{X}) \). A subspace lattice \( \mathcal{L} \) on a Banach space \( \mathbb{X} \) is a collection of closed (under norm topology) subspaces of \( \mathbb{X} \) which is closed under the formation of arbitrary intersection and closed linear span (denoted by \( \lor \)), and which includes \( \{0\} \) and \( \mathbb{X} \). For a subspace lattice \( \mathcal{L} \), we define \( \text{Alg} \mathcal{L} \) by

\[
\text{Alg} \mathcal{L} = \{ T \in \mathcal{B}(\mathbb{X}) \mid T(N) \subseteq N \text{ for all } N \in \mathcal{L} \}.
\]

A totally ordered subspace lattice \( \mathcal{N} \) on \( \mathbb{X} \) is called a nest and \( \text{Alg} \mathcal{N} \) is called a nest algebra. When \( \mathcal{N} \neq \{\{0\}, \mathbb{X}\} \), we say that \( \mathcal{N} \) is non-trivial. It is clear that if \( \mathcal{N} \) is trivial, then \( \text{Alg} \mathcal{N} = \mathcal{B}(\mathbb{X}) \). Denote \( \text{Alg}_{\mathcal{F}} \mathcal{N} := \text{Alg} \mathcal{N} \cap \mathcal{F}(\mathbb{X}) \), the set of all finite rank operators in \( \text{Alg} \mathcal{N} \) and for \( N \in \mathcal{N} \), let \( N_- = \lor \{ M \in \mathcal{N} \mid M \subset N \} \). The identity element of a nest algebra will be denoted by \( I \). An element \( P \) in a nest algebra is called a non-trivial idempotent if \( P \neq 0, I \) and \( P^2 = P \).

Let \( \mathcal{N} \) be a non-trivial nest on a Banach space \( \mathbb{X} \). If there exists a non-trivial idempotent \( P \in \text{Alg} \mathcal{N} \) with range \( P(\mathbb{X}) \in \mathcal{N} \), then we have \( (I - P)(\text{Alg} \mathcal{N})P = \{0\} \) and hence

\[
\text{Alg} \mathcal{N} = P(\text{Alg} \mathcal{N})P + P(\text{Alg} \mathcal{N})(I - P) + (I - P)(\text{Alg} \mathcal{N})(I - P)
\]

as sum of linear spaces. This is so-called Peirce decomposition of \( \text{Alg} \mathcal{N} \). The sets \( P(\text{Alg} \mathcal{N})P \), \( P(\text{Alg} \mathcal{N})(I - P) \) and \( (I - P)(\text{Alg} \mathcal{N})(I - P) \) are closed.
in $\text{Alg}\mathcal{N}$. In fact, $P(\text{Alg}\mathcal{N})P$ and $(I - P)(\text{Alg}\mathcal{N})(I - P)$ are Banach sub-algebras of $\text{Alg}\mathcal{N}$ whose unit elements are $P$ and $I - P$, respectively and $P(\text{Alg}\mathcal{N})(I - P)$ is a Banach $(P(\text{Alg}\mathcal{N})P, (I - P)(\text{Alg}\mathcal{N})(I - P))$-bimodule. Also $P(\text{Alg}\mathcal{N})(I - P)$ is faithful as a left $P(\text{Alg}\mathcal{N})P$-module as well as a right $(I - P)(\text{Alg}\mathcal{N})(I - P)$-module. For more information on nest algebras, we refer to [3].

A subspace lattice $\mathcal{L}$ on a Hilbert space $\mathbb{H}$ is called a commutative subspace lattice, or a CSL, if the projections of $\mathbb{H}$ onto the subspaces of $\mathcal{L}$ commute with each other. If $\mathcal{L}$ is a CSL, then $\text{Alg}\mathcal{L}$ is called a CSL algebra. Each nest algebra on a Hilbert space is a CSL-algebra.

2. Main results

In order to prove our results we need the following result.

THEOREM 2.1 ([8]). Let $\mathbb{A}$ be a Banach algebra with unity $1$, $\mathbb{X}$ be a Banach space and let $\phi : \mathbb{A} \times \mathbb{A} \to \mathbb{X}$ be a continuous bilinear map with the property that

$$a, b \in \mathbb{A}, \; ab = 1 \Rightarrow \phi(a, b) = \phi(1, 1).$$

Then

$$\phi(a, a) = \phi(a^2, 1)$$

for all $a \in \mathbb{A}$.

PROPOSITION 2.2. Let $\mathbb{A}$ be a Banach algebra with unity $1$, and $\mathbb{M}$ be a unital Banach left $\mathbb{A}$-module. Let $\delta : \mathbb{A} \to \mathbb{M}$ be a continuous linear map. If $\delta$ is left derivable at $1$, then $\delta$ is a Jordan left derivation.

Proof. Since $1 \cdot 1 = 1$, it follows that $\delta(1) = 2\delta(1)$ and therefore $\delta(1) = 0$. Define a continuous bilinear map $\phi : \mathbb{A} \times \mathbb{A} \to \mathbb{M}$ by $\phi(a, b) = a\delta(b) + b\delta(a)$. Then $\phi(a, b) = \phi(1, 1)$ for all $a, b \in \mathbb{A}$ with $ab = 1$, since $\delta$ is left derivable at 1. By applying Theorem 2.1 we obtain $\phi(a, a) = \phi(a^2, 1)$ for all $a \in \mathbb{A}$. So,

$$\delta(a^2) = 2a\delta(a) \quad (a \in \mathbb{A}).$$

□
Corollary 2.3. Let $\mathbb{A}$ be a Banach algebra with unity $1$, and $\mathbb{M}$ be a unital Banach left $\mathbb{A}$-module. Let $x, y \in \mathbb{A}$ with $x + y = 1$ and let $\delta : \mathbb{A} \to \mathbb{M}$ be a continuous linear map. If $\delta$ is left derivable at $x$ and $y$, then $\delta$ is a Jordan left derivation.

Proof. For $a, b \in \mathbb{A}$ with $ab = 1$, we have $abx = x$ and $aby = y$. Since $\delta$ is left derivable at $x$ and $y$, it follows that

$$\delta(x) = \delta(abx) = a\delta(bx) + bx\delta(a)$$

and

$$\delta(y) = \delta(aby) = a\delta(by) + by\delta(a).$$

Combining the two above equations, we get that

$$\delta(1) = \delta(x + y) = a\delta(bx) + bx\delta(a) + a\delta(by) + by\delta(a) = a\delta(b) + b\delta(a),$$

i.e. $\delta$ is left derivable at $1$. It follows from Proposition 2.2 that $\delta$ is a Jordan left derivation.

Remark 2.4. If $\mathbb{A}$ is a CSL-algebra or a unital semisimple Banach algebra, then by [12] and [14] every continuous Jordan left derivation on $\mathbb{A}$ is zero. Hence it follows from Proposition 2.2 that every continuous linear map $\delta : \mathbb{A} \to \mathbb{A}$ which is left derivable at $1$ is zero.

The following is our main result.

Theorem 2.5. Let $\mathcal{N}$ be a nest on a Banach space $X$ such that there exists non-trivial idempotent $P \in \text{Alg} \mathcal{N}$ with range $P(X) \in \mathcal{N}$. If $\delta : \text{Alg} \mathcal{N} \to \text{Alg} \mathcal{N}$ is a continuous left derivable map at $P$, then $\delta$ is a Jordan left derivation.

Proof. For a notational convenience, we denote $\mathbb{A} = \text{Alg} \mathcal{N}$, $\mathbb{A}_{11} = \text{P} \hat{\mathbb{A}} P$, $\mathbb{A}_{12} = \text{P} \hat{\mathbb{A}} (I - P)$ and $\mathbb{A}_{22} = (I - P) \hat{\mathbb{A}} (I - P)$. As mentioned in the introduction $\mathbb{A} = \mathbb{A}_{11} + \mathbb{A}_{12} + \mathbb{A}_{22}$. Throughout the proof, $a_{ij}$ and $b_{ij}$ will denote arbitrary elements in $\mathbb{A}_{ij}$ for $1 \leq i, j \leq 2$.

First we show that $\delta(P) = 0$. Since $P^2 = P$, we have $2P\delta(P) = \delta(P)$. It follows from equation $2P\delta(P) = \delta(P)$ that $P\delta(P) = 0$ and it implies that $\delta(P) = 0$.

We complete the proof by verifying the following steps.
Step 1. \( P\delta(a_{11}^2)P = 2a_{11}P\delta(a_{11})P \) and \( P\delta(a_{11}^2)(I-P) = 2a_{11}P\delta(a_{11})(I-P) \).

For any \( a_{11}, b_{11} \) with \( a_{11}b_{11} = P \), we have

\[
(2.1) \quad a_{11}\delta(b_{11}) + b_{11}\delta(a_{11}) = \delta(P).
\]

Multiplying this identity by \( P \) both from the left and from the right, we find

\[
a_{11}P\delta(b_{11})P + b_{11}P\delta(a_{11})P = P\delta(P)P \quad (a_{11}b_{11} = P).
\]

Define a continuous linear map \( d: \mathbb{A}_{11} \to \mathbb{A}_{11} \) by \( d(a_{11}) = P\delta(a_{11})P \). By above identity \( d \) is left derivable at \( P \). Hence by Proposition \( 2.2 \), \( d \) is a Jordan left derivation, which implies

\[
P\delta(a_{11}^2)P = 2a_{11}P\delta(a_{11})P \quad (a_{11} \in \mathbb{A}_{11}).
\]

By multiplying \((2.1)\) by \( P \) from the left and by \((I-P)\) from the right, we arrive at

\[
a_{11}P\delta(b_{11})(I-P) + b_{11}P\delta(a_{11})(I-P) = P\delta(P)(I-P) \quad (a_{11}b_{11} = P).
\]

Define a continuous linear map \( D: \mathbb{A}_{11} \to \mathbb{A}_{12} \) by \( D(a_{11}) = P\delta(a_{11})(I-P) \). It is easy to see that \( D \) is a left derivable at \( P \). It follows from Proposition \( 2.2 \) that \( D \) is a Jordan left derivation. Thus,

\[
P\delta(a_{11}^2)(I-P) = 2a_{11}P\delta(a_{11})(I-P) \quad (a_{11} \in \mathbb{A}_{11}).
\]

Step 2. \( P\delta(a_{22}) = 0 \).

Since \((P + a_{22})P = P\), we have

\[
(P + a_{22})\delta(P) + P\delta(P + a_{22}) = \delta(P).
\]

From \( \delta(P) = 0 \) we get

\[
P\delta(a_{22}) = 0.
\]

Step 3. \( P\delta(a_{12}) = 0 \).

Applying \( \delta \) to \((P + a_{12})P = P\), we get

\[
(P + a_{12})\delta(P) + P\delta(P + a_{12}) = \delta(P).
\]

Since \( \delta(P) = 0 \), it follows that

\[
P\delta(a_{12}) = 0.
\]
Step 4. \((I - P)\delta(a_{11})(I - P) = 0\).

For any \(a_{11}, b_{11}\) with \(b_{11}a_{11} = P\), we have \((I - P + b_{11})a_{11} = P\) and hence
\[
(I - P + b_{11})\delta(a_{11}) + a_{11}\delta(I - P + b_{11}) = \delta(P).
\]

Multiplying this identity by \(I - P\) both from the left and from the right we arrive at
\[
(I - P)\delta(a_{11})(I - P) = 0.
\]

Since any element in a Banach algebra with unit element is a sum of its invertible elements ([1]), by the linearity of \(\delta\) and above identity we have
\[
(I - P)\delta(a_{11})(I - P) = 0
\]
for all \(a_{11} \in \mathbb{A}_{11}\).

Step 5. \((I - P)\delta(a_{12})(I - P) = 0\).

Since \((P - a_{12})(I + a_{12}) = P\), it follows that
\[
(P - a_{12})\delta(I + a_{12}) + (I + a_{12})\delta(P - a_{12}) = \delta(P).
\]

Multiplying this identity by \(I - P\) both from the left and from the right and using the fact that \(\delta(P) = 0\), we find
\[
(I - P)\delta(a_{12})(I - P) = 0.
\]

Step 6. \((I - P)\delta(a_{22})(I - P) = 0\).

Applying \(\delta\) to \((P + a_{12})(P - a_{12}a_{22} + a_{22}) = P\), we see that
\[
(P + a_{12})\delta(P - a_{12}a_{22} + a_{22}) + (P - a_{12}a_{22} + a_{22})\delta(P + a_{12}) = \delta(P).
\]

Now, multiplying this identity from the left by \(P\), from the right by \(I - P\) and by Steps 2,3 and 5 and the fact that \(\delta(P) = 0\), we get \(a_{12}(I - P)\delta(a_{22})(I - P) = 0\). Since \(a_{12} \in \mathbb{A}_{12}\) is arbitrary, we have \(\mathbb{A}_{12}((I - P)\delta(a_{22})(I - P)) = \{0\}\). From the fact that \(\mathbb{A}_{12}\) is faithful as right \(\mathbb{A}_{22}\)-module, we arrive at
\[
(I - P)\delta(a_{22})(I - P) = 0.
\]

Since \(ab = PaPbP + PaPb(I - P) + Pa(I - P)b(I - P) + (I - P)a(I - P)b(I - P)\), for any \(a, b \in \mathbb{A}\), by Steps 1–6, it follows that \(\delta\) is a Jordan left derivation. \(\square\)
Our next result characterizes the linear mappings on $Alg\mathcal{N}$ which are generalized left derivable at $P$.

**Theorem 2.6.** Let $\mathcal{N}$ be a nest on a Banach space $X$ such that there exists a non-trivial idempotent $P \in Alg\mathcal{N}$ with range $P(X) \in \mathcal{N}$. If $\delta : Alg\mathcal{N} \to Alg\mathcal{N}$ is a continuous generalized left derivable map at $P$, then $\delta$ is a generalized Jordan left derivation.

**Proof.** Define $\Delta : Alg\mathcal{N} \to Alg\mathcal{N}$ by $\Delta(a) = \delta(a) - a\delta(1)$. It is easy to see that $\Delta$ is a continuous left derivable map at $P$. By Theorem 2.5, $\Delta$ is a Jordan left derivation. Therefore

$$\delta(a^2) = \Delta(a^2) + a^2\delta(1)$$

$$= 2a\Delta(a) + a^2\delta(1)$$

$$= 2a(\delta(a) - a\delta(1)) + a^2\delta(1)$$

$$= 2a\delta(a) - a^2\delta(1)$$

for all $a \in Alg\mathcal{N}$. So $\delta$ is a generalized Jordan left derivation. $\square$

Since every continuous Jordan left derivation on a CSL algebra is zero (12), we have the following result.

**Corollary 2.7.** Let $\mathcal{N}$ be a non-trivial nest on a Hilbert space $\mathbb{H}$. Let $P$ be a non-trivial idempotent in $Alg\mathcal{N}$ with range $P(\mathbb{H}) \in \mathcal{N}$ and $\delta : Alg\mathcal{N} \to Alg\mathcal{N}$ be a continuous linear map.

(i) If $\delta$ is left derivable at $P$, then $\delta$ is zero.

(ii) If $\delta$ is generalized left derivable at $P$, then $\delta(a) = a\delta(1)$ for all $a \in Alg\mathcal{N}$.

**Proof.** (i) Since every continuous Jordan left derivation on a CSL algebra is zero (12), by Theorem 2.5, $\delta$ is zero.

(ii) By Theorem 2.6, $\delta$ is a generalized Jordan left derivation, so the mapping $\Delta : Alg\mathcal{N} \to Alg\mathcal{N}$ defined by $\Delta(a) = \delta(a) - a\delta(1)$ is a continuous Jordan left derivation. Therefore $\Delta = 0$ and hence $\delta(a) = a\delta(1)$ for all $a \in Alg\mathcal{N}$. $\square$

Now, we characterize (generalized) left Jordan derivations which are continuous in the strongly operator topology, but in order to prove our result we must assume an additional (mild) condition concerning the nest $\mathcal{N}$. At present we have no counter-example, so it remains an open problem if this additional condition can be omitted.

The idea of the proof of Proposition 2.8 (i) comes from [2].
Proposition 2.8. Let $\mathcal{N}$ be a nest on a Banach space $\mathbb{X}$, with each $N \in \mathcal{N}$ complemented in $\mathbb{X}$ whenever $N_\perp = N$. Let $\delta : \text{Alg}\, \mathcal{N} \to \text{Alg}\, \mathcal{N}$ be a continuous linear map in a strong operator topology.

(i) If $\delta(P) = 2P\delta(P)$ for every idempotent $P$ in $\text{Alg}\, \mathcal{N}$, then $\delta = 0$.

(ii) If $\delta(P) = 2P\delta(P) - P\delta(I)$ for every idempotent $P$ in $\text{Alg}\, \mathcal{N}$, then $\delta(a) = a\delta(I)$ for all $a \in \text{Alg}\, \mathcal{N}$.

Proof. (i) For arbitrary idempotent operator $P \in \text{Alg}\, \mathcal{N}$, by hypothesis we have $\delta(P) = 2P\delta(P)$. It follows from equation $2P\delta(P) = \delta(P)$ that $P\delta(P) = 0$ and it implies that $\delta(P) = 0$.

Notice that $\text{Alg}_{\mathbb{F}} \mathcal{N}$ is contained in the linear span of the idempotents in $\text{Alg}\, \mathcal{N}$ (see [11]), which implies that $\delta(F) = 0$ for all finite rank operator $F$ in $\text{Alg}\, \mathcal{N}$. Since $\delta$ is continuous under the strong operator topology and $\text{Alg}_{\mathbb{F}} \mathcal{N}^{\text{SOT}} = \text{Alg}\, \mathcal{N}$ (see [13]), we find that $\delta(a) = 0$ for all $a \in \text{Alg}\, \mathcal{N}$.

(ii) Define $\Delta : \text{Alg}\, \mathcal{N} \to \text{Alg}\, \mathcal{N}$ by $\Delta(a) = \delta(a) - a\delta(I)$. It is easy to see that $\Delta$ is a continuous left map satisfying $\Delta(P) = 2P\Delta(P)$ for every idempotent $P$ in $\text{Alg}\, \mathcal{N}$. So by (i) we have $\Delta = 0$ and hence $\delta(a) = a\delta(I)$ for all $a \in \text{Alg}\, \mathcal{N}$. □

It is obvious that the nests on Hilbert spaces, finite nests and the nests having order-type $\omega + 1$ or $1 + \omega^*$, where $\omega$ is the order-type of the natural numbers, satisfy the condition in Proposition 2.8 automatically.

References


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